

Mathematical Concepts and Methods for Intermediate Microeconomics

Some of the more basic mathematical concepts you will need to review for this class are the slope and intercept formula for a linear function. How do changes in the formula affect the shape of the line? Is a slope of -2 steeper or flatter than a slope of -5 ? What is the slope of a horizontal line (zero) and a vertical line (infinity).

We will also consider the concept of equilibrium, which involves the solution of a number of equations simultaneously (i.e. finding values for several variables which solve several equations at once). The most obvious example of solving for equilibrium in economics is finding the price and quantity at which supply equals demand.

You will need to be comfortable with basic differentiation rules and procedures. We will use derivatives to talk about increasing or decreasing functions and for optimizing (finding the minimum or maximum of a function using the first order and second order conditions). What follows below is a review of important concepts on optimization.

Optimization:

The major mathematical concept that we will use the most in Intermediate Microeconomic Theory is the optimum of a function. Optima are either maxima or minima. Most individual household or firm decisions in economics involve maximization (of utility, profits) or minimization (of expenditure, cost) and these decisions also often involve constraints that need to be taken into consideration (certain amount of income to spend, cost levels, etc.).

Functions with One Variable

Start with a function like $y=f(x)$, or $\Pi = f(Q)$. The derivative, which will be denoted as dy/dx or $d\Pi/dQ$ or $f'(x)$, measures the slope of the function at a certain point (i.e. for a particular value of x or Q).

When $d\Pi/dQ > 0$, Π is an increasing function of Q , Π moves in the same direction with Q .
When $d\Pi/dQ < 0$, Π is a decreasing function of Q , Π moves in opposite direction from Q .
When $d\Pi/dQ = 0$, the function has a stationary point (either maximum or minimum).

Example: $\Pi = \Pi(Q) = -2Q^2 + 24Q + 5$

$d\Pi/dQ = -4Q + 24$, so this function is increasing for $-4Q + 24 > 0$ or when $Q < 6$.

The function is decreasing for $-4Q + 24 < 0$ or when $Q > 6$.

The function has a stationary point at $-4Q + 24 = 0$ or when $Q = 6$.

Stationary Points: Maxima or Minima?

How can you tell whether a stationary point is a maximum or a minimum of your function? You need to look at how the derivative is changing at the stationary point. You need to look at the derivative of the derivative, or the second derivative of the function.

The second derivative is denoted: $\frac{d^2\Pi}{dQ^2}$ or $\frac{d}{dQ} \frac{d\Pi}{dQ}$ or $\Pi''(Q)$.

When $\Pi''(Q) > 0$, then $\Pi'(Q)$ is moving in the same direction as Q (the slope of the function is increasing as Q increases).

When $\Pi''(Q) < 0$, then $\Pi'(Q)$ is moving in the opposite direction from Q (the slope of the function is decreasing as Q increases).

When $\Pi''(Q) = 0$, then $\Pi'(Q)$ is switching from negative to positive (or positive to negative) as you increase Q and there is an inflexion point in $\Pi(Q)$.

To determine whether you have a maximum or a minimum you need to check the second derivative of the function at that value for which you found the stationary point. If $f'' < 0$, the stationary point is a maximum. If $f'' > 0$, the stationary point is a minimum. These are the Second Order Conditions for a maximum and a minimum. (Functions which have maxima look like hills. Functions which have minima look like valleys).

Example: In the Π function above we have a stationary point at $Q = 6$. Check $\Pi''(Q)$ at $Q = 6$. $\Pi''(Q) = \frac{d}{dQ}(-4Q + 24) = -4 < 0$ which means that the function $\Pi(Q)$ is maximized at $Q = 6$.

General Rules for Optimizing $y=f(x)$:

x is an optimum if:

$f'(x) = 0$ is the first order condition (FOC) for optimum

and $f''(x) < 0$ is the second order condition (SOC) for a maximum

or $f''(x) > 0$ is the second order condition (SOC) for a minimum

Functions with More than One Variable

Most of the problems we will tackle in Intermediate Microeconomic Theory will involve more complex functions than the ones described above. We will need to use techniques from multivariate calculus to deal with this type of function. Now, we have $y = f(x, z)$ instead of $y = f(x)$. (The Utility function is often expressed as a function of quantities of different goods people can purchase. Output is expressed as a function of the quantities of different inputs required in the production).

