GROWTH RATES AT BGP OF ORIGINAL SYSTEM

**Remark:** Use Eq. 10 + Definition of BGP

**Def:**
- At BGP all variables grow at constant rates.
  \[ \hat{k} \text{ BGP is constant} \]

Eq. (7) is:
\[ \hat{y}_t = R \cdot \frac{y_t}{k_t} - s \]

\[ \Rightarrow \frac{y_t}{k_t} \text{ is a constant} \]

- Using Def. Growth Rates
\[ \frac{y_t}{k_t} \rightarrow \text{BGP constant} \]

- Using "Hat Algebra"
\[ y_t - \hat{k} \text{ BGP} = 0 \]
\[ \Rightarrow \hat{y}_t = \hat{k} \text{ BGP} \]

\[ \Rightarrow \hat{y}_t = \hat{k}_t \text{ BGP} \quad (10) \]
Taking growth rates of (1):

(1) \[ Y_t = k_t^L \cdot L_t \]

Using hat at every where

(11) \[ \hat{Y}_t = \alpha \hat{K}_t + (1-\alpha) \hat{L}_t \]

At BGP by (10): \[ \hat{K}_t = \hat{Y}_t \]

So we write (11) as:

\[ Y_t = \alpha \cdot \hat{K}_t + (1-\alpha) \cdot \hat{L}_t \]

Then using (10):

\[ \hat{Y}_t = \hat{K}_t = \eta \]

How about consumption at BGP?

\[ C_t = (1-\delta) \cdot \eta \]

\[ C_t = (1-\delta) \cdot \hat{Y}_t \]
USING MAT ALGEBRA:
\[
\hat{C}_t = (1 - \alpha) + \gamma \hat{B}_t = \gamma \hat{Y}_t
\]

\[
\frac{C_t}{B_t} = \gamma \rightarrow \frac{\hat{C}_t}{\hat{B}_t} = \gamma
\]

\[8 \text{ BY (12)} \]

CONCLUSION:

AT BGP:

(13)

\[
\begin{align*}
\frac{\hat{Y}_t}{L_t} & = \frac{\hat{C}_t}{\hat{B}_t} = \frac{\hat{C}_t}{\hat{C}_t} = \frac{\hat{C}_t}{\hat{C}_t} = 1 \\
\end{align*}
\]

PER CAPITA?

\[
\frac{\hat{Y}_t}{L_t} < 1 \\
\]

OUTPUT PER WORKER DOES NOT GROW (i.e. IS CONSTANT AT THE BGP)!

NEGATIVE RESULT!
Modify the system to work with one that has a S.S.

- Choose new variables
- Compute new "production function"
- Compute "law of motion" (\( \dot{Y} \) and or \( \frac{d}{dt} \)) of new variables

Since we have (13) we choose to work with "per worker/capita" magnitudes of modified/new variables

\[ Y_t = \frac{Y_t}{L_t} \quad k_t = \frac{k_t}{L_t} \quad c_t = \frac{c_t}{L_t} \]

Why? Because (13) tells us that \( \dot{Y_t} = \dot{k_t} = \dot{c_t} = 0 \)

\( \Rightarrow \) BGP of the modified system is a steady state!
Compute new production function

Use (1) & divide both sides by $L_t$

$$y_t = \frac{Y_t}{L_t} = \frac{k_t \cdot L_t^{1-\delta}}{L_t} = \frac{k_t}{L_t} \cdot L_t^{1-\delta} = \left(\frac{k_t}{L_t}\right)^{\alpha} = M_{kt} = f(y_t)$$

(20)

$$y_t = f(y_t) = M_{kt}$$

AGGREGATE PROD.
FUNCTION IN PER WORKER/CAPITA

Compute law of motion of $M_t$ using:

- Definition of $M_t = \frac{k_t}{L_t}$
- Hat algebra & eq. (4).
- Eq. (7): $k_t = a \cdot \frac{y_t}{k_t} - S$
- Additional algebra to transform equation into one with only new variables.
\[ \vec{r}_t = \left( \frac{k_t}{L_t} \right) = \frac{k_t}{L_t} - \frac{L_t}{L_t} = \frac{k_t}{L_t} - 1 \]

Using (7)

\[ \vec{r}_t = \frac{2}{k_t} \left( \frac{y_t - \delta}{L_t} \right) - \gamma = \frac{2y_t}{k_t} - (\delta + \gamma) \]

Dividing numerator and denominator of first term by \( L_t \)

\[ \boxed{ \frac{2}{k_t} \left( \frac{y_t}{L_t} \right) - (\delta + \gamma) = \frac{2y_t}{k_t L_t} - (\delta + \gamma) } \]  

Summary

1. (20) \( \delta_t = f(\vec{r}_t) = \frac{\delta_t}{\delta_t} \)
2. (21) \( \vec{r}_t = \frac{2y_t}{k_t L_t} - (\delta + \gamma) \)

Since \( \vec{r}_t = \vec{r}_t \cdot \vec{r}_t \Rightarrow \)

1. (22) \( \vec{m}_t = \frac{\alpha_y}{\delta_t} = (\delta + \gamma) \vec{r}_t \)
\[ C_t \leq 0 ? \]
\[ C_t = (1 - \alpha). Y_t \implies \]
\[ \frac{C_t}{L_t} = (1 - \alpha) \frac{Y_t}{L_t} \]
\( \Rightarrow \)
\[ \begin{align*}
(23) \quad & C_t = (1 - \alpha). Y_t \\
(20) \quad & Y_t = h_t^\alpha = f(h_t) \\
(21) \quad & h_t = R \cdot \frac{h_{t-1}}{h_t} - (S+3) \\
(22) \quad & h_t = R g_t - (S+3) h_t \\
(23) \quad & C_t = (1 - \alpha). Y_t 
\end{align*} \]

Using (20) & (21) we get
\[ h_{t-1} = R \cdot h_t^\alpha - (S+3)h_t \]

\( \Rightarrow \)
\[ \begin{align*}
(24) \quad & h_{t-1} = R \cdot h_t^\alpha - (S+3)h_t \\
(25) \quad & C_t = (1 - \alpha). Y_t 
\end{align*} \]
AT $h_t^* : \Rightarrow \dot{h}_t^* = 0 \Rightarrow h_t^* \text{ will not move!}$

IF $h_t < h_t^* : \Rightarrow \dot{h}_t > 0 \Rightarrow h_t \uparrow$

IF $h_t > h_t^* : \Rightarrow \dot{h}_t < 0 \Rightarrow h_t \downarrow$