- Market/Economics behind growth models
- R&D/Externalities.

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**More on R&D**

- Externalities < Positive Negative
- ≠ Private rewards
  - Society

---

- True/Average evolution of technology may be ≠ than the one perceived by the agent.

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**Set up for model with endogenous tech. change**

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**General things**

- Product of final good: LY
- Labor

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Evolution of technology: A_t (aggregate)
FUNCTION OF LA & PARAMETERS/FUNCTIONS

\( A_t = \text{function of } LA \)

"STANDING ON SHOULDERS OF GIANTS" (POSITIVE EXTERNALITY)

"STEPPING ON TOES" (NEGATIVE EXTERNALITY)

NEED: TRACTABLE: SOLVING THE MODEL IS POSSIBLE + BGP? SIMULATIONS

MARKET FEASIBLE/MICROFOUNDATIONS

AGGREGATE MODEL

1. \( Y_t = k_t \cdot (A_t \cdot L_{yt})^{1-\lambda} \)
2. \( k_t = r_k \cdot Y_t - d \cdot k_t \)
3. \( \hat{k}_t = r_k \cdot Y_t - d \cdot \hat{k}_t \)
4. \( L_y + L_A = L, \hat{L}_t = \gamma \)
5. \( \hat{A}_t = \delta \cdot A_{t-1} \cdot L_{A_t} \)

\[ \hat{A}_t = \delta \cdot A_{t-1} \cdot L_{A_t} \]

PARAMETERS

\( 0 < \delta < 1 \) (DEPRECIATION)
\( 0 < r_k < 1 \) (PHYSICAL INVESTMENT RATE)
\( 0 < \gamma < 1 \)
\( \lambda > 0 \)
\( 0 < \delta < 1 \)
\[ A_t = \frac{S \cdot L A_t}{A_t^{1-\phi}} \]

**Idea is:**

- + Returns to N&D \((0 < \lambda < 1)\)
  on "stepping on toes"
- \(\phi > 0 \Rightarrow \text{standing of shoulders...}"

**But**

- \(\phi < 1 \Rightarrow \)
  "fishing" closer to frontier it becomes harder to move forward.

**Notation:**

\[ R_y = \frac{L_y}{L} = \% \text{ of workers in final good production} \]

\[ R_R = \frac{L_A}{L} = \% \text{ of workers in red sector.} \]

Since \(L_A + L_y = L \Rightarrow \frac{L_A}{L} + \frac{L_y}{L} = 1\)

\[ \Rightarrow R_R + R_y = 1 \]

\[ \Rightarrow R_y = (1 - R_R) \]
We will assume \( q_A, q_Y \) constants in the first analysis.

We can still get policy implications.

**Summary**

1. \( y_t = k_t^{1-q} (A_t^{-1} y_t)^{1-q} = k_t^{1-q} (A_t^{-1} (1-q) y_t) \)
2. \( \hat{R}_t = \frac{\alpha_k}{k_t} \cdot \frac{y_t}{k_t} - d \)
3. \( \hat{A}_t = \frac{1}{A_t^{-1} - \phi} \)
4. \( \hat{A}_t = \frac{\hat{A}_t (1 - \alpha)}{A_t^{-1} - \phi} \)
5. \( y_t = \hat{A}_t = \frac{1}{\hat{A}_t} \)
6. \( \hat{L}_t = \hat{A}_t = \hat{A}_t = \hat{L}_t \)

**Next Steps:**

- Characterize BGP
- Modified System
- Two Important Diagrams
- Disturbances to BGP
- Approaches
CHARACTERIZE BCP

- DEF.: AT BCP:

$R_t$, $A_t$, $L_t$, $C_{At}$, $C_{L_t}$

CONSTANT

USE THIS + EQUATIONS (1) - (9) TO GET

$T_t$ BCP
$A_t$ BCP
$T_t$ BCP

As function of parameters:

$R_k$, $\lambda$
$\eta$
$S$
$d$
$\phi$
Using BGP definition

\[ E(3) \Rightarrow \frac{Y_e}{k_t} \text{ constant} \]

\[ \Rightarrow \left( \frac{Y_e}{k_t} \right)_{\text{BCP}} = 0 \Rightarrow Y_e = k_t \]

Using BGP definition

\[ E(6) \Rightarrow \hat{A}_{\text{BCP}} \text{ constant} \]

\[ \Rightarrow \left( \frac{S \hat{A}_t}{A_t} \right)_{\text{BCP}} \text{ constant} \]

Since \( S \) constant

\[ \Rightarrow \left( \frac{S \hat{A}_t}{A_t} \right)_{\text{BCP}} = \left( A_t \right)_{\text{BCP}} \text{ constant} \]

Using that algebra:

\[ \lambda \left( L_{\text{At}} \right) = (1-\phi) \left( A_t \right)_{\text{BCP}} \]

\[ \Rightarrow \hat{A}_{\text{BCP}} = \frac{\lambda \eta}{1-\phi} \]
We need to take/use hat algebra on (1):

\[ y_t = k_t^2 + (A_t \cdot L y_t)^{-1} \]

\[ \Rightarrow = \alpha k_t + 1 - \alpha [A_t^2 + \frac{1}{\alpha}] \]

\[ = \alpha k + (1 - \alpha) [A_t^2 + \frac{1}{\alpha}] \]

At BGP by Eq. (20): \[ k_{BCP} = y_{BCP} \]

\[ \Rightarrow A^2_{BCP} (1 - \alpha) = (1 - \alpha) [A^2_{BCP} + \frac{1}{\alpha}] \]

Using (21):

\[ A^2_{BCP} = \frac{1}{1 - \phi} \]

