

Core Selection in Auctions and Exchanges*

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Abstract

A design's ability to implement core outcomes is of central importance in practice. It is well known that such implementation is possible in those one-sided environments with the substitutes property. This paper characterizes divisible good environments in which core-selecting design is possible, offering a unified treatment of auctions where trade is necessarily one-sided and those where agents and the auctioneer can both buy and sell. We show that heterogeneity in either pre-auction marginal utilities or substitution patterns (as captured by utility Hessians) can independently challenge core selection. In particular, core selection may fail even with a single good and even when revenue is positive. We introduce a joint condition on preferences and equilibrium allocations which ensures that outcomes are in the core. Our results point to the alignment between the bidders' incentives to substitute and the trades necessary to realize the available surplus as the key to core selection.

JEL CODES: D44, D47. KEYWORDS: Core-selecting auction; Package auction; Package exchange; Divisible-good auction; Complements; Substitutes; Vickrey auction.

1 Introduction

The last decade has seen significant progress concerning allocation problems, in which traders seek to buy or sell multiple heterogeneous goods. Special attention has been given to a design's ability to select a core outcome – a strong efficiency criterion and a key challenge for practical market design. When efficient implementation is the objective, the literature

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has generally considered the Vickrey mechanism, either directly or as a benchmark.¹ Core selection ensures that no coalition of participants (the bidders and the auctioneer) can do better by trading on its own. Its absence leads to incentive problems that can compromise virtually any aspect of design, including efficiency, revenue, and participation. These issues (and the Vickrey auction, in general) are well understood in *one-sided* auctions (where all bidders are interested in buying) for *indivisible* goods. In such settings, the literature has established a close link between core selection and a goods substitutes property. In many resource allocation problems, however, bidders are interested in selling or buying various goods. Likewise, the auctioneer may specifically desire to reallocate goods among the bidders, or wish to acquire certain goods and sell others. Indeed, reallocating the spectrum controlled by television stations into uses of higher-value is part of the the objective of the Federal Communications Commission in the U.S. spectrum auction scheduled for 2015.² Moreover, as is the case with spectrum, electricity, and emission permits, bidders typically have complex preferences over the goods being auctioned, and the same goods may be considered substitutes by some bidders and complements by others;³ in particular, accommodating complementarities is often essential. In his Fisher-Schultz lecture, Milgrom (2007) emphasized that the theory of package exchanges has few predictive results. For markets with multiple heterogeneous divisible goods, this paper characterizes the possibilities and limits for a core-selecting Vickrey design in such *two-sided* settings – where bidders and the auctioneer are able to both buy and sell and bidders have heterogeneous preferences – such as auctions in which bidders enter with non-zero endowments of the goods auctioned, or reallocating goods among the bidders is part of the auctioneer’s objective.

MAIN RESULTS. Two-sidedness has important implications for core selection. To begin, since gains from trade among agents may exist even in the absence of the auctioneer’s quantity vector, a two-sided setting creates the possibility of additional deviations by coalitions that do not include the auctioneer. We show, however, that the Vickrey auction is immune to such deviations; that is, participation is not only individually but *coalitionally* rational. For core-selection in two-sided divisible settings, we provide examples demonstrating that Vickrey payoffs may not lie in the core even when goods are substitutes for all bidders. In fact, core

¹ In fact, Goeree and Lien (2014) show that any core selecting auction is a Vickrey auction. Specifically, they show that any Bayesian incentive compatible core selecting mechanism that obeys participation constraints is a Vickrey auction. This result is proven for indivisible goods, but can easily be shown to hold for our divisible setting as well.

² See <http://www.fcc.gov/incentiveauctions>. In applications like spectrum auctions, the auctioneer is not necessarily constrained to allocate each bidder a nonnegative quantity of each good; bidders may wish to switch to a different frequency or exit a market.

³ For instance, spectrum licenses differ in their geographical coverage and technical characteristics such as interference with adjacent frequency bands; electricity contracts differ in duration and location characteristics; government bonds have different maturities; and emission permits are issued for different time periods and pollutants.

selection may fail even when a single good is being auctioned – that bidders hold non-zero endowments of the goods auctioned, for instance, may critically affect its efficiency. On the other hand, auctions in which each bidder’s preferences exhibit rich complementarities may yield a core outcome.

We establish a core selection condition as a *joint* restriction on allocations and demands. It involves the existence of implicit *packages* (bundles of goods) which are substitutes for all bidders.⁴ Mathematically, the condition requires that each bidder’s allocation must be contained in the cone generated by the packages. Intuitively, the joint condition captures that underlying the absence of incentives to deviate by coalitions of the bidders and the auctioneer is an order of packages, *common* among bidders, that aligns the bidders’ incentives to substitute with the trades necessary for the surplus in the auction, created by the participants’ pre-auction endowments and the quantity auctioned, to be realized. Our core selection condition reduces to the classic substitutes condition when the packages are just the primitive goods and the auction is one-sided. In general, the condition ensures that a two-sided auction for goods *functions like a one-sided auction for packages*.

We then ask what core selection requires in terms of the primitives. Whether the Vickrey auction leads to core outcomes depends on how different bidders’ substitution patterns (as captured by utility convexities (Hessians)) and marginal utilities at the initial allocation are. With sufficient heterogeneity in substitution patterns in bidder preferences, core selection fails *independently* of heterogeneity in marginal utility at the initial (pre-auction) allocations; and with sufficient heterogeneity in the initial allocations, core selection fails *independently* of heterogeneity in the substitution patterns. A preference condition establishes when we can find packages that are substitutes. We can then characterize a set of auctioned quantity vectors for which core selection holds. This design aspect – that the core-selection property depends on the quantity vector – is unique to (perfectly and imperfectly) divisible goods. Hence, insofar as the quantity vector is a choice variable for the auctioneer, it can be selected to align the trades necessary to realize the surplus with the bidders’ incentives to substitute, regardless of the heterogeneity in marginal utilities at the initial allocations. These results identify settings where a designer’s choice of the Vickrey auction will not cause the incentive problems associated with the absence of core selection.

Taken together, the core-selection results and the converses we introduce offer two general insights. First, it is not substitutability *per se* that enables core selection – the Vickrey auction yields a core outcome for a rich class of bidder valuations (utility Hessians) which

⁴ Authors in the literature on equilibrium existence with imperfectly divisible goods have considered substitutes under a particular reordering of the space of goods. Examples include the gross substitutes and complements condition of Sun and Yang (2006); its close relative, the full substitutability condition of Ostrovsky (2008) and Hatfield et al. (2013); and, more generally, the basis changes employed in Baldwin and Klemperer (2014). We show that the gross substitutes and complements condition is of limited use for core selection in the one-sided auction environment on which the literature has focused. (See also Example 4.)

may exhibit complementarities and substitutabilities and vary across bidders. Rather, a homogeneity property underlying the preference substitution patterns is. With it, each bidder’s quantity demanded changes in a common set of directions in response to a price change. Second, unlike one-sided (divisible or indivisible) auctions, substitutability (in goods or packages) is not sufficient on its own for core selection; instead, an additional condition is needed – a positive package allocation – which in turn restricts the direction of the quantity auctioned.

Our analysis of substitutability is cast in the language of cones, which we use to provide a unified treatment of one- and two-sided divisible-good settings, and analytic and geometric arguments for all main results. Generated by packages, cones can induce partial orders in which the substitutes property holds, which in turn allow us to identify the notion of substitutability relevant for understanding core selection in divisible-good markets (i.e., package substitutability) in terms of the primitives; identify its homogeneity across bidders as central to, but insufficient for, core selection in two-sided markets; and conveniently think about design.

Although the Vickrey auction possesses key advantages (truthful reporting as a dominant strategy, efficient allocation), when core selection fails, its revenue performance strictly below the smallest auctioneer’s revenue at any core allocation, even when all bids are high, is seen as a significant limitation in practice (e.g., Ausubel and Milgrom (2006)). In the U.S. spectrum auctions, for example, both efficiency and revenue are mandates. However, we show that in two-sided divisible good settings, the Vickrey mechanism may outperform the popular uniform-price design in revenue terms, particularly in environments where it is likely to be core-selecting.

The literature on package allocation problems for indivisible goods recognizes two central complexity challenges. First, finding core outcomes is NP-hard in the general case. In addition, with K goods, a bidder may be asked to report values for $2^K - 1$ (non-empty) packages – unrealistic, even if K is small.⁵ For package allocation problems with divisible goods, the difficulty of computing core outcomes is largely absent, but the problem of asking bidders to consider a high-dimensional bid space is worsened: bidders may be asked to submit demand schedules for all goods as a function of all prices, $\mathbb{R}^K \rightarrow \mathbb{R}^K$, or, equivalently, their valuations over the entire space of feasible allocations. However, the demand structure (common package substitutability) we identify as facilitating core selection can enable a reduction in the dimensionality of the bid space by removing the requirement that agents make fully contingent bids. Instead, the designer can equivalently ask bidders to submit bids ($\mathbb{R} \rightarrow \mathbb{R}$) in K independent Vickrey auctions for packages. This is ultimately useful to the extent that these packages have a meaning in practical applications. We argue that this is

⁵ De Vries and Vohra (2003) survey complexity issues associated with package designs.

the case in many high-stakes applications, including spectrum and electricity auctions.

RELATED LITERATURE. The literature on core selection has largely focused on settings with indivisible goods. This paper provides analysis of core selection in environments with multi-unit demands. An important predecessor is Milgrom and Strulovici (2009), who examine one-sided divisible good auctions. For divisible and indivisible goods, our results highlight the central role that two-sidedness and heterogeneity in bidder preferences play for core selection.⁶

In response to the low revenue problem of the Vickrey auction, the literature has explored *core-selecting auctions* (Day and Milgrom (2008)), which allow for compromising on strategy-proofness to attain revenue objectives by modifying transfers. Variants of core-selecting auctions include menu auctions (the pay-as-bid (Bernheim and Whinston (1986))), the ascending proxy auction (Ausubel and Milgrom (2002), Parkes and Ungar (2000)), the assignment auction (Milgrom (2009)), and the mechanisms resulting from the core computations in Day and Raghavan (2007), Day and Cramton (2012), Erdil and Klemperer (2010), and Ausubel and Baranov (2010).⁷ Milgrom and Segal (2014) require strategy-proofness but not the efficiency of the outcome given the bids.⁸

In environments with indivisible or imperfectly divisible goods, the literature has demonstrated that substitutes properties are useful.⁹ In a recent contribution, Baldwin and Klemperer (2014) provide a geometrically beautiful characterization of equilibrium existence in a very general framework. Like us, the authors also allow for negative allocations in indivisible-good settings, allow for complements, and recognize that when the substitutes condition holds for a transformed problem under a change of basis, many results continue to hold. They focus on existence; as we show, core selection requires conditions more stringent than substitutes when reallocation is feasible or when the property holds under a change of basis.

⁶ We do not require that bidders are *single-minded* (i.e., that they wish to acquire one good or bundle of goods, and care only about winning or losing and the payment).

⁷ In light of the Vickrey auction's poor revenue performance outside of settings where goods are substitutes, the core-selecting auctions are viewed as a potential alternative to the Vickrey design in practice, if the problem of communication complexity can be addressed (e.g., Day and Milgrom (2008)). Articles on settings with divisible goods include Ausubel (2006), who proposed a dynamic auction design that achieves Pareto efficiency, and Ausubel and Cramton (2004) who examine design properties of simultaneous clock auctions.

⁸ We do not require that bidders are *single-minded* (i.e., that they wish to acquire one good or bundle of goods, and care only about winning or losing and the payment).

⁹ Integer programming problems, being over a nonconvex set, are not always amenable to the strong duality properties on which classical results for divisible goods are dependent. Variations of the substitutes property can get around this problem; indeed, M^\sharp -concavity, which is a strengthening of the substitutes property, guarantees that strong duality holds – see Murota (2003). Such properties guarantee existence of competitive equilibrium in indivisible Walrasian economies (Gul and Stacchetti (1999)), stable outcomes in matching models (Hatfield and Milgrom (2005), Ostrovsky (2008), Hatfield et al. (2013)), in addition to ensuring that the Vickrey auction is core-selecting (Ausubel and Milgrom (2002)).

Additionally, while the notion of demand types in Baldwin and Klemperer (2014) allows both substitutes and complements, the condition that the authors place on demand types in perfectly competitive environments with imperfectly divisible goods to get existence (not an order-theoretic result) is not order-theoretic. Our analysis of core selection (an order-theoretic result) points to the importance of an order-theoretic property of preferences, which admits goods complementarity and substitutability, and explores the role of (heterogeneity in) bidder preferences.

STRUCTURE OF THE PAPER. Section 2 introduces the setting. Section 3 discusses the link between core selection and bidder submodularity in two-sided settings. Section 4 develops the main results: the motivating examples and characterization of core selection in two-sided settings. Section 5 presents the converses. Section 6 establishes complexity reduction and revenue results. Section 7 discusses extensions. All proofs are contained in the Appendix.

2 Setting

An auctioneer has a vector $Q \in \mathbb{R}^K$ of K perfectly divisible goods to sell to N heterogeneous agents who may participate in this auction. Throughout, we denote the set and its cardinality by the same symbol. Each agent i has a valuation for quantity obtained in the auction¹⁰ q_i and payments (transfers) to the auctioneer x_i given by $u_i(q_i) - x_i$, where $u_i : \mathbb{R}^K \rightarrow \mathbb{R}$ is twice continuously differentiable and strictly concave and the images of the marginal utilities, $\nabla u_i(\mathbb{R}^K)$, are the same for all agents.¹¹ With respect to applications, our setting makes two innovations. We do not restrict the feasible allocations to be non-negative. This allows accommodation of environments in which bidders and the auctioneer can buy or sell; for instance, when bidders have non-zero endowments or short-selling is allowed. Additionally, this allows us to consider settings in which the auctioneer seeks to reallocate goods among the bidders, purchase goods (a procurement auction¹²) or has no quantity for sale (a proper exchange).

We refer to an auction in which each agent is only able to purchase goods from the auctioneer (who is looking to sell them) or only able to sell goods to the auctioneer (who is

¹⁰ We can think of u_i as a reduced form describing the agents' initial endowments $\{q_i^0\}_i$ and preferences over total quantities \hat{u}_i ; then $u_i(q_i) \equiv \hat{u}_i(q_i^0 + q_i) - \hat{u}_i(q_i^0)$ and, in particular, $u_i(0) = 0$. We work directly with $u_i(q_i)$, since the core-selection properties depend only on $u_i(q_i)$. Our information assumptions are minimal; we only assume that agents know their own preferences u_i .

¹¹ This ensures existence of a competitive equilibrium allocation in finite quantities and that aggregate demand is everywhere well-defined. This is stronger than necessary for equilibrium existence, but allows us to focus on core selection using the language of demand and supply without running into issues related to invertibility of demands and supplies.

¹² Procurement auctions, while one-sided, require a more careful definition of core selection – see Section 3.

looking to buy them in a procurement auction) as *one-sided*. Formally, a one-sided auction is one where bidders' auction allocations q_i are constrained to lie in the positive or negative orthant.

(TWO-SIDED) VICKREY AUCTION. The most widely studied mechanism for efficient resource allocation in transferable utility environments such as ours is the (generalized) Vickrey auction. When $Q = 0$, the two-sided auction becomes what has been referred to as a *Vickrey exchange*.