These functions are difficult to draw (they are at least three dimensional) so we use a mapmaker's technique. Pick a given level of the dependent variable and then plot combinations of the independent variables that give you that level. This process is how we get indifference curves and isoquants. It is like putting a mountain into two dimensions; the construction will be helpful to us in illustrating optima in a multivariate world

How to compute the optimum: keep thinking about the mountain analogy. Finding an optimum requires getting to the top of the mountain and making sure you are at the highest point in all directions. The point you find must be an optimum on the north/south axis as well as on the east/west axis. You use partial derivatives to do this.

When taking a partial derivative, you take the derivative of the function with respect to ONE of the independent variables while holding all of the other independent variables constant (you treat them as if they were constant). You can take a partial derivative of a multivariate function for each of the independent variables. Essentially, you can hold one

direction constant and look at what's happening in the other direction (i.e. find the optimum) and then hold the other direction constant and find the optimum in the first direction. The true "top of the mountain" is that point at which you have a maximum in all directions.

Partial derivatives of a function like $y = f(x, z)$ are written as $\partial y/\partial x$ and $\partial y/\partial z$ and can be interpreted as "the derivative of y with respect to x holding everything else constant." (This interpretation is just like the economic use of the phrase *ceteris paribus*). To take a partial derivative with respect to one of the independent variables, just treat all the other independent variables as constants (as if they were some fixed number).

Example (multi-product firm) : $\Pi = \Pi(Q, T) = -2Q^2 + 2QT - 3T^2 + 10Q$
 $\partial \Pi/\partial Q = -4Q + 2T + 10 = \Pi_Q = \Pi_1$

$$\partial \Pi/\partial T = 2Q - 6T = \Pi_T = \Pi_2$$

You can also take second partial derivatives to see how the first partial change when the independent variables change. You get more second partials than first partials because you can take the partial $\partial \Pi/\partial Q$ or $\partial \Pi/\partial T$ with respect to Q and T .

$$\frac{\partial}{\partial Q} \frac{\partial \Pi}{\partial Q} = \frac{\partial^2 \Pi}{\partial Q^2} = \Pi_{QQ} = \Pi_{11} = -4$$

$$\frac{\partial}{\partial T} \frac{\partial \Pi}{\partial T} = \frac{\partial^2 \Pi}{\partial T^2} = \Pi_{TT} = \Pi_{22} = -6$$

$$\frac{\partial}{\partial Q} \frac{\partial \Pi}{\partial T} = \frac{\partial^2 \Pi}{\partial Q \partial T} = \Pi_{QT} = \Pi_{21} = 2$$

$$\frac{\partial}{\partial T} \frac{\partial \Pi}{\partial Q} = \frac{\partial^2 \Pi}{\partial T \partial Q} = \Pi_{TQ} = \Pi_{12} = 2$$

Note that the "cross partials" are the same. In general, $f_{ij} = f_{ji}$.

You can use the partials to find optima just like we did before. This time you need to have a stationary point in all directions at the same time. The first order conditions for an optimum in this case are that $\Pi_1 = 0$ and $\Pi_2 = 0$ at the same time. (You need to have values for Q and T that solve both of these equations at the same time.)

Solve the two first-order conditions simultaneously:

$$\Pi_1 = -4Q + 2T + 10 = 0$$

$$\Pi_2 = 2Q - 6T = 0$$

Pick one of the equations and get an expression for T in terms of Q (or Q in terms of T):

$$10 - 4Q + 2T = 0 \Rightarrow 10 + 2T = 4Q \Rightarrow T = 4/2 Q - 10/2 = 2Q - 5.$$

Substitute this into the other equation to get an actual value for Q (or T):

$$2Q - 6(2Q - 5) = 0 \Rightarrow 30 - 12Q + 2Q = 0 \Rightarrow 30 - 10Q = 0 \text{ so } Q = 30/10 = 3.$$

Substitute the value for Q (or T) into the first equation that you used to get a value for the other variable:

$$-4(3) + 2T + 10 = 0 \Rightarrow -2 + 2T = 0 \text{ so } T = 1.$$

This function has a stationary point at $Q = 3, T = 1$.

Determining whether this stationary point is a maximum, minimum, or reflection (saddle) point will depend on the second derivatives.

Second Order Conditions (Multivariate Case):

$\Pi_{11} < 0$ and $\Pi_{22} < 0 \Rightarrow$ slopes are decreasing in both directions from the stationary point. This is part of the condition for a maximum.

$\Pi_{11} > 0$ and $\Pi_{22} > 0 \Rightarrow$ slopes are increasing in both directions from the stationary point. This is part of the condition for a minimum.

