\[ \begin{align*}
Y_t &= M_t \cdot L_t \Rightarrow Y_t = \frac{M_t}{L_t} = \frac{Y_t}{L_t} \\
K_t &= L_t \Rightarrow K_t = \frac{L_t}{L_t} = \frac{Y_t}{L_t}
\end{align*} \]

\[ \text{Growth Rate of } Y_t: \dot{Y}_t = \dot{Y}_t + \eta \\
\text{Assumed } \eta \text{ is large enough so that } \dot{Y}_t = \frac{\dot{M}_t}{L_t} + \eta > 0 \\
\text{And hence } \dot{Y}_t < \eta \\
\text{If instead } \eta \text{ is very small}
\]

\[ \text{Path of } Y_t \text{ by } \dot{Y}_t \\
\text{Assuming we have } Y_t > 0 \]
Example 2:

Assume country is at BGP at t:
There is a permanent P in the savings rate r.
Show how the following things look/change:

\( h^*, y^*, h_t, y_t, c_t \)
\( h_t, g_t, c_t \)
\( y_t, \eta_t \)

Modified system Eq. (24):

\[
\dot{h}_t = -2 \cdot h_{t-1} - (d + n)
\]

At \( t \): \( \dot{h}_t > 0 \) \( \Rightarrow \) \( h_t \) P until it reaches \( h^* \) (the new S.S. level)
$y_t = x_t^\mu \Rightarrow \hat{y}_t = \hat{x}_t^\mu$

$\lambda_t = (1-\rho) y_t = (1-\rho) x_t^\mu \Rightarrow \hat{\lambda}_t = \hat{x}_t^\mu$

But we do not know if

$C_t^* \leq C_t^*$

It will depend on the relationship between $\omega$ and $\rho$

If $\omega > \rho$: $\hat{\omega} \Rightarrow C_t^* < C_t^*$

If $\omega < \rho$: $\hat{\omega} \Rightarrow C_t^* > C_t^*$

Very small

Will revisit next week
PATHS OF $Y_t$ & $\hat{Y}_t$

$Y_t = \frac{M_t}{C_t}$

$\hat{Y}_t = \frac{\hat{Y}_t + n}{t}$

**Homework:** PATHS FROM INITIAL SITUATION TO BGP

$h_0 = 16$

$h^* = 25$

$h_t = 25$

$h^*_t = 20$

$h^*_t = 10$
EXECUTIVE SIMULATIONS

$\lambda = 0.35$
$\delta = 0.02$
$k_0 = 1 \quad c_0 = 10$
$
\rho = 0.10$

$\eta = 2$

US : 0.10
FOREIGN_1 : 0.05
FOREIGN_2 : 0.15
\( Y_t = K_t^\alpha \cdot (A_t \cdot L_t)^{1-\alpha} \)

\( \hat{A}_t = g \quad \text{EXOG. RATE OF TECH. CHANGE} \)

\( \hat{L}_t = h \)

\( \dot{K}_t = \frac{2 \cdot Y_t - f \cdot K_t}{S-I} \quad \text{VAR: Y_t, K_t, L_t, Ct, ... , A_t} \)

\( \dot{K}_t = \frac{2 \cdot Y_t - s}{K_t} \quad \text{PARAM: g, d, f, s, a, n} \)

\text{TASKS:}
- CHARACTERIZE BGP
- MODIFY SYSTEM
- DISTURBANCES/ADJ. TO BGP.
Characterize BGP Growth Rates

- Growth rates of variables at BGP

- At BGP all growth rates constant

\[ \frac{y_t}{k_t} \text{ constant} \Rightarrow y_{t BGP} = \frac{1}{k_{t BGP}} y_t \]

- What is that rate?

Take growth rates of \((30)\)

\[ \dot{y}_t = \dot{k}_t + \frac{y_t}{k_t} \cdot (A_t \cdot L_t) \]

\[ = \dot{k} + (1-\dot{k}) (A_t + L_t) = \dot{k} (1-(\dot{k}+\dot{l}+\dot{m})) \]

i.e.

\[ \dot{y}_t = \dot{k} + (1-\dot{k}) (\dot{g} + \dot{n}) \]

\[ \frac{y_{t BGP}}{k_{t BGP}} = \frac{1}{k_{t BGP}} y_t \]

\[ y_{t BGP} = \dot{k} + (1-\dot{k}) (\dot{g} + \dot{n}) \]

\[ y_{t BGP} (1-\dot{k}) = (k_{t BGP}) (\dot{g} + \dot{n}) \]

\[ \dot{y}_{t BGP} = \frac{1}{k_{t BGP}} y_{t BGP} = \dot{g} + \dot{n} \]

\[ \Rightarrow y_{t BGP} = \frac{k_{t BGP} \dot{g} + \dot{n}}{\dot{k} + \dot{l} + \dot{m}} \]

This equation guides us in the choice of modified variables.
i.e. need to select new variables so that its system will have a steady state (that coincides with BGP of original system)

\[ Y_t = k_t = g + n = A_t + L_t = A_t \cdot L_t \]

Notice:

\[ Y_t = k_t = g + n = A_t + L_t = A_t \cdot L_t \]

Using that algebraic reverse

\[ \frac{Y_t}{A_t \cdot L_t} \] will be constant at BGP.