A *two-sided (demand-implemented)*¹³ Vickrey auction of K perfectly divisible goods asks bidders to submit (net) demand curves $b_i : \mathbb{R}^K \rightarrow \mathbb{R}^K$ that are C^1 and strictly downward-sloping (i.e., have negative definite Jacobians). We assume that if only one agent were present, the auction is cancelled and no goods are allocated;¹⁴ thus, in the analysis to follow, there are at least two participating bidders. Otherwise, aggregating the demands of $j \neq i$, gives the (inverse) residual supply curve $p = R_i(q_i)$ of bidder i ; bidders receive the market-clearing quantity, i.e., the unique solution to $R_i(q_i) = b_i(q_i)$, and are charged a payment of $\int_0^{q_i} R_i(\mathbf{r}) \cdot d\mathbf{r}$ (Figure 1). Hence, a bidder's payoff from receiving q_i is

$$\int_0^{q_i} (\nabla u_i(\mathbf{r}) - R_i(\mathbf{r})) \cdot d\mathbf{r}.$$

Geometrically, residual supply is the gradient of the value function of other agents in the coalition – the marginal cost of reallocating q_i to i ; it is the price that would clear the market for $Q - q_i$ were i not present.¹⁵ If bidder i could optimize ex post, his first-order condition would equalize his marginal utility and the marginal payment:

$$\nabla u_i(q_i) - R_i(q_i) = 0.$$

Hence, submitting the inverse demand function $b_i(q_i) = \nabla u_i(q_i)$ is a weakly dominant strat-

¹³ It is useful to understand the Vickrey auction by thinking about its implementation by asking agents to submit demand functions $\mathbb{R}^K \rightarrow \mathbb{R}^K$, as we do here in our characterization of the payment rule. Given strict concavity, this is equivalent to implementation by requiring agents to submit utility functions $\mathbb{R}^K \rightarrow \mathbb{R}$, as is common in the literature.

¹⁴ In one-sided settings, the convention in the literature is to assume that if only one bidder is present, he receives the entire quantity being auctioned for free. In a two-sided setting where $Q < 0$, this would imply that a single bidder whose preferences satisfy free disposal is better off in autarky than participating in the Vickrey auction. Therefore, we assume that with only one bidder present, the auctioneer abandons the Vickrey mechanism, since the knowledge of the agent's preferences that it is designed to reveal is then unnecessary to determine the efficient allocation. This is without loss for our results when $Q > 0$.

¹⁵ The aggregate quantity supplied to i at p is $Q - \sum_{j \neq i} b_j^{-1}(p)$. Given the assumption that the $\nabla u_j(\mathbb{R}^K)$ are the same, this will be a surjective and upward-sloping function on $\nabla u_j(\mathbb{R}^K)$ under truthful reporting, and we can define $R_i : \mathbb{R}^K \rightarrow \mathbb{R}^K$ by $R_i(q_i) = \left(Q - \sum_{j \neq i} (\nabla u_j)^{-1} \right)^{-1}(q_i)$. To complete the definition of the game, we assume that if any (inverse) residual supply R_i is not well-defined, the auction is cancelled; this has no effect on incentive compatibility.

egy for i , since it causes the ex post FOC to be satisfied for *any* realization of residual supply.

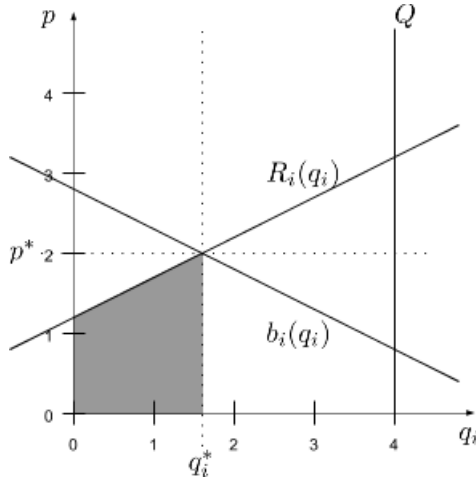


FIGURE 1: VICKREY AUCTION PAYMENT, ALLOCATION, RESIDUAL SUPPLY

Notes: The Vickrey auction can be implemented by soliciting downward-sloping (net) demand schedules and clearing the market as in a standard uniform price auction, but charging bidders the area under their residual supply curves, eliminating bidders' price impact and incentivizing them to bid sincerely. Bidder i 's residual supply aggregates the other agents' quantity demanded, for every price and mirrors the result about the vertical line at Q . The equilibrium auction allocation q_i^* is then determined by the point where i 's bid schedule intersects residual supply. Bidder i 's payment corresponds to the area below his residual supply curve from 0 to q_i^* ; this is the opportunity cost of not allocating q_i^* to the other agents.

Together with market clearing, the bidders' demand functions $b_i(q_i)$ give us the Kuhn-Tucker conditions for the grand coalition's optimization problem, and thus the divisible-good Vickrey auction yields an efficient allocation. The reader may wonder: why run a two-sided auction instead of two one-sided Vickrey auctions, whose properties are well understood in the literature? We will show that while resulting in the same allocation, the standard design would compromise the auctioneer's revenue (Section 6.2).

COALITIONAL VALUE FUNCTION AND THE CORE. To characterize the *core* of the environment described above, let us label the auctioneer agent 0 and, for any *coalition* $W \subseteq N$, define the *surplus function* $v : 2^N \times \mathbb{R}^K \rightarrow \mathbb{R}$ as the maximum value of the goods allocation among the members of W ,

$$v(W, Q) = \max_{\{q_i\}_{i \in W}} \sum_{i \in W} u_i(q_i) \text{ s.t. } \sum_{i \in W} q_i = Q.$$

For each i , the *efficient allocation* $q_i^* : 2^{N+1} \times \mathbb{R}^K \rightarrow \mathbb{R}^K$ is one that maximizes the total

utility of the bidders,

$$\{q_i^*(W, Q)\}_{i \in W} = \arg \max_{\{q_i\}_{i \in W}} \sum_{i \in W} u_i(q_i) \text{ s.t. } \sum_{i \in W} q_i = Q.$$

The surplus function allows us to characterize agents' payoffs in the Vickrey auction as their *marginal product to the grand coalition*, $v(N) - v(N \setminus i)$.^{16,17} Now define the *coalitional value function* $V : 2^{N+1} \times \mathbb{R}^K \rightarrow \mathbb{R}$ by

$$V(W, Q) \equiv \begin{cases} v(W, Q), & 0 \in W \\ v(W, 0), & 0 \notin W \end{cases}.$$

The *core* is the set of payoff profiles such that no coalition can be better off by abandoning the mechanism and trading on its own. A payoff profile π is in the core for participants Z if $\sum_{i \in Z \cup 0} \pi_i = v(Z)$ and $\sum_{i \in W} \pi_i \geq V(W)$ for all $W \subset Z \cup 0$. If $\sum_{i \in W} \pi_i < V(W)$ we say that W *blocks* π . Note that we are not assuming that $v(W, Q) = 0$ if $0 \notin W$; that is, that goods cannot be redistributed among the bidders without the auctioneer. In the two-sided auction, unless reallocation of initial endowments among agents is unnecessary (e.g., initial allocation is efficient) or infeasible (e.g., the auction is one-sided) and the only surplus comes from allocating Q , agents may be able to realize gains from trade without benefit of the auctioneer's quantity vector.

3 Core Selection and Bidder Submodularity

Perhaps the main question that has returned attention to the Vickrey auction over the past decade concerns the conditions under which it is *core-selecting*, i.e., it yields a payoff profile that lies in the core, ensuring that no coalition can profitably persuade the auctioneer to cancel the auction and award them the good. This has been shown to reduce to the question of whether the coalitional value function $v(W, Q)$ has decreasing differences in its first argument under the usual set order \subseteq (Ausubel and Milgrom (2002)): $v(W, Q)$ is *bidder-*

¹⁶ This formulation is more commonly used in the literature, and indeed, as alluded to in an earlier footnote, the Vickrey auction can be equivalently implemented by soliciting agents' valuations, allocating the quantity vector efficiently, and charging a payment of $u_i(q_i^*(W, Q)) - v(N) + v(N \setminus i)$.

¹⁷ It is useful to recall why this payment rule results in a payoff of $\pi_i = v(N, Q) - v(N \setminus i, Q)$: $R_i(q_i)$ is the marginal cost to the coalition $N \setminus i$ of assigning q_i to i , $-\nabla v(N \setminus i, Q - q_i)$. Hence,

$$v(N \setminus i, Q) - \int_0^{q_i^*(N, Q)} R_i(\mathbf{r}) \cdot d\mathbf{r} = v(N \setminus i, Q - q_i^*(N, Q)) \Leftrightarrow \int_0^{q_i^*([0, N])} R_i(\mathbf{r}) \cdot d\mathbf{r} = v(N \setminus i, Q) - v(N \setminus i, Q - q_i^*(N, Q)).$$

Since $v([0, N], Q) = v(N \setminus i, Q - q_i^*(N, Q)) + u_i(q_i^*(N, Q))$, we have $\pi_i = q_i(q_i^*(N, Q)) - \int_0^{q_i^*(N, Q)} R_i(\mathbf{r}) \cdot d\mathbf{r} = v(N, Q) - v(N \setminus i, Q)$.

submodular for the quantity vector Q if, for all coalitions $W \subseteq N$ that include bidder j , $v(W, Q) - v(W \setminus j, Q)$ does not increase when more bidders are added to W . Intuitively, bidders are more valuable when added to smaller coalitions than when added to larger coalitions. Unlike one-sided auctions, in two-sided settings, to ensure core selection, we need to consider coalitions that do not include the auctioneer: bidders may have a profitable joint deviation to cease their participation in the auction and trade among themselves. As it turns out, however, bidders always prefer to participate in the Vickrey auction than negotiate on their own.

Lemma 1 (Coalitional Rationality of Participation). *Coalitions which do not involve the auctioneer never block the Vickrey payoff profile.*

This new result shows that participation in the two-sided Vickrey auction is not only individually rational, but *coalitionally* rational as well. For intuition, observe that setting $Q = 0$ and considering the grand coalition yields the following corollary:

Corollary 1 (Vickrey (1961)). *The auctioneer’s revenue in a Vickrey exchange is never positive.*

This classic no budget balance result by Vickrey (1961) implies that when $Q = 0$, and thus the mechanism is only reallocating, agents are collectively better off than without the auctioneer, as the auctioneer effectively subsidizes them. Lemma 1 generalizes this insight, showing that *every* coalition is so subsidized for *any* Q . This need not be read as a negative statement about Vickrey revenue: indeed, our analysis in section 6.3 will imply that when heterogeneity in the agents’ marginal utilities at their initial endowments (i.e., with $q_i = 0$) is sufficiently small relative to Q , the auctioneer will make a profit (indeed, he will receive greater revenue than he would with the common uniform-price auction design).

The many reasons to care whether a design is core selecting are well understood from one-sided indivisible good settings (e.g., Ausubel and Milgrom (2002), Milgrom (2007)). Those most frequently mentioned are that (a) it prevents manipulation by the bidders – shill bidding is unprofitable (Yokoo and Matsubara (2004), Day and Milgrom (2008)); (b) the auctioneer’s revenue is monotone in the number of bidders; and (c) no coalition of losing bidders can profitably collude. Note that the last of these does not readily carry over to the divisible-goods case, where the idea of a “losing bidder” is unnatural – typically, all bidders are allocated some quantity – and collusion among bidders can be profitable. For the others, however, we can draw similar conclusions about two-sided divisible good auctions.

Lemma 2 (Implications of Core Selection). *The Vickrey auction is core selecting regardless of which agents participate if, and only if, $v(\cdot, Q)$ is submodular.*

Hence, if the Vickrey auction is core selecting for any participating agents,

- (i) The auctioneer’s revenue is increasing in the number of participating bidders;
- (ii) If the potential shill bidder is not the only participant in the auction, shill bidding is unprofitable regardless of which agents participate.¹⁸

Since, in keeping with classical auction theory, we do not in general assume that the auctioneer has preferences over the goods he sells, the definition of the core requires the auctioneer to receive positive revenue so that $\{0\}$ does not block the payoff profile. This is unimportant in one-sided settings with $Q > 0$, where the auctioneer always receives non-negative revenue. To study core-selection in two-sided settings and in one sided settings for procurement auctions ($Q < 0$), we will also consider weaker conditions under which the payoff profile is *unblocked by all* $W \neq \emptyset$. We say that the Vickrey auction is *core-selecting with respect to bidders* if the payoff profile is unblocked by every coalition except for $\{0\}$, the auctioneer alone.

Corollary 2. *The Vickrey auction is core-selecting with respect to bidders regardless of which agents participate if, and only if, $v(\cdot, Q)$ is submodular on the sublattice $2^{N \setminus i} \times \{i\}$ for each i .*

Hence, if the Vickrey auction is core selecting with respect to bidders for any participating agents, when at least two bidders participate,

- (i) The auctioneer’s revenue is increasing in the number of participating bidders;
- (ii) Shill bidding is unprofitable regardless of which agents participate.

Saying that $v(\cdot, Q)$ is submodular on each $2^{N \setminus i} \times \{i\}$ is equivalent to saying that $v(W, Q) + v(Z, Q) \geq v(W \cup Z, Q) + v(W \cap Z, Q)$ for all W, Z such that $W \cap Z \neq \emptyset$.¹⁹

4 Core Selection with Divisible Goods

This section and the next contain the main results of the paper. We begin with examples that illustrate the key new aspects of core selection with divisible goods.

4.1 Motivating examples

The literature has established a close connection between core selection (bidder submodularity)²⁰ and the substitutes property of agents’ demands for goods.

¹⁸ More precisely, ‘unprofitable shill bidding’ means that if v is submodular for any utility profile $\{u_i\}_{i=1}^N$ in some type space U , then mimicking any set of agents with any types in U is unprofitable.

¹⁹ This corollary then follows from the observation that for potential blocking coalitions other than the auctioneer alone, none of the terms in the proof of the first part take the empty set as an argument. The same is true in the second part when at least two bidders participate.

²⁰ Bidder submodularity is necessary when the objective is core selection for any agents as opposed to a given set of participants.

Definition 1. Goods are (*weak*) *substitutes* if the quantity each agent demands of any good is nondecreasing in the price of every other good.

Do we have reasons to think that core selection might *not* hold with divisible goods? Existing results show that when goods are substitutes and allocations are restricted to be nonnegative, bidder submodularity holds and, thus, the Vickrey auction is core-selecting (e.g., Milgrom and Strulovici (2009, Theorem 31)). We first show in Example 1 that, in the context of a broader set of applications, this restriction on feasible allocations is crucial for core selection: when the bidders and/or the auctioneer can buy and sell, the Vickrey outcome may not lie in the core *even when substitutability is trivially satisfied* (e.g., in settings with only one good). In the examples that follow, we will be working with quadratic valuations

$$u_i(q_i, \theta_i, S_i) = \theta'_i q_i - \frac{1}{2} q'_i S_i q_i, \quad (1)$$

where S_i is positive definite, for which the coalitional value function and equilibrium quantities are given in closed form.²¹

Lemma 3 (Coalitional Value Function: Quadratic Utility). *With quadratic valuations, the coalitional value function is*

$$v(W, Q) = \frac{1}{2} \sum_{i \in W} \theta'_i S_i^{-1} \theta_i - \frac{1}{2} \left(\sum_{i \in W} S_i^{-1} \theta_i - Q \right)' H(W) \left(\sum_{i \in W} S_i^{-1} \theta_i - Q \right),$$

where $H(W) \equiv \left(\sum_{j \in W} S_j^{-1} \right)^{-1}$ is a harmonic mean and the competitive equilibrium quantity of bidder i is $q_i^*(W, Q) = S_i^{-1} \theta_i - S_i^{-1} H(W) \left(\sum_{j \in W} S_j^{-1} \theta_j - Q \right)$.