The final condition will guarantee that you are not at a saddle point $\Pi_{11} \Pi_{22} - \Pi_{12}^2 > 0$

General Rules for Optimizing $y = f(x, z)$:

$y_x = 0$ and $y_z = 0$ are FOCs for optimum

$y_{xx} < 0$ and $y_{zz} < 0$ and $y_{xx} y_{zz} - y_{xz}^2 > 0$ are SOC for a MAX

$y_{xx} > 0$ and $y_{zz} > 0$ and $y_{xx} y_{zz} - y_{xz}^2 > 0$ are SOC for a MIN

CONSTRAINTS: Many of the problems that we will want to solve involve constraints on the function we want to maximize. (Utility is maximized taking the budget constraint into account; profits may have to be maximized taking a production cap into account.) Finding an optimum in a case like this is like dealing with an uncrossable fence on the mountain. You want to get as high as you can on the mountain except that you can't cross the fence.

In our profit function example, suppose that the firm had a production cap imposed by an outsider like the government so that it could produce no more than $Q + T = 2$. We generally rewrite constraints so that they show something equal to zero. Rewritten, the constraint looks like $2 - Q - T = 0$.

We need a "trick" to help us maximize profits subject to the constraint. The problem is that to find an optimum we need all the first partials to equal zero and we also need the constraint to equal zero. This process would leave us with 3 equations to solve with only 2 independent variables. (If you know any linear algebra, you should realize that this won't work.) The "trick" we will use is called the method of Lagrange. We create a new variable λ called the Lagrangean Multiplier and use it to combine the objective function and the constraint.

Create $\mathcal{L} = f(x, z) + \lambda \cdot \text{constraint}$

Then $\mathcal{L} = \Pi(Q, T) + \lambda(2 - Q - T)$.

Notice that when the constraint holds (so that $2 - Q - T = 0$) Then $\mathcal{L} = \Pi(Q, T)$. If you maximize assuming that the constraint holds, then you are also maximizing $\Pi(Q, T)$. For the purposes of the problem, you treat λ as another independent variable.

Example: Maximize $\mathcal{L} = -2Q^2 + 2QT - 3T^2 + 10Q + \lambda(2 - Q - T)$.

The First order conditions for an optimum are:

$$\partial \mathcal{L} / \partial Q = -4Q + 2T + 10 - \lambda = 0$$

$$\partial \mathcal{L} / \partial T = 2Q - 6T - \lambda = 0$$

$$\partial \mathcal{L} / \partial \lambda = 2 - Q - T = 0$$

The third FOC just guarantees that the constraint holds. Also, the other partials include λ but you can still solve for values of Q and T that solve the equations. The easiest way to do this is to eliminate λ by solving for λ in the FOCs and then setting them equal to each other. Otherwise, the process is just like what is described above.

Solve for λ in the first two FOCs: $-4Q + 2T + 10 = \lambda$. Set them equal and get the expression for Q or T):

$$-4Q + 2T + 10 = 2Q - 6T \Rightarrow 6Q = 10 + 8T \text{ so that } Q = 10/6 + 8/6T.$$

Substitute this expression into the constraint to get a value for T (the other variable), then plug that value back into the expression for Q (the first variable) to get a value there. This will give you the values of Q and T for which this function has a stationary point. (If you do it here, you should find that $Q = 13/7$ and $T = 1/7$.)

There are also SOCs that are needed to determine whether the stationary point is a maximum or a minimum. For the constrained case, we are really testing to see if we have a “quasi-concave” function (for a maximum) or “quasi-convex” function (for a minimum). The SOCs are very unpleasant looking. (For a maximum of the function $y = f(x, z)$ the SOC requires that $f_{11}f_2^2 - 2f_{12}f_1f_2 + f_{22}f_1^2 > 0$. This condition is a condition on the “bordered Hessian” matrix of first and second order partial derivatives of the function.) You can assume that the functions you use in constrained optimization problems satisfy all of the correct SOCs. I will only give you the right kinds of functions in this course but it is important that you realize that not all functions will “work”.

Finally, you can also solve for a value for λ in any constrained optimization problem.

The variable λ represents the effect on Π (in this case) of a slight change in the constraint. Essentially, λ tells you how Π changes when you relax the constraint a little (e.g., let the constant get a little bigger). One could interpret λ as the marginal profit of an increase in output here. In the consumer theory case, where you are maximizing utility subject to the budget constrained, λ represents the marginal utility of an addition to income.