Since

\[ \left( \frac{Y_t}{A_t \cdot L_t} \right) = Y_t - (A_t + L_t) \]

Label:

\[ \frac{Y_t}{A_t \cdot L_t} \] is output per unit of "effective labor" i.e. \( A_t \cdot L_t \) is "effective labor"

At BGP:

\[ \frac{Y_t}{A_t \cdot L_t} = \frac{K_t}{A_t \cdot L_t} = \frac{C_t}{A_t} \]

\[ \frac{M_t}{A_t} \]

\[ \frac{J_t}{A_t} \]

\[ \frac{C_t}{A_t} \]

Constant!
THEN AT BGP

\[
\left( \frac{M_t}{A_t} \right)_{BGP} = \left( \frac{K_t}{A_t} \right)_{BGP} = \left( \frac{C_t}{A_t} \right)_{BGP} = 0
\]

\[
\Rightarrow \begin{cases} 
A_t = B_t = 0 \\
K_t = 0 \\
C_t = 0 
\end{cases}
\]

OUTPUT PER WORKER AT BGP GROWS AT RATE OF EXOGENOUS TECH. CHANGE.

WORK WITH MODIFIED SYSTEM TO CHARACTERIZE PATHS ALONG BGP OF ORIGINAL; EVALUATE DISTURBANCES & PATHS TO BGP FROM EQUILIBRIUM CONDITIONS.

MODIFIED SYSTEM

VARIABLES:

\[
\tilde{M}_t = \frac{Y_t}{A_t \cdot L_t} \quad \tilde{K} = \frac{K_t}{A_t \cdot L_t} \quad \tilde{C}_t = \frac{C_t}{A_t \cdot L_t}
\]

VARIABLES IN "EFFECTIVE UNITS OF LABOR".

NOTICE:

\[
\tilde{M}_t = \frac{M_t}{A_t} \quad \tilde{C}_t = \frac{C_t}{A_t}
\]
Using Hat Algebra:

\[ \hat{y}_t = \hat{y}_t - \hat{A}_t = \hat{y}_t - g \]

\[ \hat{x}_t = \hat{x}_t - g \]

\[ \hat{e}_t = \hat{e}_t - g \]

\[ \hat{\gamma}_t = \hat{\gamma}_t + g \]

\[ \hat{\phi}_t = \hat{\phi}_t + g \]

\[ \hat{e}_t = \hat{e}_t + g \]

Set up on modified system

Goal:

\[ \hat{y}_t = \text{function} \left( \hat{x}_t, \text{parameters} \right) \]

\[ \hat{e}_t = \text{function} \left( \hat{e}_t, \text{parameters} \right) \]

Aggregate Prod function:

\[ \hat{y}_t = \frac{Y_t}{A_t \cdot L_t} = \frac{k_t \cdot (A_t \cdot L_t)^{1-\lambda}}{A_t \cdot L_t} = \frac{k_t (A_t \cdot L_t)^{1-\lambda}}{(A_t L_t)^{1-\lambda}} \]

Using (30)

\[ = \left( \frac{K_t}{A_t \cdot L_t} \right)^{\lambda} \cdot \left( \frac{A_t L_t}{A_t L_t} \right)^{1-\lambda} = 1 \]

\[ \hat{y}_t = \hat{x}_t \]

(41) \[ \hat{y}_t = \hat{x}_t \]
LAW OF MOTION OF $\mathbf{x}_t$

Compute: How?
- Use definition of $\mathbf{x}_t$
- Use hat algebra
- Use (34)
- Divide num. & denom. of 1st term of (34)
  by $A_t \cdot L_t$
  - Use (41)

$$\mathbf{x}_t = \frac{k_t}{A_t \cdot L_t}$$

$$\implies \hat{\mathbf{x}}_t = k_t - (A_t \cdot L_t) = k_t - (g + \kappa)$$

Using (34)

$$\hat{\mathbf{x}}_t = \frac{\mathbf{r} \cdot y_t}{k_t} - s - (g + \kappa)$$

$$= \frac{\mathbf{r} \cdot y_t}{A_t \cdot L_t} - \frac{k_t}{A_t \cdot L_t}$$

$$= \frac{\mathbf{r} \cdot \hat{\mathbf{x}}_t}{k_t} - (g + \kappa)$$

Using (41)

$$\hat{\mathbf{x}}_t = \frac{\mathbf{r} \cdot \hat{\mathbf{x}}_t}{k_t} - (g + \kappa)$$

(45)
\[ \begin{align*}
(41) & \quad \hat{h}_t = \hat{h}_t \\
(45) & \quad \hat{h}_t = r \hat{h}_t - (s + p + q) \hat{h}_t \\
(46) & \quad \hat{h}_t = r \hat{h}_t - (s + p + q) \hat{h}_t \\
(41) & \quad \check{h}_t = \hat{h}_t
\end{align*} \]
Disturbances 2 Adjustment to BGP  

For ~ Variables

\[ z_{zt} = \delta + \phi + \theta \]

Levels at BGP

At BGP (SS or Rodikien):

\[ \hat{h}_t = 0 \quad \text{Then (SS)} \Rightarrow \]

\[ \hat{h}_t = 0 = 2 \hat{h}_t - (s + \phi + \theta) \]

\[ \Rightarrow 2 \hat{h}_t = s + \phi + \theta \]

\[ \Rightarrow \hat{h}_t = \frac{s + \phi + \theta}{2} \]

Using (41):

\[ y_t = \hat{h}_t = \frac{s + \phi + \theta}{2} \]