In particular, note that when $\theta_i = \theta$ for each $i \in N$, $v(W)$ simplifies to $\theta' Q - \frac{1}{2} Q' H(W) Q$.²²

Example 1 (No Core Selection with a Single Divisible Good). An auctioneer has $Q = 1$ units of a divisible good for sale in a Vickrey auction. Consider bidders 1 and 2 with

²¹ The quadratic model is the commonly studied environment in the divisible good auctions and microstructure literature; the innovation here is allowing for heterogeneity in the utility convexities $\{S_i\}_{i=1}^N$. By considering agents' preferences over auction allocations u_i instead of over total allocations $\hat{u}_i(q_i) \equiv \hat{\theta}'_i q_i - \frac{1}{2} q'_i S_i q_i$, (1) reduces heterogeneity in endowments $\{q_i^0\}_{i=1}^N$ and the constant term of marginal utility into heterogeneity in the latter alone (cf. Ft. 10): by letting $\theta_i = \hat{\theta}_i - S_i q_i^0$, we have

$$u_i(q_i, \theta_i, S_i) = \hat{u}_i(q_i + q_i^0, \hat{\theta}_i, S_i) - \hat{u}_i(0, \hat{\theta}_i, S_i) = \hat{\theta}'_i (q_i + q_i^0) - \frac{1}{2} (q_i + q_i^0)' S_i (q_i + q_i^0) - \left(\hat{\theta}'_i q_i^0 - \frac{1}{2} q_i^0{}' S_i q_i^0 \right).$$

Hence, in the quadratic model, agent i is characterized by the tuple of characteristics (θ_i, S_i) (or equivalently, $(\hat{\theta}_i, q_i^0, S_i)$), where S_i is positive definite.

²² Also note that the difference-in-difference $v(W \cup Z) + v(W \cap Z) - v(W) - v(Z)$ is just $\frac{1}{2} Q' (H(W) + H(Z) - H(W \cap Z) - H(W \cup Z)) Q$, and that the central matrix in this expression is the difference-in-difference of the inverse slopes of coalitions' aggregate demands.

valuations $u_i(q_i) = 4q_i - \frac{1}{2}q_i^2$ and bidder 3 with valuation $u_3(q_3) = 2q_3 - \frac{1}{2}q_3^2$. Then, using Lemma 3, allocations are

$$\begin{aligned} q_1^*({1, 2, 3}, Q) &= q_2^*({1, 2, 3}, Q) = 4 - \frac{1}{3}(4 + 4 + 2 - 1) = 1 & q_3^*({1, 2, 3}, Q) &= -1 \\ q_1^*({1, 3}, Q) &= q_2^*({2, 3}, Q) = \frac{3}{2} & q_3^*({1, 3}, Q) &= -\frac{1}{2} \\ q_1^*({1, 2}, Q) &= q_2^*({1, 2}, Q) = \frac{1}{2} & q_3^*({2, 3}, Q) &= -\frac{1}{2}, \end{aligned}$$

and surpluses $v(W, Q)$ are

$$\begin{aligned} v({1, 2, 3}, Q) &= \frac{1}{2}(16 + 16 + 4) - \frac{1}{6}(4 + 4 + 2 - 1)^2 = \frac{9}{2} & v({1, 3}, Q) &= v({2, 3}, Q) = \frac{15}{4} \\ v({1, 2}, Q) &= \frac{15}{4} & v({1}, Q) &= \frac{7}{2}. \end{aligned}$$

Thus, the sum of Vickrey payoffs for the coalition of the auctioneer and bidder 1 is $\pi_0 + \pi_1 = -v({0, 1, 2, 3}) + v({0, 1, 2}) + v({0, 1, 3}) = 3 < v({0, 1})$ and so the coalition $\{0, 1\}$ blocks the Vickrey payoff imputation.²³

We can glean from the literature the fact that if each of the agents' auction allocations were positive – that is, if the auction behaved like a one-sided one – then the payoff imputation would be in the core. Instead, bidder 3 ends up selling some of her initial endowment in the Vickrey auction. Since the Vickrey payment rule ensures that the auctioneer pays the bidder more for this amount than he ultimately receives from the other bidders from selling it, this makes accepting an offer to cancel the auction and avoid reallocation attractive; the auctioneer could award all of the quantity to bidder 1 to receive a payment larger than the Vickrey revenue.²⁴ However, if the *marginal* utilities at zero auction allocation were identical (e.g., if initial allocations and marginal utilities at zero total allocation were identical), reallocation would be unnecessary and the outcome would be in the core (cf. Theorem 3).

Conversely, goods that are *not* substitutes are generally viewed as a challenge to core selection. Example 2 shows that the absence of substitutes, even for all bidders, does not preclude Vickrey payoffs from being a core point.

²³ To the best of our knowledge, we are the first to recognize the challenges that reallocation presents for core selection. The divisible goods results of Milgrom and Strulovici (2009) focus on one-sided auctions and do not consider quantity vectors outside the positive orthant, as do the indivisible goods results of Ausubel and Milgrom (2002). More specifically, in the argument for divisible goods in Milgrom and Strulovici (2009, Theorem 31), one step involves showing that decreasing differences in $v(\cdot, Q)$ are implied by $v(W, Q) - v(W, Q')$, $Q' \geq Q$, being nondecreasing in W . This implication follows from the grand coalition receiving fewer goods in aggregate when an additional agent participates, which is true only when that additional agent's allocation is positive.

²⁴ Note that the bidder who successfully forms a coalition with the auctioneer that blocks the Vickrey payoff need not be the one whose valuation is the highest. In Example 1, if the marginal utility intercepts of bidders 1 and 2 were $4 - \varepsilon$ and $4 + \varepsilon$, respectively, then $\{0, 1\}$ would still form a blocking coalition.

Example 2 (Core Selection with Complement and Substitute Goods Valuations). Suppose a seller wishes to auction vector $[1 \ 1]'$ of two goods using a Vickrey auction. Consider bidder 1 with valuation

$$[1, 1]q_i - \frac{1}{2}q_i' \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix} q_i,$$

and bidders 2 and 3 with valuations

$$[1, 1]q_i - \frac{1}{2}q_i' \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix} q_i.$$

Goods are complements for bidder 1, but substitutes for bidders 2 and 3. Let us calculate allocations $q_i^*(W, Q)$,

$$\begin{aligned} q_2^*({1, 2, 3}, Q) = q_3^*({1, 2, 3}, Q) &= \begin{bmatrix} 0.2857 \\ 0.2857 \end{bmatrix} & q_1^*({1, 2, 3}, Q) &= \begin{bmatrix} 0.4286 \\ 0.4286 \end{bmatrix} \\ q_2^*({1, 2}, Q) = q_3^*({1, 3}, Q) &= \begin{bmatrix} 0.4 \\ 0.4 \end{bmatrix} & q_1^*({1, 2}, Q) &= \begin{bmatrix} 0.6 \\ 0.6 \end{bmatrix} \\ q_2^*({2, 3}, Q) = q_3^*({2, 3}, Q) &= \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} & q_1^*({1, 3}, Q) &= \begin{bmatrix} 0.6 \\ 0.6 \end{bmatrix}, \end{aligned}$$

and surpluses $v(W, Q)$,

$$\begin{aligned} v(\{1\}, Q) &= \frac{3}{2}, & v(\{2\}, Q) &= v(\{3\}, Q) = \frac{5}{4}, \\ v(\{2, 3\}, Q) &= \frac{13}{8}, & v(\{1, 2\}, Q) &= v(\{1, 3\}, Q) = \frac{17}{10}, & v(\{1, 2, 3\}, Q) &= \frac{25}{14}. \end{aligned}$$

Then bidder 1 receives a payoff of $\frac{9}{56}$, bidders 2 and 3 receive a payoff of $\frac{3}{35}$, and thus the auctioneer receives $\frac{25}{14} - \frac{9}{56} - \frac{6}{35} = \frac{407}{280}$; this is positive, so the auctioneer alone does not block the Vickrey payoff profile. Trivially, no coalition of all but one agent blocks the Vickrey payoffs. We also have $\frac{3}{2} - \frac{9}{56} - \frac{407}{280} = -\frac{4}{35} < 0$ and $\frac{5}{4} - \frac{3}{35} - \frac{407}{280} = -\frac{81}{280} < 0$, so no coalition of one bidder and the auctioneer blocks the Vickrey payoffs. Since we know from Lemma 1 that no coalition which does not involve the auctioneer blocks the Vickrey payoff profile, it follows that the Vickrey payoffs are in the core. Clearly, bidders 2 and 3 could have the same complement valuation as bidder 1, and the Vickrey auction would still be core-selecting.

Why does this example work? The necessity of the substitutes condition for the Vickrey payoffs to lie in the core has been demonstrated in a variety of environments. We will show that it is not the presence of complementarities *per se*, but instead a type of heterogeneity in *substitution patterns* – the second-order derivatives of agents' utility functions – that can challenge core selection. Anticipating our results, there is a sense in which substitution patterns are symmetric across bidders in Example 2: the two utility Hessian matrices commute,

since both have the same eigenvectors $[1 \ 1]'$, $[-1 \ 1]'$.²⁵ The commutativity property allows transforming the problem from one with heterogeneous substitutability and complementarity between goods to one with substitutability between packages and, as in the example, achieves core selection. Importantly, the ability to transform the problem into one with strict substitutability does *not* generally, by itself, render the Vickrey auction core-selecting. Rather, what matters is that the substitution patterns are not too different across agents, as the next example demonstrates.

Example 3 (No Core Selection with Substitutes and Efficient Initial Allocation).

An auctioneer wishes to sell 1 unit of good 1 and 0 units of good 2 using a two-sided Vickrey auction; that is, he wishes to auction the vector $Q = [1, 0]$.²⁶ Three bidders with the following valuations participate

$$[10 \ 10]q_1 - \frac{1}{2}q_1' \begin{bmatrix} 0.25 & 0.19 \\ 0.19 & 1.49 \end{bmatrix} q_1, \quad [10 \ 10]q_2 - \frac{1}{2}q_2' \begin{bmatrix} 5.8 & 0 \\ 0 & 5.8 \end{bmatrix} q_2, \quad [10 \ 10]q_3 - \frac{1}{2}q_3' \begin{bmatrix} 3.07 & 3.47 \\ 3.47 & 4.47 \end{bmatrix} q_3.$$

Note that endowments and utility functions are such that no gains to trade exist prior to the auction (i.e., marginal utilities at zero trade are the same across bidders). By Lemma 3, since the initial allocation is efficient, if

$$Q'H(\{0, 1, 2\})Q + Q'H(\{0, 1, 3\})Q < Q'H(\{0, 1\})Q + Q'H(\{0, 1, 2, 3\})Q,$$

then

$$\pi_0 + \pi_1 = -v([0, 3]) + v([0, 2]) + v(\{0, 1, 3\}) < v(\{0, 1\})$$

and the coalition $\{0, 1\}$ blocks the Vickrey allocation. This is exactly what happens, since

$$Q'(H(\{0, 1\}) + H(\{0, 1, 2, 3\}) - H(\{0, 1, 2\}) - H(\{0, 1, 3\}))Q = Q' \begin{bmatrix} -0.0015 & 0.0146 \\ 0.0146 & 0.2398 \end{bmatrix} Q = -0.0015.$$

In this example, a positive quantity is auctioned and goods are substitutes for all bidders. Thus, if the new quantity did not require reallocation of the goods among the bidders and only selling, we know from the literature that core selection would hold. Because the bidders' substitution patterns are heterogeneous, the new supply of good 1 creates gains to trade for the bidders from reallocating good 2 and, despite the absence of gains to trade before the auction, the reallocation is sufficient for core selection to break down.

²⁵ Two matrices A and B commute if $AB = BA$. Diagonalizable matrices A and B commute if, and only if, they have the same eigenspace.

²⁶ Let us remark for a future reference that the logic of this example can be mimicked for the case when the auctioneer wishes to exchange 0.5 units of good 1 for 0.9 units of good 2 using a two-sided Vickrey auction; that is, he wishes to auction the vector $Q = [0.5, -0.9]$.

These examples demonstrate that, in a two-sided Vickrey auction, goods substitutability is neither necessary (Example 2) nor sufficient (Examples 1 and 3) for a core-selecting design. Moreover, the heterogeneity in substitution patterns, as captured by utility convexities (Example 3), and *pre-auction gains from trade*, as determined jointly by marginal utilities at agents' initial allocations (Example 1), can separately challenge core selection.²⁷ In our subsequent analysis, we work to characterize these challenges, and in so doing establish the reasons that the Vickrey payoffs are in or outside the core in the above examples.

We characterize the sense in which whether the Vickrey auction fails to be core-selecting depends on heterogeneity in substitution patterns, both across bidders and across allocations in the domain of $u_i(\cdot)$. In Section 4.2, we identify the notion of substitutability that is relevant for core selection in two-sided divisible good auctions; in Section 4.3, we add a condition on allocations.

4.2 Generalizing the substitutes property

The demand-theoretic substitutes property is a monotonicity property in the partial order \geq , relating changes in price to changes in quantity. When agents' preferences satisfy the substitutes condition, there are two related consequences:

- When the price of good ℓ increases, agents buy more of good j .
- When price vector for goods changes in a certain direction, agents' demand vectors for goods each move in a certain similar direction.

The literature has focused on the former, as is natural in one-sided auctions; indeed, as will be apparent later, in this case the weaker second condition is of limited use. The latter is, however, important in two-sided auctions, where we show that one can get additional mileage toward understanding when core selection holds from its weaker requirements.²⁸ Explaining the content of the latter at the primitive level is the focus of this section. The crux of our argument is that it is primarily the second of these that gives the substitutes condition its power, and hence that the choice to order the space of goods by \geq does not constrain us when proving core selection. We show that one can use the substitutes monotonicity property in *any* lattice order on \mathbb{R}^K to achieve core selection. In particular, we can endow \mathbb{R}^K with a partial order generated by *packages* – bundles of goods – analogously to how \geq is generated

²⁷ That is, with sufficient heterogeneity in substitution patterns, core selection fails *regardless* of homogeneity in marginal utility at the initial endowment (Example 3); and with sufficient heterogeneity in the marginal utilities, core selection fails *regardless* of homogeneity in the substitution patterns (Example 1). Clearly, core selection can obtain with small enough heterogeneity in substitution patterns across bidders – e.g., when the common eigenvectors in Example 2 are perturbed slightly – and similarly for heterogeneity in marginal utilities at agents' initial allocations.

²⁸ We generalize the substitutes (monotonicity) property by dropping the first of these – the quantifier on goods.

by goods. We show that when we can find such packages which are substitutes, they specify a change of basis which, when applied in either a one- or two-sided auction, allows us to aggregate bidders' indirect utility functions into a submodular objective function for the social planner's dual minimization problem, just as the usual substitutes condition does. We also give conditions on the primitives that establish precisely when we can find packages that are substitutes.

To develop these ideas, it will be useful to work with cones generated by packages; these cones will in turn generate partial orders in which the substitutes property holds. Recall that a *cone* C is a subset of \mathbb{R}^K such that for every vector $c \in C$, we also have $ac \in C$ for all $a > 0$. The *dual cone* of C , denoted C^* , is the set of all vectors in $x \in \mathbb{R}^K$ such that $x \cdot c \geq 0$ for all $c \in C$. Geometrically, C^* is the set of all vectors that form an acute angle with every vector in C . A cone is *pointed* if $-C \cap C = \{0\}$. A pointed cone C defines a partial order \succeq_C on \mathbb{R}^K : $x \succeq_C x'$ if $x - x' \in C$. A set of K linearly independent vectors $\mathcal{C} = \{c_k\}_{k=1}^K$ is a *basis* for the cone C if C is the set of positive linear combinations of vectors in \mathcal{C} , i.e. $C = \{\sum_{k=1}^K c_k \lambda_k \mid \lambda_k \geq 0 \forall k\}$.