Result:

\[ \hat{A}_{BCP} = A_{BCP} \]

\[ \hat{A}_{BCP} = \frac{1}{1 - \phi} \]

What is the modified system?

i.e. Find a system that has a s.s. coinciding with
BCP of Original system

Proposed variables for modified system:

\[
\frac{Y_t}{A_t \cdot L_t} = \frac{K_t}{A_t \cdot L_t} \]

where

\[
Y_t = Y_t \quad L_t = L_t \quad K_t = K_t
\]

Check that at BGP of original the modified has a S.S.S.

\[
\hat{M_t} = \hat{Y_t} - \hat{A_t} - \hat{L_t} = \hat{Y_t} - \hat{A_t} - \hat{L_t}
\]

At BGP:

\[
\begin{align*}
\hat{M_t} &= \hat{A_t} - \hat{A_t} - \hat{L_t} - \hat{A_t} - \hat{L_t} - \hat{A_t} - \hat{L_t} \\
&= 0
\end{align*}
\]

Remarks:

At BGP

\[
\hat{M_t} = \left( \frac{\hat{Y_t}}{\hat{L_t}} \right) = \hat{Y_t} \quad \text{where}\]

\[
\frac{\hat{Y_t}}{\hat{L_t}} = \frac{\hat{A_t}}{\hat{A_t}}
\]

\[\text{rate of tech. change}\]

6) Notice larger \( \gamma \Rightarrow \hat{A_t} \] BCP why? since \( \frac{\hat{Y_t}}{\hat{L_t}} = \frac{\hat{A_t}}{\hat{A_t}} \) constant
MORE PEOPLE $\Rightarrow$ MORE RED WORKERS $\Rightarrow$ TECHNOLOGY

c) ALSO $\phi A \Rightarrow A_{t}^{BCP}$
   "LESS "STEPPING ON TOES"

d) $S$ DOES NOT INFLUENCE $A_{t}^{BCP}$

e) $\phi \Rightarrow 1 - \phi + A_{t}^{BCP}$

---

**Modified System**

$$\tilde{N}_{t} = \frac{Y_{t}}{A_{t}L_{t}}$$

**System**

$$\tilde{L}_{t} = \frac{k_{t}}{A_{t}L_{t}}$$

**Prod. Function**

Divide (1) by $A_{t}L_{t}$

$$\frac{Y_{t}}{A_{t}L_{t}} = \frac{k_{t}^{L_{t}} (A_{t}L_{t})^{1 - L_{t}}}{A_{t}L_{t}} = \frac{k_{t}^{L_{t}} (A_{t}L_{t})^{1 - L_{t}}}{A_{t}L_{t}}$$

$$\frac{N_{t}}{L_{t}} = \frac{\tilde{N}_{t}^{L_{t}}}{L_{t}} = \frac{\tilde{N}_{t}^{L_{t}}}{L_{t}}$$

$$\frac{N_{t}}{L_{t}} = \frac{\tilde{N}_{t}^{L_{t}}}{L_{t}} = \frac{\tilde{N}_{t}^{L_{t}}}{L_{t}}$$

$$\tilde{N}_{t}^{L_{t}} = \frac{\tilde{N}_{t}^{L_{t}}}{L_{t}}$$

$$\tilde{N}_{t}^{L_{t}} = \frac{\tilde{N}_{t}^{L_{t}}}{L_{t}}$$
LAW OF MOTION OF $\hat{x}_t$

\[
\hat{x}_t = \left( \frac{K_t}{A_t L_t} \right) \hat{x}_t - (A_t + \hat{I}_t)
\]

Using (3):

\[
= q_k \left( \frac{y_t}{A_t L_t} \right) \hat{x}_t - d - A_t - \gamma
\]

Using (30):

\[
= q_k \left( \frac{\hat{x}_t}{\hat{x}_t} \right) - \left[ d + A_t + \gamma \right]
\]

\[
\hat{x}_t = q_k \left( \frac{x_t}{x_t} \right) - \left[ d + A_t + \gamma \right]
\]

EVERYWHERE
\[ (22) \quad \hat{A}_t = \delta \frac{L_A}{A_t^{1-\phi}} \]

**AT BGP:**

\[ \hat{A}_{\text{BGP}} = \frac{\lambda}{1-\phi} \]

---

**Summary: Modified**

(20) \[ \hat{S}_t = \hat{r}_t \times (1-2\eta)^{1-\phi} \]

(21) \[ \hat{r}_t = r_t \times (1-2\eta)^{1-\phi} - [d + \hat{A}_{t+1}] \]

(5) \[ \hat{A}_t = \delta \frac{L_A}{A_t^{1-\phi}} \]

BGP of modified is a S. S. S.

*Why?*

- AT BGP, \[ \hat{A}_{\text{BGP}} = \frac{\lambda}{1-\phi} \]

(i.e. is a constant)
SUMMARY MODIFIED SYSTEM

(30) \[ \hat{\gamma}_t = \lambda (1 - \psi \beta)^{-1} \]

(31) \[ \hat{\gamma}_t = \varrho \hat{\gamma}_{t-1} (1 - \psi \beta)^{-1} - [d + \hat{A}_t] \]

(32) \[ \hat{A}_t = \frac{S \lambda \alpha}{A_t^{1-\phi}} \]

WHAT IS TRUE AT S.S. OF MODIFIED SYSTEM/BGP OF ORIGINAL?

\[ \hat{\gamma}^{BCO} = 0 \]

\[ \hat{A}^{BCO} = \frac{\lambda \gamma}{1-\phi} \]

IMPORTANT DIAGRAM:

SEE APPENDIX B FROM WEB HANDOUT FOR PROOF OF \[ \hat{\gamma}_t \]