For any basis \mathcal{C} of K linearly independent packages, the cone C they generate orders the space of allocations by the quantity of these packages they contain; likewise, the dual cone C^* orders the space of prices by how expensive these packages are. Given a basis of packages \mathcal{C} , let T_C^{-1} be their associated (pullback) change of basis matrix, that is, the matrix whose columns are the elements of \mathcal{C} . This gives alternative ways to define a cone and its dual

$$\begin{aligned} C &= \{x \mid T_C x \geq 0\} = \{T_C^{-1} x \mid x \geq 0\} \\ C^* &= \{T_C' x \mid x \geq 0\} = \{x \mid T_C^{-1'} x \geq 0\} \end{aligned}$$

and so $C = C^{**}$. T_C , the (polyhedral) generator for C , maps quantity vectors to their representation as a vector of packages, and T_C^{-1} maps price vectors to their representation as a vector of package prices. We can also see from the above that the rows of T_C form a basis \mathcal{C}^* for C^* ; since $T_C T_C^{-1} = I_K$, this \mathcal{C}^* consists of vectors that are orthogonal to all but one of the vectors of \mathcal{C} – the package that it is the price of. Thus, when we talk about the price of a package increasing, we mean that the (goods) price vector moves in the direction in C^* orthogonal to each of the other packages in \mathcal{C} .²⁹ In the remainder of the paper, when we talk about packages \mathcal{C} we mean a *basis* of packages \mathcal{C} , and when we talk about a cone C

²⁹ It is important for our analysis that the packages form a basis; otherwise the packages would define a partial order, but not a lattice order and there would be no corresponding lattice order on prices. It is immediate that when cone C is generated by a basis, $(\mathbb{R}^K, \succeq_C)$ is a lattice. Clearly, $x \vee_C y = T_C^{-1}(T_C x \vee T_C y)$ and $x \wedge_C y = T_C^{-1}(T_C x \wedge T_C y)$. This is important because we are interested in the submodularity of the indirect utility function for packages; for substitutes to guarantee this, we need the space of prices (which we need the indirect utility function to be submodular in) $(\mathbb{R}^K, \succeq_{C^*})$ to be a lattice. A finitely generated cone C is a lattice cone if, and only if, it is generated by a cone basis consisting of K linearly independent vectors (e.g., Aliprantis and Tourky (2007)).

we mean a cone generated by such packages.

We argue that the key to the aggregation properties underlying the core selection results (i.e., bidder-submodularity) is that quantity demanded across bidders changes in a *common* set of directions in response to a price change. This leads us to the idea of package substitutability. Given packages \mathcal{C} , consider bidder i 's utility maximization problem over packages

$$\max_{q_i^C} u_i(T_C^{-1}q_i^C) - p_C \cdot q_i^C \equiv \Pi_i^C(p_C),$$

where the price vector p_C specifies the price of each package and the quantity vector q_i^C is a vector of packages. When the packages are goods, this is the usual indirect utility function $\Pi_i(p)$. It is immediate that $T_C^{-1}q_i^C(p_C) = q_i(T_C'p_C)$ and, equivalently, $q_i^C(T_C^{-1}p) = T_Cq_i(p)$.

Definition 2. *Packages are substitutes* if, for any k , the quantity $\{q_i^C(p_C)\}_k$ of package c_k demanded by i weakly increases when the price of any other package $j \neq k$ increases.

Intuitively, to characterize what package substitutability means in terms of primitives, we can relate demand behavior to the properties of the second order derivatives of the utility function through bidders' first-order conditions.

Lemma 4 (Goods Substitutability as a Condition on Primitives). *Goods are substitutes for i if, and only if, i 's indirect utility function Π_i is submodular in \geq , or equivalently, for all q_i , $((D^2u_i(q_i))^{-1} + \alpha I)\mathbb{R}_+^K \subseteq \mathbb{R}_+^K$ for some α .*

Applying the inverse function theorem to i 's first-order condition tells us that $Dq_i(p) = (D^2u_i(q_i))^{-1}$; when this matrix's off-diagonal entries are nonnegative, then quantity demanded of each good is weakly increasing in the prices of the other goods. In the statement of the Lemma, we express this geometrically as a mapping property of the matrix: $(D^2u_i(q_i))^{-1}$, plus some positive diagonal matrix αI , maps positive vectors to positive vectors. Theorem 1 gives a similar statement about the substitutes property for packages \mathcal{C} .

Theorem 1 (Package Substitutability as a Condition on Primitives). *Packages \mathcal{C} are substitutes for i if, and only if, Π_i^C is submodular in \geq , or equivalently i 's indirect utility function Π_i is submodular in \succeq_{C^*} , or equivalently, for all q_i , $((D^2u_i(q_i))^{-1} + \alpha T_C^{-1}T_C^{-1}')C^* \subseteq C$.*

This time, the inverse function theorem tells us that $Dq_i^C(p_C) = T_C(D^2u_i(T_C^{-1}q_i^C))^{-1}T_C'$; the nonnegativity of the off-diagonal elements of this matrix can again be expressed geometrically as a mapping property, this time between the package price cone C^* and the package cone C , as we do in the statement of the theorem. Unlike the cone generated by goods (the positive orthant), however, the package cone and its dual need not be the same, so the mapping property needed is slightly different.

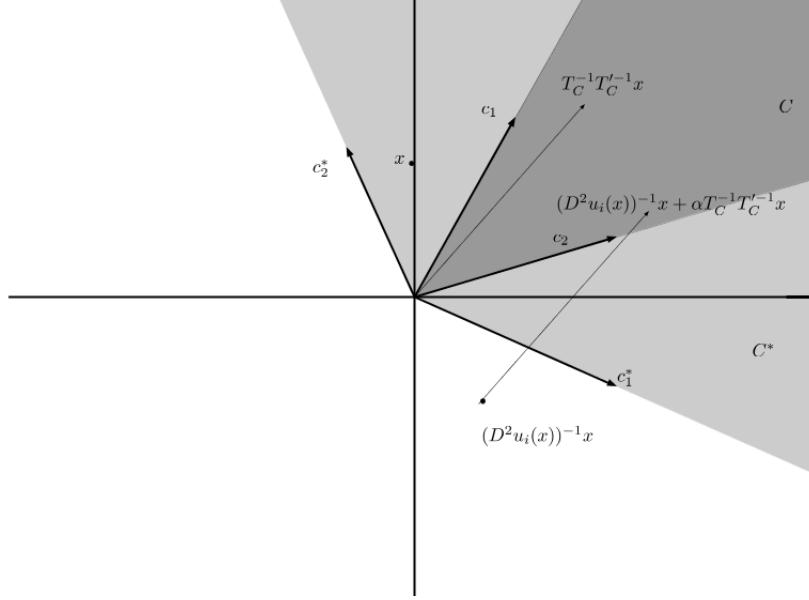


FIGURE 2: CONES AND SUBSTITUTES

Notes: The figure depicts packages c_1 and c_2 , the cone C they generate, its dual C^* , and the edges of C^* , which are the prices of packages c_1 and c_2 , respectively. The substitutes condition says that the inverse Hessian matrix maps a point x in C^* to some point y in C , minus a scalar multiple of its dual representation $T_C^{-1}T_C'^{-1}x$.

Thus, like goods substitutability, package substitutability also corresponds to decreasing differences in indirect utility, but in the partial order defined by package prices rather than by prices of goods. Theorem 1 completely characterizes the set of utility functions for which packages are substitutes; they correspond to (and, hence, are as restrictive as) the *submodularity of the indirect utility function in the package price partial order*.

What exactly are these packages in real-world terms? Consider selling two goods, wireless spectrum in Los Angeles and Chicago. Bidder demand for a unit of package of $[1 \ 1]'$ corresponds to buying a unit of spectrum in each market, while a unit of package of $[1 \ -1]'$ corresponds to buying a unit of spectrum in Los Angeles and selling a unit in Chicago. Bidders might differ in which market they value spectrum in relatively more; we need not assume that everyone's utility over packages is increasing in the same direction. Similarly, in electricity auctions, the goods could correspond to electricity in different locations and package $[1 \ -1]'$ to a contract to transmit electricity from LA to Chicago. Alternatively, the package can represent contracts to move a TV station between different frequencies. We later discuss that the choice to characterize allocations as bundles of *goods* rather than *packages* is not only unimportant for core selection, but need not constrain implementation either. Generalizing substitutes from goods to packages by changing how we think of 'more' is thus natural in some applications when we know how agents are likely to substitute between these

packages.

Let us make two final observations about the notion of substitutes. First, the demand-theoretic notion of *package* substitutability is not just a property of an agent’s valuation; rather, it is about the interaction of an agent’s utility and the choice set.^{30,31} In addition, for the core selection analysis, the substitutes condition only needs to hold in a region of the space of goods where the auction allocation will be guaranteed by assumption to lie: precisely in the cube $C \cap (Q - C)$, since allocations lie in C and that the auction clears with allocations summing to Q (feasibility). We will say *packages \mathcal{C} are substitutes on X* for bidder i if for a demanded quantity vector in a set X , a price change sufficiently small for the demand to still be in X causes it to increase; more precisely, that is, using Theorem 1, if $((D^2u_i(q_i))^{-1} + \alpha T'_C T_C)C^* \subseteq C$ for all $q_i \in X$.

4.3 Bidder submodularity when packages are substitutes

As Examples 1 and 3 show, goods substitutability is neither sufficient nor necessary for core selection – allocations matter separately. Clearly, the same argument applies with package substitutability, and hence the condition in Theorem 1 does not guarantee core selection. That is, unlike one-sided settings, the submodularity of bidders’ indirect utility functions (i.e., goods or package substitutability) is not sufficient for the submodularity of the coalitional value function (i.e., core selection). As further hinted by the examples, (the direction of) the quantity vector Q matters. Theorem 2 establishes a joint condition on package substitutability and allocations as a condition for core selection.³²

³⁰ The Supplementary Material (Lemma B.1, Corollary B.2) gives a characterization of substitutes in one-sided auctions in terms of valuations.

³¹ At the boundary of the choice set in a one-sided auction, what ‘substitutes’ mean at the primitive level (i.e., $u_i(\cdot)$) does not change, but what ‘packages are substitutes’ means in terms of primitives changes. Namely, unlike in the interior, when demand is on the choice set’s boundary, there are directions in which demand is constrained not to move. Since the directions in which demand can change do not necessarily correspond to the span of a subset of the packages, an agent may only be able to increase quantity demanded of one package by changing the quantity he demands of the others. Hence, the conditions on the primitives that would cause him to do so in response to some change in price will in general differ, and the map between the properties of demand and the primitives is discontinuous at the boundary of the choice set. This is important only when we are interested in the substitutes property for packages, not goods (and, hence, is not a consideration in Milgrom and Strulovici (2009)). Clearly, in the indivisible goods literature, the distinction does not arise. Let us note that this observation is also relevant for the analysis in Baldwin and Klempner (2014): constraining the allocation may change the (unimodal) demand types of an agent with the same valuation.

³² The proof of Theorem 2 differs from the arguments used in the core selection result for one-sided divisible good auctions by Milgrom and Strulovici (2009, Theorem 31), based on the submodularity of the indirect utility function, employing which requires more assumptions. In a two-sided setting, submodularity of the social planner’s dual objective function in (package) prices and bidders does not follow directly from submodularity of the bidders’ indirect utility functions. Instead, we must also require that bidders’ allocations (of packages) will be positive.

Theorem 2 (Core Selection: Package Substitutability and Allocations). *For a given Q , the Vickrey auction is core-selecting with respect to bidders for all profiles of participants if there exist packages \mathcal{C} which are substitutes on $C \cap (Q - C)$ for all bidders, nonnegative quantities of which are allocated to each bidder, for each coalition $W \subset N$; that is, $q_i^*(W, Q) \in C$, for all i , where C is the cone generated by \mathcal{C} . If, in addition, for each W , the derivative of $v(W, Q)$ in the direction of each package in \mathcal{C} is positive, then the Vickrey auction is core selecting for all profiles of participants.*

Relative to the literature, Theorem 2 extends sufficient conditions for core-selecting design to auctions in which bidders and the auctioneer can all buy and sell. The condition that the allocation must be contained in the cone generated by the packages – i.e., $q_i^*(W, Q) \succeq_C 0$, for all i and W – reduces to the classic substitutes condition when the packages are just the primitive goods and the auction is one-sided, since in this case allocations are automatically in the positive orthant. When the auction is two-sided or the packages which are substitutes are not simply goods, the quantity of goods need not always be non-negative in the efficient allocation, but the theorem requires that the quantity of packages must be. That is, bidders are all “net package buyers.” Note that the direction in which Q lies is essential for this, as it determines whether allocating nonnegative quantities of the packages to each bidder is even possible. When the auctioneer can choose Q subject to a cost of production or an engineering constraint, he can make core selection easier by choosing $Q \in C$. Additionally, since the quantity vector Q plays a role in determining both the area $C \cap (Q - C)$ and the auction allocations, the designer’s choice of Q can decide whether both the package substitutability and allocation conditions are satisfied, since the former need only apply on the consumption vectors up to Q – precisely on $C \cap (Q - C)$. We examine this in Section 5.2.

Why is this additional condition necessary? The allocation and derivative conditions ensure that, in two different respects, *the two-sided auction for goods functions like a one-sided auction for packages*. First, the allocation condition ensures that we can restrict attention to the cone C – the positive orthant in the *package* space. Second, the derivative condition ensures that packages function like *goods* for each coalition in that more packages is better for each, at least around Q . As we show, this ensures that the auctioneer will receive non-negative payments for packages from all bidders, just as in a one-sided auction.

Intuitively, underlying the absence of incentives to deviate by coalitions of the bidders and the auctioneer in one- and two-sided settings is an order of packages, common among bidders, that (i) aligns the bidders’ incentives to substitute with the ways in which *gains from*

Additionally, the authors require that goods are substitutes – and hence, that bidders’ indirect utility functions are submodular – everywhere. In a two-sided setting, making this argument with respect to a package order would require considering prices outside of the positive orthant, because the meets or joins in the package price order of prices inside the positive orthant may lie there. We show, however, that it is only necessary to require substitutability where the allocations are assumed to lie – namely, on $C \cap (Q - C)$.

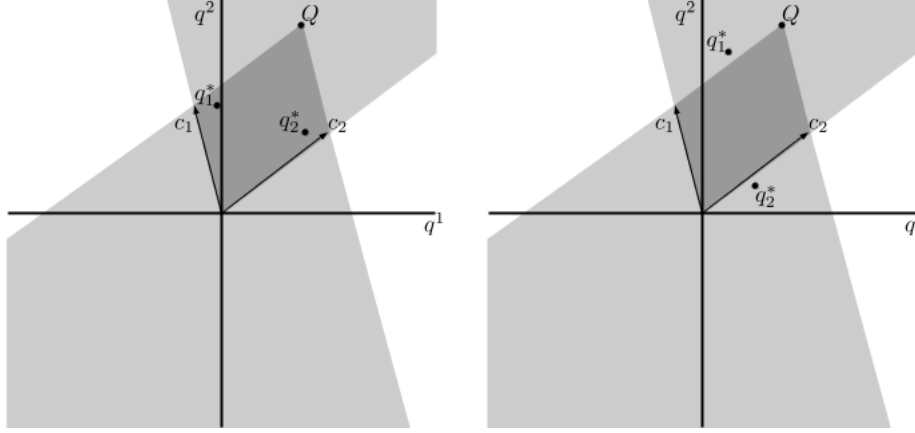


FIGURE 3: ALLOCATION CONDITION FOR THEOREM 2

Notes: Figure 3 shows the allocation condition of Theorem 2 with two bidders and two goods. In the left panel, both bidders receive allocations in C , and the condition is satisfied. In the right panel, this is no longer true.

trade in the auction, created by the available quantity Q and the differences between agents' marginal utilities at their initial endowments, can be realized; and (ii) defines packages that all coalitions desire, so that the auctioneer receives positive revenue.³³ Note that the reordering logic used in Theorem 2 does not require reallocation of initial endowments among the bidders to be feasible: the package substitutability and allocation conditions give core selection also if the auctioneer cannot reallocate goods among the bidders (i.e., is restricted to run a one-sided auction).

Let us revisit the examples from Section 4.1 to see which conditions for core selection in Theorem 2 are violated; Figure 3 illustrates. In Example 1, marginal utilities differ at zero auction allocation (i.e., pre-auction gains from trade are non-zero) and core selection fails with one good. The only possible cones are the half-lines \mathbb{R}_+ and \mathbb{R}_- ; hence, for the condition from Theorem 2 to hold, all bidders would have to be net buyers or net sellers. In Example 2, core selection holds with complements and substitutes. As the core-selection condition in Theorem 2 entails, through a change of basis, one can find a common set of packages that are substitutes such that the direction defined by the auction quantities ought to be aligned with that of (package) substitutability. In Example 3, core selection fails with goods that are substitutes for all bidders and no gains from trade before the auction. This

³³ From Lemma 2 and Corollary 2, we know that the difference between bidder-submodularity everywhere and on the sublattices $2^{N-1} \times \{i\}$ is that positive revenue holds in the former so that the auctioneer does not block the payoff profile. The derivative condition on $v(\cdot)$ in Theorem 2 (new, compared to single-object or, more generally, one-sided auctions), when combined with the concavity of $v(\cdot)$, helps ensure that revenue is non-negative by guaranteeing that $v(W, Q) \geq 0$ for each W . This is necessary (but not, on its own, sufficient) for revenue to be positive for all profiles of participants: if some coalition W with $v(W, Q) < 0$ participated, revenue $v(W, Q) - \sum_{i \in W} (v(W, Q) - v(W \setminus i, Q))$ would be negative, since the sum terms are the agents' Vickrey payoffs, which are always positive.

is because the substitution patterns among the agents are sufficiently different to create a need to reallocate goods with $Q \geq 0$.

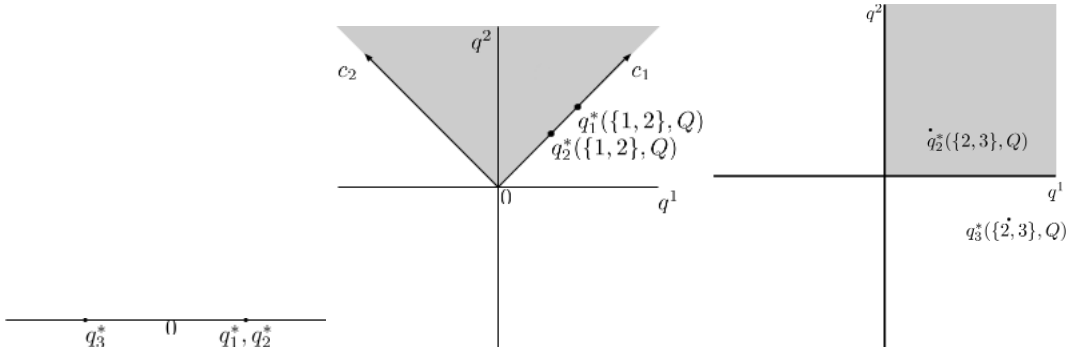


FIGURE 4: CORE-SELECTION CONDITION IN EXAMPLES

Notes: These three panels illustrate Theorem 2 in Examples 1-3. First panel, Example 1: No cone in \mathbb{R} can contain both auction allocations $q_3^*(\{1, 2, 3\}, Q)$ and $q_1^* = q_2^*(\{1, 2, 3\}, Q)$. Second panel, Example 2: Each of the quantity vectors lies in the cone formed by the common eigenvectors $c_1 = [1 \ 1]'$ and $c_2 = [-1 \ 1]'$, namely, along the line c_1 . Third panel, Example 3: Auction allocation $q_3^*(\{2, 3\}, Q)$ is not in the positive orthant, \mathbb{R}_+^2 ,

CORE SELECTION BEYOND INDIVISIBLE-GOOD, ONE-SIDED AUCTIONS. Our results highlight two general observations about core selection in *divisible* good, *two-sided* auctions.

First, relative to (divisible or not) one-sided auctions, it is non-trivial in auctions where bidders are able to both buy and sell that the gains from trade in the auction cause the equilibrium allocations to lie in a cone which satisfies the package substitutability property for each agent. Clearly, in one-sided auctions with substitutes, the alignment between substitution and gains from trade is assured; the core selection property is implied by the aggregation of substitutes alone.

Additionally, with divisible goods (i.e., multi-unit demands), substitutability is inessential for core selection in the Vickrey auction; instead, the relevant substitutability property is that for packages. Examples of *package* substitutability and the basis change idea have recently appeared in the literature: the *gross substitutes and complements* (GSC) property (Sun and Yang (2006)), *full substitutes* (Ostrovsky (2008), Hatfield et al. (2013)) and the notion of demand types in Baldwin and Klemperer (2014), which all admit complementarities in goods. The authors apply these notions to establish equilibrium existence, whereas this paper points to the usefulness of a basis change in identifying the key conditions underlying core selection in the Vickrey auction. As we have shown, however, substitutability under a change of basis alone is insufficient for core selection in two-sided divisible good settings, as the equilibrium allocations must also lie in the cone (i.e., in the image of the positive orthant under that change of basis).³⁴ Example 4 illustrates how Theorem 2 contributes in

³⁴ The basis change also suggests useful design properties (Section 6.1) and allows a unified treatment of

the context of GSC which, to the best of our knowledge, has not been studied in relation to core selection.

Example 4 (Core Selection and Gross Substitutes and Complements). GSC implies that there are two sets of goods K_1 and K_2 , and that all goods in each set K_l are substitutes for other goods in K_l but are complements for goods in K_{-l} (e.g., left and right shoes, production inputs, spectrum licenses). It is plain to see (and has also been noted by Baldwin and Klemperer (2014)) that GSC is equivalent to substitutes under the basis change $T_{GSC} = I_{K_1} \oplus -I_{K_2}$ (i.e., goods in K_1 and “negative goods” in K_2 are substitutes). Unfortunately, it is only possible for GSC to work in Theorem 2 to prove that the Vickrey outcome is core-selecting when $Q_k < 0$ for each $k \in K_2$, since otherwise someone must receive a nonpositive allocation of $q_i^{GSC} = T_{GSC}^{-1}q_i$. In fact, even when the auctioneer seeks to acquire goods in K_2 , GSC can be insufficient to ensure core selection. Suppose the auctioneer wishes to sell $Q = [2.6082 \ 0.9203 \ -0.0410]'$ to three bidders with the quadratic valuations

$$\begin{aligned}
 u_1(q_1) &= \begin{bmatrix} 42.4 & 21.6 & 1.6000 \end{bmatrix} q_1 - \frac{1}{2} q_1' \begin{bmatrix} 9 & 2.52 & -3.1 \\ 2.52 & 2.11 & -1.355 \\ -3.1 & -1.355 & 3.6 \end{bmatrix} q_1 \\
 u_2(q_2) &= \begin{bmatrix} 59.2 & 21 & 3.2 \end{bmatrix} q_2 - \frac{1}{2} q_2' \begin{bmatrix} 14 & 2.08 & -2.22 \\ 2.08 & 3.54 & -2.37 \\ -2.22 & -2.37 & 3.14 \end{bmatrix} q_2 \\
 u_3(q_3) &= \begin{bmatrix} 42.4 & 23.8 & 2.2 \end{bmatrix} q_3 - \frac{1}{2} q_3' \begin{bmatrix} 13.2 & 6.74 & -5.1 \\ 6.74 & 6.19 & -3.03 \\ -5.1 & -3.03 & 4.14 \end{bmatrix} q_3.
 \end{aligned}$$

That is, goods 1 and 2 are substitutes for each other and complements for good 3. The bidders’ inverse Hessian matrices $-S_1^{-1}$, $-S_2^{-1}$ and $-S_3^{-1}$ are given by, respectively,

$$\begin{bmatrix} -0.1928 & 0.1631 & -0.1047 \\ 0.1631 & -0.7629 & -0.1467 \\ -0.1047 & -0.1467 & -0.4231 \end{bmatrix}, \begin{bmatrix} -0.0811 & 0.0187 & -0.0432 \\ 0.0187 & -0.5754 & -0.4210 \\ -0.0432 & -0.4210 & -0.6668 \end{bmatrix}, \begin{bmatrix} -0.2155 & 0.1631 & -0.1461 \\ 0.1631 & -0.3752 & -0.0737 \\ -0.1461 & -0.0737 & -0.4754 \end{bmatrix}.$$

Since these are also the Jacobian matrices of demand, we can see that each agent’s preferences exhibits gross substitutes and complements; that is, the packages $\mathcal{C}_{GSC} = \{[1 \ 0 \ 0], [0 \ 1 \ 0], [0 \ 0 \ -1]\}$ are substitutes for these agents. However, even though $Q \in \mathcal{C}_{GSC}$, we cannot use

the core selection conditions for one- and two-sided settings. The full power of the basis change is in two-sided settings. In one-sided auctions, the ability to repackage goods when considering their substitutability is less likely to be useful for core selection in situations where the usual substitutes would not be, because the basis change would have to generate a cone with a significant overlap with the positive orthant, where the allocations lie. The overlap entails that, for one-sided auctions to have good core selection properties, goods must at least be *nearly* substitutes. Since a basis change is needed that maps to substitutes, complementarities will generally be challenging.

Theorem 2, since agent 2 does not receive an allocation in the GSC cone, C_{GSC} , irrespective of which other agents participate in the auction: However, consider the packages $\mathcal{C} = \{[0.8998 \ -0.1367 \ 0.5011]'$, $[0.0683 \ 1.0023 \ -0.3417]'$, $[0.1822 \ -0.6606 \ -0.9112]'\}$. A simple calculation shows that each $-T_C S_i^{-1} T_C'$ has positive off-diagonal entries, so \mathcal{C} are *substitutes* for each agent. Then, we have $q_i^*(W, Q) \in C$ for each $W \subseteq \{0, 1, 2, 3\}$. (We report the calculation and quantities in the Supplementary Material.) It then follows from Theorem 2 that $v(\cdot, Q)$ is submodular and, hence, the auction is core-selecting.

5 When Can One Expect Core Selection?

Theorem 2 pointed to the alignment between the cone generated by substitutable packages and the ways in which gains from trade in the auction are realized as the key to core selection. Intuitively, then, any conditions on primitives that guarantee its usefulness will restrict heterogeneity in bidder preferences. Theorem 3 presents one such condition: gains from trade before the auction are absent, and the utility Hessians commute for all bidders and quantities.

Theorem 3 (Core Selection: Sufficient Conditions on the Primitives I). *Suppose the bidders' initial endowments are efficiently allocated. Packages \mathcal{C} which are substitutes for all bidders and such that $q_i^*(W, Q) \in C$ for all $W \subseteq N$ and any Q exists if the Hessians $D^2 u_i(q_i)$ commute for all q_i and all i . In particular, the vectors in \mathcal{C} are an eigenbasis for the Hessians $D^2 u_i(q_i)$.*

This is intuitive: commutativity, equivalent to simultaneous diagonalizability, captures a homogeneity property of substitution patterns across bidders. It captures that there is some set of *implicit* packages in which agents' utility is *separable*:

Lemma 5 (Eigenvector Condition for Substitution Symmetry). *Utility $u_i(\cdot)$ is separable in packages \mathcal{C} if, and only if, their scalar multiples form an orthonormal eigenbasis for utility Hessians $D^2 u_i(q_i)$ for each q_i .*³⁵

This explains why, contrary to what one might expect based on the literature, the Vickrey auction is core-selecting in the environment of Example 2: the agents' Hessians $-S_i$ commute, and their marginal utilities at zero are identical (and so their initial endowments are efficiently allocated among them).

More generally, we can canonically classify these eigenvector packages in a way that makes plain what they represent: we can think of vectors which have both positive and

³⁵ E.g., quadratic models where the Hessians are constant, or environments with Hessians which have the same eigenvectors for all q_i but different eigenvalues.

negative entries as *switching packages*, in addition to at most one³⁶ vector in the interior of either the positive or the negative orthant, which can be interpreted as a *buying package* or a *selling package*, respectively. In addition, there can be goods or bads (as, e.g., in the GSC cone C_{GSC} – see Example 4). For instance, in spectrum auctions, the buying and switching vectors can correspond to a license for nationwide coverage and one to relocate coverage between the East and West coast. In Example 2, c_1 is a buying vector and c_2 is a switching vector.

It is important to note that these packages need not be explicitly incorporated into the design for our results to hold; they remain valid whether the auctioneer chooses to solicit bids for goods or for any arbitrary bundles, and we think of them in a strictly “as if” sense. However, they will be useful in some cases for implementation (see Section 6).

5.1 The role of preference heterogeneity

Theorems 2 and 3 motivate the converses to our core selection results. Indeed, in Examples 1 and 3, we showed that, in two-sided auctions, when pre-auction marginal utilities or substitution patterns differ significantly across agents, core selection may break down. We show that either type of heterogeneity in the environment can separately undermine the possibility of a core selecting design. Proposition 1 demonstrates that in fact, for *any* (possibly homogeneous) profile of bidders’ utility functions over goods, there exist heterogeneous initial endowments such that the Vickrey outcome is not in the core for some profile of participants.

Proposition 1 (No Core Selection: Heterogeneity in Initial Endowments I). *Given Q , for any utility profile $\{u_i\}_{i \in N}$, $N \geq 3$, constant vectors $\{t_i\}_{i \in N}$ exist such that for the utility profile $\{u_i(q_i + t_i)\}_{i \in N}$, the Vickrey outcome is not in the core with respect to bidders for some profile of participants.*

Likewise, given some profile of utility functions in which the initial endowment is not efficiently allocated, there exists some quantity vector Q for which the Vickrey auction is not core selecting:

Proposition 2 (No Core Selection: Heterogeneity in Initial Endowments II). *For any utility profile $\{u_i\}_{i \in N}$ such that $\nabla u_\ell(0) \neq \nabla u_j(0)$ for some $j, \ell \in N$, there is a Q such that the Vickrey outcome is not in the core with respect to bidders for some profile of participants.*

³⁶ If one eigenvector is in the interior of the positive orthant, all other eigenvectors must be switching vectors, because they are orthogonal to it and any two vectors in the positive orthant form an acute angle unless they are both on its boundary. If none of the eigenvectors is in the positive or negative orthants, each is a switching vector.

As Theorem 3 shows, conditional on the efficiency of the pre-auction allocation, commutativity (homogeneity) is sufficient for core selection, for any quantity vector Q . Non-commuting utility Hessians or nonzero pre-auction gains to trade itself do *not* necessarily break core selection, as Theorem 2 shows – the outcome can be in the core for some quantity vectors Q ; sufficient homogeneity in utility Hessians rather than separability is what matters. Isolating its effect by assuming that the initial allocation is efficient, when is homogeneity in Hessians insufficient for core selection? The next result characterizes the answer in a counterpart of Proposition 1. While we don't have a general converse, we provide one for the quadratic case: there, whenever there are no pre-auction gains to trade and the bidders' substitution patterns (Hessians) are sufficiently heterogeneous, there is a uniform-across-bidders rotation of the substitution patterns for which core selection breaks.

Proposition 3 (No Core Selection: Heterogeneity in Substitution Patterns). *Assume that agents have quadratic valuations $u_i(q_i) = \theta'_i q_i - \frac{1}{2} q'_i S_i q_i$ and $\theta_i = \theta$ for all i . For any Q , there exists a matrix A such that for the utility profile $\{\theta'_i q_i - q'_i A' S_i A q_i\}_{i \in N}$, the Vickrey outcome is not in the core for some profile of participants if $S_\ell^{-1} H(Z \setminus \ell + Z \setminus j) S_j^{-1}$ has a negative eigenvalue for some coalition $Z \subset N$ and agents $\ell, j \in Z$.*

To explain the role of the negative eigenvalue, let us first observe that in the quadratic environment with efficiently allocated initial endowments, nonnegativity of the difference-in-difference $v(Z, Q) + v(Z \setminus \{\ell \cup j\}, Q) - v(Z \setminus \ell, Q) - v(Z \setminus j, Q)$ is equivalent to its being non-increasing in the magnitude of Q , since it is a constant times the square of that magnitude. This difference-in-difference is just the incentive of coalition $0 \cup Z \setminus \{\ell \cup j\}$ to deviate and cancel the auction. From the discussion about common θ_i following Lemma 3, whenever the difference-in-difference of the inverses of the slopes of the coalitions' aggregate demand curves³⁷ $G(Z \setminus \ell, Z \setminus j) \equiv (H((Z \setminus \ell) \cup (Z \setminus j)) + H((Z \setminus \ell) \cap (Z \setminus j)) - H(Z \setminus \ell) - H(Z \setminus j))$, a *deviation matrix*, has a negative eigenvalue, there is some direction (indeed, some *set* of directions) in which this incentive to deviate is *increasing*. If Q lies in one of those directions from the origin, then bidder submodularity breaks.³⁸

Example 5 (No Core Selection and Additive Valuations). One of the two best known no-core selection results for indivisible goods shows that when one agent's preferences do not satisfy the substitutes condition, there are some *additive* valuations for the other participating agents such that the auction is not core-selecting (Ausubel and Milgrom (2002)). What are additive valuations in our divisible good, strictly concave setting? One might mean that

³⁷ Or equivalently, of the slopes of the marginal coalitional value curves – see Section 5.2.

³⁸ Mathematically, commutativity implies a lattice order on the Hessians. Without commutativity, the order of the positive definite Hessians is incomplete, and in the deviation matrix, the ranking of the inverse aggregate demand slope matrices defined by the harmonic means then depends on the vector Q they are multiplied with; if the matrices commute, the ranking is independent of Q (Theorem 3).

the valuation of any quantity vector is the sum of the values of the goods which comprise it, or add the stronger requirement that the utility function's second derivatives are the same for each good, so that the agents' marginal utility schedules for them differ by a constant. That is, with divisible goods, additivity could mean that valuations are separable in goods or that, in addition, bringing an agent's allocation of any good k from q to q' changes her marginal utility for good k by the same amount as bringing her allocation of any other good l from q to q' changes her marginal utility for l .

However, when we add the requirement that the marginal utilities at zero auction allocation are equal across agents, then with the second definition, Theorem 3 says that we will *always* have core selection when $N - 1$ agents have additive preferences, again *regardless of the first agent's preferences*. Why? Because with this definition, the utility Hessian of each agent with additive preferences is always a scalar multiple of the identity matrix, and hence commutes with *any* $K \times K$ matrix; and since all but one agent has additive preferences, it follows that all agents' Hessians commute. This contrast between our result and that of Ausubel and Milgrom (2002) is due to the lack of a restriction of allocations to binary vectors – an agent can be allocated more than one or less than zero units of a good. Again, with divisible goods, substitutability matters as a *common monotonicity* condition; then, the result analogous to that in the literature (i.e., that core selection may fail when an agent has preferences which do not satisfy substitutes) is Proposition 3.

Finally, let us note that the challenges for core selection that the analysis of this section brings attention to are not due to non-concavities or non-monotoncities in individual utility functions. Instead, well-behaved individual preferences can result in the aggregation of agents' valuations yielding a coalitional value function which is nonconcave (non-submodular) on the space of bidders when their substitution patterns or initial marginal utilities are sufficiently heterogeneous.

5.2 The role of quantity auctioned and market size

The above converses to our results characterizing environments where the Vickrey auction yields core outcomes demonstrate that heterogeneity in either pre-auction marginal utilities (and hence the presence of gains to trade before the auction) or in substitution patterns (as determined by utility convexities) can challenge core selection – for a fixed number of participants N and given the quantity vector Q . This section shows first that the profitability of a multilateral deviation need not vanish as the number of participating bidders N becomes large; and second that, to the extent that Q is a choice variable (e.g., if the auctioneer aims for revenue maximization subject to a cost function or a technical constraint), the impact of heterogeneity in the initial allocations (or pre-auction marginal utilities) on core selection can, in fact, be mitigated by choosing the quantity vector Q large enough in an appropriate

direction. Thus, core selection can be consistent with potentially large heterogeneity in the initial allocations. Indeed, a new aspect of design in divisible good settings is that the Vickrey auction's core-selecting property depends on the number of bidders as well as the quantity vector, separately from agents' primitive preferences.

Are the conditions for core selection easier to satisfy in large markets (i.e., $N \rightarrow \infty$)? One might expect that, holding Q constant, each agent's marginal product to the grand coalition will vanish in the limit, and so the auctioneer will capture all surplus, rendering the Vickrey payoffs ϵ -close to the core. This would indeed be the case if there was no need to reallocate goods among the bidders (e.g., in one-sided auctions, or with no bidders wanting to sell). It turns out that, in a two-sided auction, the auctioneer will not capture all surplus, in general.³⁹

Example 6 (Core Selection in Large Auctions). Suppose bidders have quadratic utilities with $S_i = S$ for all i . This simplifies calculations considerably, implying $H(W) = \frac{1}{|W|}S$, and using Lemma 3, we can then compute the sum of the bidders' Vickrey payoffs:⁴⁰

$$\sum_{i \in N} \pi_i = \frac{N}{2(N-1)} \sum_{i \in N} \theta_i' S^{-1} \theta_i - \frac{1}{2(N-1)} \left(\sum_{j \in N} \theta_j \right)' S^{-1} \left(\sum_{j \in N} \theta_j \right) + \frac{1}{2(N-1)} Q' S Q.$$

Unless $\{\theta_i\}_{i \in N}$ are identical, this need not become small as N becomes large. In fact, with i.i.d. private values $\{\theta_i\}_{i \in N}$, the Vickrey payoffs will be positive in expectation: letting $\{\lambda_k\}_{k \in K}$ be the (orthogonal) eigenvectors of S associated with unit eigenvalues,

$$E \left(\sum_{i \in N} \pi_i \right) = \frac{N}{2} \sum_{k \in K} \text{Var}(\lambda_k' \theta_i) + \frac{1}{2(N-1)} Q' S Q.$$

Suppose that instead of encouraging bidder participation, the auctioneer wishes to increase the auctioned quantity. Core selection can continue to hold; however, not any increase will work. One insight from Theorem 2 is that the elements of Q must change in such a way that the equilibrium allocation vectors q_i^* are increasing in the order defined by the packages.

Proposition 4 (Bidder-Submodularity and Quantity Auctioned). *The equilibrium package allocation vectors $T_C q_i^*(W, Q)$ are each increasing in the direction x if, and only if, $x \in \bigcap_{i \in W} \left(\sum_{j \in W} D^2 u_j (q_j^*(W, Q))^{-1} \right) D^2 u_i (q_i^*(W, Q)) C$.*

It follows that the designer can choose Q appropriately without risk of violating the $q_i^*(W, Q) \succeq_C 0$ condition for core selection in Theorem 2 (that is, without causing the

³⁹ The converses for core selection in the literature typically hold N fixed and allow a rich class of potential valuations. This resonates with our converses in the previous section: While it is not true that a small perturbation in valuations will take us outside the core, if the number of *potential* bidders grows large, but the number of participating agents does not, bidder submodularity is harder to satisfy.

⁴⁰ The Supplementary Material provides the derivation.

auction to cease functioning like a one-sided auction for packages) if, and only if, the cone intersection

$$\bigcap_{i \in W} \left(\sum_{j \in W} D^2 u_j (q_j^*(W, Q))^{-1} \right) D^2 u_i (q_i^C(p_C)) C$$

is nonempty. If the auctioneer wishes this to be true for all coalitions W , then clearly, the intersection over all $W \subseteq N$ of these intersections must be nonempty as well.

Of special note is the fact that this intersection is obviously nonempty if the substitution patterns intersection $\bigcap_{i \in W} D^2 u_i (q_i^*(W, Q)) C$ is. In particular, cones C generated by packages \mathcal{C} which are substitutes are likely to result in this intersection being nonempty, since this implies that each $D^2 u_i(\cdot)$ maps C to $-C^*$;⁴¹ additionally, when \mathcal{C} are goods, the intersection is just that of the cones generated by the columns of the Hessians $D^2 u_i (q_i^*(W, Q))$.

We can be specific about where Q needs to be: Applying the theorem for $Q = 0$ and equal marginal utilities at zero (so that the auction allocations are zero), and changing Q in a direction in which Proposition 4 shows will cause them to increase in the order defined by the packages \succeq_C , yields the following corollary.

Corollary 3 (Core Selection: Sufficient Conditions on the Primitives II). *Suppose that $\nabla u_i(0) = \nabla u_j(0)$ for all $i, j \in N$. If*

$$Q \in F(C) \equiv \bigcap_{W \subseteq N} \bigcap_{\{x_i\}_{i \in W} \in \mathbb{R}^{|W|K}} \bigcap_{\ell \in W} \left(\sum_{\ell \in W} D^2 u_j (x_j)^{-1} \right) D^2 u_i (x_i) C,$$

then $q_i^*(W, Q) \in C$ for all $W \subseteq N$.

$F(C)$ is the set of directions in which Q can move such that all bidders receive a larger amount of each package that generates C . This result also gives another set of sufficient conditions on primitives for core selection: knowing that \mathcal{C} are substitutes for each bidder and marginal utilities are equal at zero, if $Q \in F(C)$, Corollary 3 and Theorem 2 imply that the auction is core selecting with respect to bidders. That is, the allocations are in the cone and there is no need to compute them, provided that Q is in the cone $F(C)$; unlike Theorem 3, separability is not required.

When it is a choice variable, the quantity auctioned gives a powerful instrument to assure core selection (that is, to make the auction function like a one-sided auction for packages).⁴²

⁴¹ From Theorem 1, \mathcal{C} are substitutes if, and only if, $T_C (D^2 u_i (T_C^{-1} q_i^C))^{-1} T_C'$ has nonnegative off-diagonal entries (i.e., is the negative of an M -matrix) and so its inverse $T_C'^{-1} (D^2 u_i (T_C^{-1} q_i^C)) T_C^{-1}$ has negative entries (since M -matrices have positive inverses).

⁴² For instance, choosing the quantity vector when a range of spectrum and location combinations is feasible given engineering and interference restrictions or when the amount of nonmarket spectrum to make available is not fixed (the case in the 2015 FCC auction) can help ensure that a Vickrey auction is core selecting.

The next result give conditions under which Q can be increased to bring allocations into the cone C . This holds regardless of the presence of pre-auction gains from trade.

Corollary 4 (Core Selection: Sufficient Conditions on the Primitives III). *For any quantity vector Q and any $\Delta Q \in F(C + \epsilon T_C^{-1} \mathbf{1})$ for some $\epsilon > 0$, there exists a scalar $a > 0$ such that $q_i^*(W, Q + a\Delta Q) \in C$ for all W .*

In particular, regardless of heterogeneity in marginal utilities at agents' initial allocations, whenever packages C are substitutes and $F(C + \epsilon T_C^{-1} \mathbf{1})$ is nonempty for some $\epsilon > 0$, Theorem 2 and Corollary 4 combine to show that there exists ΔQ such that for $Q + \Delta Q$, the Vickrey auction is core-selecting with respect to bidders. Note that this is particularly likely to hold in the quadratic environment, where the agents' Hessian matrices are constant. In fact, Corollary 4 assures that, for general utilities, Q can be increased in *any* direction in $F(C + \epsilon T_C^{-1} \mathbf{1})$ to ensure that the bidders will consume positive amounts of the packages, as long as it is increased sufficiently. This highlights another difference with indivisible good settings: With divisible goods, so long as bidders' substitution patterns are sufficiently aligned, a market designer can weaken the strategic effects of Proposition 1 by appropriately increasing the quantity vector.

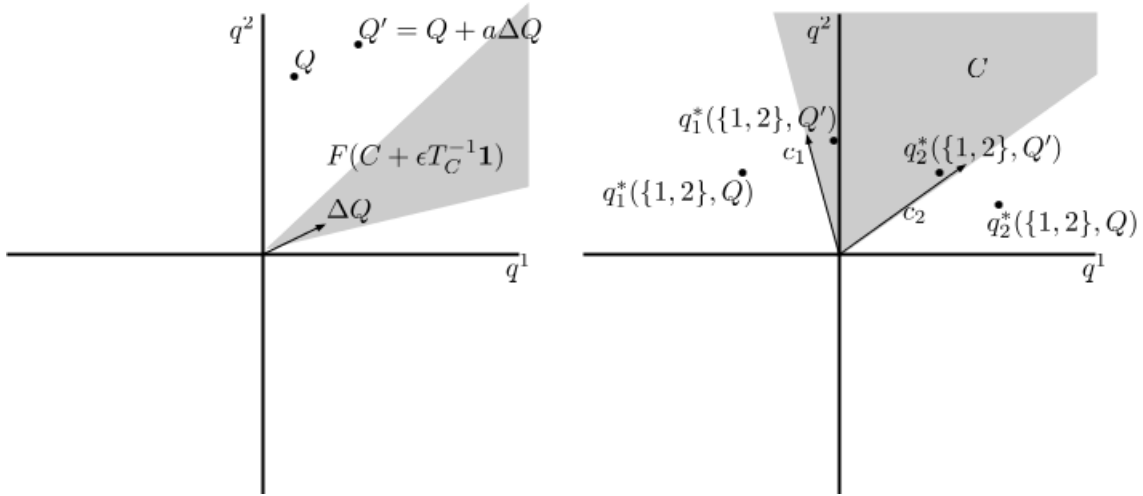


FIGURE 5: CORE SELECTION AND QUANTITY AUCTIONED

Notes: The left panel shows the cone $F(C + \epsilon T_C^{-1} \mathbf{1})$ and the quantity vector before and after the change $\Delta Q \in F(C + \epsilon T_C^{-1} \mathbf{1})$. The right panel shows the resulting change in equilibrium allocations, bringing them inside the cone C .

6 Implementation

Clearly, our analysis applies no matter how the auctioneer chooses to characterize the set of bundles on offer, whether it be in terms of goods (a convention we have so far chosen to

adopt) or in terms of an alternative set of packages. Our cone characterization of package substitutability (Theorem 1) and its application in giving sufficient conditions for core selection (Theorem 2) captures the fact that regardless of the packages employed in defining the auction, one may be able to find implicit packages through which the auctioneer can verify that core selection is to be expected. These implicit packages can be of more direct use in implementation.

6.1 Separability across packages

The requirement that bidders consider all combinations of goods when submitting their demand schedules to the auctioneer seems onerous in practice, as we mentioned in the introduction. Our analysis suggests that in the Vickrey auction, this imposition can be weakened considerably without changing the outcome of the auction when substitution patterns between goods are “the same” (i.e., commutative) across different bidders – they have the same eigenspace, and hence can be simultaneously diagonalized. Then we can write $D^2u_i(x) = T'_C M_i(x) T_C$ for each i where $M_i(x)$ is diagonal and T_C is orthogonal.⁴³ This means that we can define packages – the rows of T_C – in which each agent’s payoff is additively separable (Lemma 5). One can then run separate auctions for each of the K packages, knowing that bidders have no incentive to change their bid in one auction upon learning the outcome of another. In what follows, when agents’ utilities are additively separable in packages \mathcal{C} , assume (without loss) that the \mathcal{C} are orthonormal and write $u_i(q_i) = \sum_{k=1}^K u_i^k(\{q_i^C\}_k)$. In the Appendix, we show that when \mathcal{C} are auctioned separately, truthful bidding is still a dominant strategy.

Theorem 4 (Equivalent Implementation). *Suppose that all agents’ utilities are additively separable in the packages \mathcal{C} . When the packages \mathcal{C} are sold in separate Vickrey auctions, payments and allocation are the same in the K -good Vickrey auction, and market-clearing prices are equivalent: $p = T'_C p_C$.*

Thus, knowing that utility Hessians commute is useful – apart from permitting core selection (under the conditions in Theorem 3), by Theorem 4 and Lemma 5, commutative Hessians allow complexity reduction.⁴⁴ In some markets, indeed, one can find packages such that all bidders’ utilities are separable; e.g., when how much total spectrum one has does not affect the desirability of switching between licenses. Note that when reallocation is infeasible,

⁴³ Orthogonality follows from symmetry of the Hessians.

⁴⁴ One may wish that, additionally, the design does not require the bidders to submit more than is needed for computing the payment and the allocation – this complexity reduction can be accomplished through a dynamic implementation that induces the revelation of valuations along a path one can integrate over to recover the Vickrey payments (e.g., as in Ausubel and Cramton (2004)).

Theorem 4 breaks down (unless, of course, the packages are just the K goods). The reason is that while utilities may be separable on their domain, feasibility constraints are not.

When running *separate* auctions for packages, knowledge of the eigenbasis by the designer is critical for ensuring efficiency. Indeed, with packages other than those employed by Lemma 5, efficiency and the dominant strategy property can be compromised under auction separation.

6.2 Why run a two-sided auction instead of two one-sided auctions?

Throughout, we have assumed that a two-sided version of the Vickrey auction (that is, one that allows agents to both buy and sell) is being used – a one-sided Vickrey auction would only be constrained efficient, preventing reallocation of goods among bidders. When some agents are known to be buyers and some are known to be sellers, why not instead accommodate the designer’s objective to both buy and sell by having him run *two* standard one-sided Vickrey auctions? This may seem especially tempting in light of our conclusion that the two-sided auction presents additional challenges for core selection. Let us call such a design choice a *split auction*.

Definition 3. A *split Vickrey auction* is two separate Vickrey auctions $\mathcal{M}_+, \mathcal{M}_-$ such that

(i) (Disjoint Participation) For each $i \in N$, either $i \in N_+$ and i participates in \mathcal{M}_+ or $i \in N_-$ and i participates in \mathcal{M}_- ;

(ii) (Clearing Rule) The outcomes of the sub-mechanisms are related by $p_- = p_+$ and $Q_+ = Q - Q_-$.

If an agent is known to be a “seller,” then he belongs in N_- ; if he is known to be a “buyer,” then he belongs in N_+ . The split auction idea is exemplified by the proposed design of the 2015 FCC Incentive Auction for wireless spectrum licenses.⁴⁵ Such mechanisms have obvious disadvantages in relation to true two-sided auctions: the auctioneer needs to know ex ante who will be a buyer and who will be a seller; each agent must end up on the same side of the market for all goods; and two separate auctions must either be run simultaneously or one must be run repeatedly for different quantities of total trade. We add to these a result that a split Vickrey auction always generates less revenue than the two-sided version, despite resulting in the same allocation.

Proposition 5 (Revenue in Split vs. Two-Sided Vickrey Auction). *In a split Vickrey auction, revenue is weakly lower than in a two-sided Vickrey auction.*

⁴⁵ The proposed design is dynamically implemented with forward and reverse components, and a clearing rule that equalizes price across the two one-sided mechanisms.

6.3 Revenue

Since in divisible good settings, market clearing prices are well defined, employing a variant of the uniform-price auction would seem to be an attractive alternative. The Vickrey auction is often viewed as having unfavorable revenue properties, especially when compared to the commonly used uniform-price auction. However, this is not always the case. One can show that while the Vickrey auction creates a higher surplus compared to uniform pricing (since the latter has an inherent inefficiency with divisible goods), it gives the auctioneer a smaller share of it (Figure 6). Which effect dominates then determines the two auctions' revenue ranking.

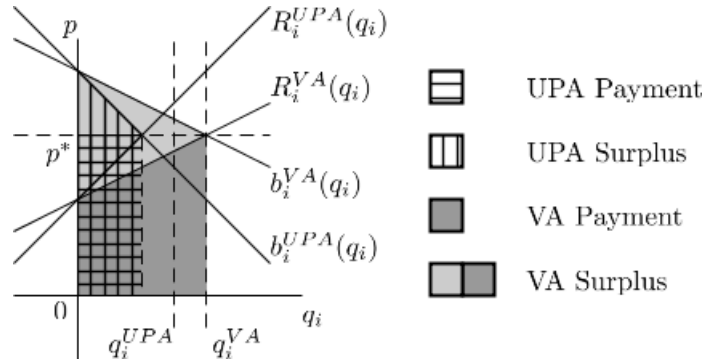


FIGURE 6: REVENUE AND SURPLUS IN VICKREY AND UNIFORM-PRICE AUCTIONS

Notes: The figure shows a bidder i 's payment and surplus generated (i.e., $u_i(q_i)$) in the Vickrey and uniform-price auctions for a single good. In the uniform-price auction, agents shade their bids b_i^{UPA} inward from their marginal utility schedules b_i^{VA} ; the market clears at the same price p^* as in the Vickrey auction, but at a quantity of smaller magnitude q_i^{UPA} .

When bidders have identical quadratic utilities – in particular, common valuations (i.e., $\theta_i = \theta$, for all i) that are unknown to the seller – and one good, Ausubel et al. (2014) show that the Vickrey auction outperforms the uniform-price auction. To allow for heterogeneity in $\{\theta_i\}_{i \in N}$, we compare the revenues in the two formats in the independent private values environment with quadratic utilities.⁴⁶ Proposition 6 shows that the Vickrey auction dominates uniform pricing in revenue terms when heterogeneity is small; thus, in the environments where core selection is more likely to be expected.

First, consider expected revenue in the Vickrey auction. When agents' substitution pat-

⁴⁶ In a working paper version of Ausubel et al. (2014), the authors compare Vickrey and UPA revenue in the independent private values case with one good. Our technical contribution is the extension to the multiple good case (a simple one, due to the package separability created by common substitution patterns). More importantly, we use the result to provide an intuitive link between the set of environments where UPA is outperformed by the Vickrey auction and those in which core selection is likely to hold.

terns are common, i.e., $S_i = S$, expected Vickrey revenue is⁴⁷

$$E(\pi_0^{VA}) = E(v(N)) - E\left(\sum_{i \in N} \pi_i\right) = -\frac{1}{2} \sum_{k \in K} \text{Var}(\lambda'_k \theta_i) - \frac{2N-1}{2N(N-1)} Q' S Q + E(\theta_i)' Q,$$

where $\{\lambda_k\}_{k \in K}$ are the (orthogonal) eigenvectors of S^{-1} associated with unit eigenvalues.

Immediately we can see that Vickrey revenue is decreasing in expected heterogeneity (i.e., the variance term). When the auctioneer buys from some agents and sells to others in the course of the auction, he is losing money on every unit he needs to reallocate (i.e., when the auctioneer is buying, he pays more than the market clearing price, whereas he receives less than that price when he is selling; see Figure 7). The need to reallocate arises precisely when valuations are heterogenous; hence, he loses money when he needs to realize gains to trade among the agents. With a large enough quantity auctioned, however, the auctioneer will make non-negative profit – see Section 5.2.

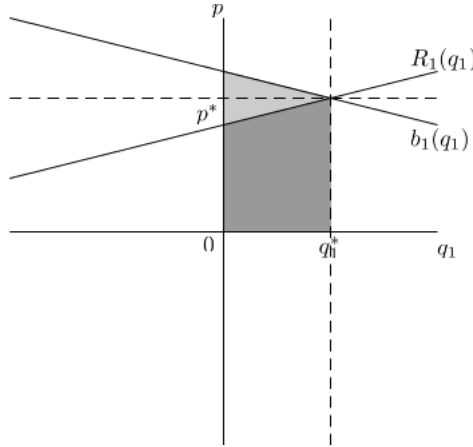


FIGURE 7: WHEN HE NEEDS TO REALLOCATE, THE AUCTIONEER LOSES MONEY

Notes: To see why reallocating means the auctioneer loses money (i.e., pays more than and receives less than price p), consider what happens when two bidders participate in a Vickrey exchange. In Figure 1, bidder 1's residual supply curve is just bidder 2's bid mirrored around the y -axis. The auction clears at q_1^* , p^* ; bidder 1 makes a payment equal to the area of the smaller, darker-shaded region, while the auctioneer must pay bidder 2 that area plus that of the larger, lighter-shaded region.

Next, let us compute equilibrium in the uniform-price auction. Assume that agents' Hessians $-S_i$ are common knowledge (an assumption which was unnecessary in the analysis of the Vickrey auction). When submitting demand schedules, bidders equalize their marginal utility with price *plus price impact* $\Delta_i q_i$. Hence, bidders' first order conditions that must be

⁴⁷ For the derivation, see the proof of Proposition 6 in the Appendix.

satisfied for every price by their demand schedules b_i are

$$\theta_i - S_i b_i(p) = p + \Delta_i b_i(p),$$

where price impact Δ_i is the slope of the inverse residual supply curve that they face, which is given by

$$R_i(q_i) = \left(\sum_{j \neq i} (\Delta_j + S_j)^{-1} \right)^{-1} \left(\sum_{\ell \neq i} (\Delta_\ell + S_\ell)^{-1} \theta_\ell + q_i - Q \right).$$

Hence, $\Delta_i = \left(\sum_{j \neq i} (\Delta_j + S_j)^{-1} \right)^{-1}$. Assuming again that $S_i = S$ for all i , we have $\Delta_i = \Delta = \frac{1}{N-2}S$. Market-clearing prices can then be computed as $\frac{1}{N} \sum_{i=1}^N \theta_i - \frac{N-1}{N(N-2)}SQ$. Thus, the expected revenue in the uniform-price auction is

$$E(\pi_0^U) = E(\theta_i)'Q - \frac{N-1}{N(N-2)}Q'SQ. \quad (2)$$

Proposition 6 (Revenue in Vickrey vs. Uniform-Price Auctions). *With quadratic utilities $u_i(\cdot)$, when $S_i = S$ for all $i \in N$ and θ_i are i.i.d., the expected revenue is higher in the Vickrey auction than the uniform-price auction if, and only if,*

$$\sum_{k \in K} \text{Var}(\lambda'_k \theta_i) < \frac{1}{(N-1)(N-2)}Q'SQ.$$

The performance of the Vickrey auction vis-à-vis the uniform-price auction in terms of revenue depends on (i) the variance of the valuations of the “implicit packages” $\{\lambda_k\}_{k \in K}$ (the quadratic utility term is constant), with a higher variance making the uniform-price auction more attractive; (ii) the number of participating bidders N , with a larger number making uniform pricing more attractive; and (iii) the sum of the quantities of the packages $\{\lambda_k / \|\lambda_k\|^2\}_{k \in K}$ supplied by the auctioneer, since these are the eigenvectors of S corresponding to unit eigenvalues, with a larger supply of them rendering the Vickrey auction more attractive.

7 Extensions

We discuss three extensions; the Supplementary Material provides details. First, our analysis so far assumes that all bidders bid for all packages. One of the enduring problems in practical auction design is that not all bidders care about all goods; i.e., we have $u_i : \mathbb{R}^{K_i} \rightarrow \mathbb{R}$, $K_i \subset K$ for some i . Theorem 2 continues to hold when this is the case so long as packages \mathcal{C} are substitutes for all bidders; the meaning of package substitutability in terms of primitives is slightly different from Theorem 1. Nontrivially, bidders need not be able to

think about how much they value these packages independently of one another;⁴⁸ that is, we can still consider a bidder’s demand for each of them even if they include goods the bidder cannot buy or sell.

Second, often, the set of feasible allocations is restricted: for instance, when reallocation of quantity among bidders is infeasible and so bidders can only buy (or in procurement auctions, can only sell). Theorem 2 still applies with an appropriately defined condition on the primitives necessary for packages to be substitutes when allocations are restricted to the positive orthant. When the packages are just goods, constraining reallocation does not cause the conditions on the primitives for substitutes to be different.

Finally, for imperfectly divisible settings (i.e., when allocations are constrained to some subset of \mathbb{Z}^K), we contribute a converse similar to Proposition 1 to illustrate that the issues due to the two-sidedness that we examine, particularly those illustrated in Example 1, are relevant beyond divisible goods. We conjecture that results analogous to ours are available in indivisible good settings, with perhaps additional restrictions on the cones used to formulate the idea of package substitutability due to the integer choice set.

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⁴⁸ In particular, this would require (and none of our results assumes this) that each bidder’s preference domain K_i is equal to the span of some subset of packages.

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Appendix⁴⁹

Proof of Lemma 1 (Coalitional Rationality of Participation): Suppose that Z is the set of agents who participate. We want to show that

$$|W|v(Z, Q) - \sum_{i \in W} v(Z \setminus i, Q) \geq v(W, 0),$$

for all W . Denote by $|W|Z$ the set of agents formed by replicating Z , $|W|$ times. Note further that $|W|v(Z, Q) = v(|W|Z, |W|Q)$: the same price clears each of the replica economies, so joining them together cannot change the efficient allocation and, thus, leaves total surplus unchanged. Now note that $v(|W|Z, |W|Q) \geq v(W, 0) + \sum_{i \in W} v(Z \setminus i, Q)$, since in the social planner’s problem for the grand coalition $|W|Z$, we could always allocate Q to the members of each $Z \setminus i$ and reallocate goods efficiently between the members of the residual coalition W . The assertion follows. \square

Proof of Lemma 2 (Implications of Core Selection): (*Submodularity of $v(\cdot)$ \Leftrightarrow core selection*) First, since 2^N is a product of chains, submodularity is equivalent to decreasing differences. Let Z be the set of participating agents. Since $\sum_{i \in Z} \pi_i = v(Z)$ by allocative efficiency, the payoff imputation is unblocked by the union of 0 and $W \subset Z$ if, and only if,

$$\begin{aligned} & v(Z, Q) - \sum_{i \notin W} \pi_i \geq v(W, Q) \quad \forall W \subset Z \Leftrightarrow v(Z, Q) - v(W, Q) \geq \sum_{i \notin W} v(Z, Q) - v(Z \setminus i, Q) \\ \Leftrightarrow & \sum_{i \notin W} v(Z \setminus \{j \notin W | j < i\}, Q) - v(Z \setminus \{j \notin W | j \leq i\}, Q) - v(Z, Q) + v(Z \setminus i, Q) \geq 0 \end{aligned}$$

⁴⁹ The cones in this paper are convex and pointed. Quasipositivity of A is equivalent to nonnegativity of the off-diagonal entries of A .

$$\Leftrightarrow \sum_{i \notin W} v(Z \setminus \{j \notin W | j < i\}, Q) + v(Z \setminus i, Q) - v((Z \setminus i) \cap (Z \setminus \{j \notin W | j < i\}), Q) - v((Z \setminus i) \cup (Z \setminus \{j \notin W | j < i\}), Q) \geq 0.$$

The sum terms are positive whenever $v(\cdot)$ is submodular in bidders. When it is not, some difference-in-difference must be positive, and the Vickrey payoff profile for some set of participants must be blocked by some coalition formed by excluding two participating bidders.

(\Rightarrow (1)) We want to show that when an agent acquires some quantity by submitting more than one bid, he can acquire the same quantity for a smaller payment by submitting one bid. If agent $i \in Z$ makes two bids b_α and b_β , which are the truthful bids of some admissible types u_α and u_β , and acquires q_α and q_β , then the market must clear at the same price as it would if agent i acquired $q_\alpha + q_\beta$ through a single bid, since the other agents must still be consuming a total of $Q - q_\alpha - q_\beta$ and R_i is injective. Then (from the calculations in footnote 17) α pays

$$x_\alpha = v((Z \cup \beta)/i) - v(Z \cup \beta/i, Q - q_\alpha) = v((Z \cup \beta)/i) - v((Z \cup \beta \cup \alpha)/i) + u_\alpha(q_\alpha)$$

and β pays

$$x_\beta = v((Z \cup \alpha)/i) - v((Z \cup \alpha)/i, Q - q_\beta) = v((Z \cup \alpha)/i) - v((Z \cup \beta \cup \alpha)/i) + u_\beta(q_\beta).$$

Now suppose agent i instead made the bid $b_{\alpha\beta} = \nabla v(\alpha \cup \beta, \cdot)$. This bid would yield i a quantity of $q_\alpha + q_\beta$ and cause him to pay

$$\begin{aligned} x_{\alpha\beta} &= v(Z/i) - v((Z \cup \alpha \cup \beta)/i, Q - q_\alpha - q_\beta) \\ &= v(Z/i) - v((Z \cup \beta \cup \alpha)/i) + v(\alpha \cup \beta, q_\alpha + q_\beta) \\ &= v(Z/i) - v((Z \cup \beta \cup \alpha)/i) + u_\alpha(q_\alpha) + u_\beta(q_\beta). \end{aligned}$$

Then, by bidder-submodularity, the following inequality holds

$$x_{\alpha\beta} - x_\alpha - x_\beta = v(Z/i) + v((Z \cup \beta \cup \alpha)/i) - v((Z \cup \beta)/i) - v((Z \cup \alpha)/i) \leq 0$$

and so it costs less to acquire $q_\alpha + q_\beta$ with one bid than with two when coalition Z/i is present. It follows by induction that any quantity that can be acquired with some collection of shill bids can be acquired more cheaply with a single bid.

(\Rightarrow (2)) When the agents in Z participate, seller revenue is

$$\sum_{i \in Z} v(Z/i) - (|Z| - 1)v(Z) = v(Z/j) + \sum_{i \in Z/j} (v(Z/i) - v(Z)),$$

choosing some $j \in Z$. Each term in the last sum increases with N , by bidder-submodularity. The revenue difference when $Z \cup \ell$ participate and when Z participate is $v(Z) - v(Z \cup \ell) + v((Z \cup \ell)/j) - v(Z/j) + \sum_{i \in Z/j} (v((Z \cup \ell)/i) - v(Z \cup \ell) - v(Z/i) + v(Z))$. Each term of the sum is weakly positive by bidder-submodularity, as is the part outside the sum. \square

For Corollary 2, note that the more stringent requirement of bidder-submodularity does not yield a stronger conclusion about shill bidding, since the empty set would only arise in

the proof if the skill bidder was the only participant in the auction, a case in which incentives to skill are different and primarily concern fooling the auctioneer into running the Vickrey auction instead of canceling it.

Proof of Lemma 3 (Coalitional Value Function): The Lagrangian is

$$\mathcal{L}(q_W, \mu) = \sum_{i \in W} \left(\theta'_i q_i - \frac{1}{2} q'_i S_i q_i \right) - \mu' \left(\sum_{i \in W} q_i - Q \right).$$

The first-order condition (which is sufficient, by concavity of the objective) is

$$\theta_i - S_i q_i = \mu, \quad \forall i \in W.$$

Solving for μ in terms of primitives by finding q_i and then summing across the i :

$$\mu = H(W) \left(\sum_{i \in W} S_i^{-1} \theta_i - Q \right).$$

Substituting back into the FOC:

$$\begin{aligned} \theta_i - S_i q_i &= H(W) \left(\sum_{j \in W} S_j^{-1} \theta_j - Q \right), \quad \forall i \in W \\ q_i &= S_i^{-1} \theta_i - S_i^{-1} H(W) \left(\sum_{j \in W} S_j^{-1} \theta_j - Q \right), \quad \forall i \in W. \end{aligned}$$

Now substituting our optimal choice of quantity back into the coalitional payoff:

$$\begin{aligned} v(W, Q) &= \sum_{i \in W} \theta'_i S_i^{-1} \theta_i - \left(\sum_{i \in W} \theta'_i S_i^{-1} \right) H(W) \left(\sum_{i \in W} S_i^{-1} \theta_i - Q \right) \\ &\quad - \frac{1}{2} \sum_{i \in W} \theta'_i S_i^{-1} \theta_i + \left(\sum_{i \in W} \theta'_i S_i^{-1} \right) H(W) \left(\sum_{i \in W} S_i^{-1} \theta_i - Q \right) \\ &\quad - \frac{1}{2} \left(\sum_{i \in W} S_i^{-1} \theta_i - Q \right)' H(W) \left(\sum_{i \in W} S_i^{-1} \theta_i - Q \right) \\ &= \frac{1}{2} \sum_{i \in W} \theta'_i S_i^{-1} \theta_i - \frac{1}{2} \left(\sum_{i \in W} S_i^{-1} \theta_i - Q \right)' H(W) \left(\sum_{i \in W} S_i^{-1} \theta_i - Q \right). \end{aligned}$$

□

Proof of Lemma 4 (Goods Substitutability as a Condition on Primitives): Agent i 's first-order condition is $\nabla u_i(q_i(p)) - p = 0$. By the implicit function theorem, then, we have $D_p q_i(p) = (D^2 u_i(q_i(p)))^{-1}$. Hence, goods are substitutes for i exactly when the off-diagonal entries of $(D^2 u_i(q_i(p)))^{-1}$ are nonnegative, that is, when $(D^2 u_i(q_i(p)))^{-1}$ is quasipositive. From the envelope theorem, $D\Pi_i(p) = q_i(p)$, so $D^2 \Pi_i(p) = -(D^2 u_i(q_i(p)))^{-1}$; it follows that $(D^2 u_i(q_i(p)))^{-1}$ is quasipositive exactly when Π_i has decreasing differences. □

Proof of Theorem 1 (Package Substitutability as a Condition on Primitives):
 ((1) \Leftrightarrow (2)) In the package pricing UMP, the agent's first-order condition is $T_C^{-1'} \nabla u_i(T_C^{-1} q_i^C) - p_C = 0$. By the implicit function theorem,

$$Dq_i^C(p_C) = (T_C^{-1'} D^2 u_i(T_C^{-1} q_i^C) T_C^{-1})^{-1} = T_C (D^2 u_i(T_C^{-1} q_i^C))^{-1} T_C'.$$

Note that the envelope theorem implies that $Dq_i^C(p_C) = -D^2 \Pi_i^C$, so submodularity of Π_i^C in \geq is equivalent to quasipositivity of $Dq_i^C(p_C)$ (and hence substitutability of packages C). Also note that since $D^2 \Pi_i(p) = -(D^2 u_i(q_i(p)))^{-1}$ from the proof of Lemma 4, the chain rule and the above calculations imply that this is also equivalent to submodularity of Π_i in \succeq_C . This is true everywhere if, and only if, for some α we have, as desired,

$$\begin{aligned} & T_C (D^2 u_i(x))^{-1} T_C' y + \alpha y \geq 0, \quad \forall y \geq 0, x \in \mathbb{R}^K \\ & \Leftrightarrow T_C (D^2 u_i(x))^{-1} z + \alpha T_C^{-1'} z \geq 0, \quad \forall z \in C^*, x \in \mathbb{R}^K \\ & \Leftrightarrow T_C (D^2 u_i(x))^{-1} z + \alpha T_C T_C^{-1} T_C^{-1'} z \geq 0, \quad \forall z \in C^*, x \in \mathbb{R}^K \\ & \Leftrightarrow (D^2 u_i(x))^{-1} z + \alpha T_C^{-1} T_C^{-1'} z \in C, \quad \forall z \in C^*, x \in \mathbb{R}^K. \end{aligned}$$

□

Proof of Theorem 2 (Core Selection: Package Substitutability and Allocations):

We want to deal with the problem where allocations are constrained to the region where we know packages C are substitutes, because we know this problem will have the same solution as that without such a constraint, since by assumption $q_i^*(W, Q) \in C \cap (Q - C)$ for all $W \subseteq N$. In order to do so, we need the following lemma, proved in the Supplementary Material, which essentially says that when packages are locally substitutes in the original, unconstrained problem, they are still (globally) substitutes in the new constrained one.

Lemma A.1 (Constrained Submodularity). *If packages are substitutes on $C \cap (Q - C)$, then*

$$\tilde{\Pi}_i^C(p_C) = \max_{q_i^C} \{u_i(T_C^{-1} q_i^C) - p_C' q_i^C \text{ s.t. } \{q_i^C\}_k \in [0, \{T_C Q\}_k]\}$$

is submodular.

Note that

$$\begin{aligned} v(W, Q) &= \max_{\{q_i\}_{i \in W}} \left\{ \sum_{i \in W} u_i(q_i) \text{ s.t. } \sum_{i \in W} q_i = Q \right\} \\ &= \max_{\{q_i \in C \cap (Q - C)\}_{i \in W}} \left\{ \sum_{i \in W} u_i(q_i) \text{ s.t. } \sum_{i \in W} q_i = Q \right\} \end{aligned} \tag{3}$$

$$= \max_{\{q_i \in C \cap (Q - C)\}_{i \in W}} \left\{ \sum_{i \in W} u_i(q_i) \text{ s.t. } \sum_{i \in W} T_C q_i = T_C Q \right\} \tag{4}$$

$$= \min_{p_C} \max_{\{q_i \in C \cap (Q - C)\}_{i \in W}} \left\{ \sum_{i \in W} u_i(q_i) - p_C \cdot \left(\sum_{i \in W} T_C q_i - T_C Q \right) \right\} \tag{5}$$

$$= \min_{p_C} \left\{ \sum_{i \in W} \max_{q_i \in C \cap (Q - C)} \{u_i(q_i) - p_C \cdot T_C q_i\} + p_C \cdot T_C Q \right\}$$

$$\begin{aligned}
&= \min_{p_C} \left\{ \sum_{i \in W} \max_{\{q_i^C\}_{k \in [0, T_C Q]}} \{u_i(T_C^{-1} q_i^C) - p_C \cdot q_i^C\} + p_C \cdot T_C Q \right\} \\
&= \min_{p_C} \left\{ \sum_{i \in W} \tilde{\Pi}_i^C(p_C) + p_C \cdot T_C Q \right\}.
\end{aligned}$$

We write the social planner's problem as (3) using our assumption about the efficient allocation. The primal (4) and dual (5) optimization problems are equivalent since the primal problem is maximization of a concave function over a convex region under a feasible linear constraint, and hence Slater's condition ensures that strong duality holds. From here we follow the proofs of Theorems 2.6.2 and 2.7.6 in Topkis (1998):

The objective $\sum_{i \in W} \tilde{\Pi}_i^C(p_C) + p_C \cdot x$ has decreasing differences in (W, p_C) : $\sum_{i \in W} (\tilde{\Pi}_i^C(p_C''') - \tilde{\Pi}_i^C(p_C''))$, $p_C''' > p_C''$, is decreasing in W since $(\tilde{\Pi}_j^C(p_C''') - \tilde{\Pi}_j^C(p_C'')) < 0$: the allocation chosen by j at p_C''' is cheaper at p_C'' (because $q_i^C(p_C''') \geq 0$) so profit must be at least as high at the smaller price vector. Hence for all $p_C, p_C' \in \mathbb{R}^K$ we have

$$\begin{aligned}
&\sum_{i \in W \cup Z} \tilde{\Pi}_i^C(p_C' \vee p_C) - \sum_{i \in W} \tilde{\Pi}_i^C(p_C) \\
&= \sum_{i \in W \cup Z} \tilde{\Pi}_i^C(p_C' \vee p_C) - \sum_{i \in W \cup Z} \tilde{\Pi}_i^C(p_C) \\
&\quad + \sum_{i \in W \cup Z} \tilde{\Pi}_i^C(p_C) - \sum_{i \in W} \tilde{\Pi}_i^C(p_C) \\
&\leq \sum_{i \in Z} \tilde{\Pi}_i^C(p_C' \vee p_C) - \sum_{i \in Z} \tilde{\Pi}_i^C(p_C) && \text{(decreasing differences)} \\
&\quad + \sum_{i \in Z} \tilde{\Pi}_i^C(p_C) - \sum_{i \in W \cap Z} \tilde{\Pi}_i^C(p_C) && (W \cup Z - W = Z - W \cap Z) \\
&\leq \sum_{i \in Z} \tilde{\Pi}_i^C(p_C) - \sum_{i \in Z} \tilde{\Pi}_i^C(p_C \wedge p_C') && \text{(Lemma A.1)} \\
&\quad + \sum_{i \in Z} \tilde{\Pi}_i^C(p_C \wedge p_C') - \sum_{i \in W \cap Z} \tilde{\Pi}_i^C(p_C \wedge p_C') && \text{(decreasing differences)} \\
&= \sum_{i \in Z} \tilde{\Pi}_i^C(p_C') - \sum_{i \in W \cap Z} \tilde{\Pi}_i^C(p_C' \wedge p_C).
\end{aligned}$$

We also have $p_C' \vee p_C - p_C = p_C' - p_C' \wedge p_C$. So for all $p_C, p_C' \in \mathbb{R}^K$,

$$\begin{aligned}
&\sum_{i \in Z} \tilde{\Pi}_i^C(p_C') + p_C' \cdot T_C Q + \sum_{i \in W} \tilde{\Pi}_i^C(p_C) + p_C \cdot T_C Q \\
&\geq \sum_{i \in W \cup Z} \tilde{\Pi}_i^C(p_C' \vee p_C) + p_C' \vee p_C \cdot T_C Q + \sum_{i \in W \cap Z} \tilde{\Pi}_i^C(p_C' \wedge p_C) + p_C' \wedge p_C \cdot T_C Q \\
&\geq v(W \cup Z, Q) + v(W \cap Z, Q).
\end{aligned} \tag{6}$$

for any W, Z such that $W \cap Z \neq \emptyset$. Taking minimums over prices on the left-hand side yields

$$v(W \cup Z, Q) + v(W \cap Z, Q) \leq v(Z, Q) + v(W, Q)$$

as desired, and $v(\cdot)$ is submodular on the sublattice $2^{N-1} \times \{i\}$ for each $i \in N$. Now we consider $W \cap Z = \emptyset$. When $p_C, p_C' \geq 0$, $p_C \wedge p_C' \geq 0$ and so from (6) we have

$$\begin{aligned}
&\sum_{i \in Z} \tilde{\Pi}_i^C(p_C') + p_C' \cdot T_C Q + \sum_{i \in W} \tilde{\Pi}_i^C(p_C) + p_C \cdot T_C Q \geq v(W \cup Z, Q) + p_C' \wedge p_C \cdot T_C Q \\
&\geq v(W \cup Z, Q) + 0 = v(W \cup Z, Q) + v(W \cap Z, Q). \tag{7}
\end{aligned}$$

By the envelope theorem, $\nabla v(W, Q) = T_C' \arg \min_{p_C} \left\{ \sum_{i \in W} \tilde{\Pi}_i^C(p_C) + p_C \cdot T_C Q \right\}$, and we

have

$$T_C^{-1}\nabla v(W, Q) = \arg \min_{p_C} \left\{ \sum_{i \in W} \Pi_i^C(p_C) + p_C \cdot T_C Q \right\} \Rightarrow T_C^{-1}\nabla v(W, Q) \in \arg \min_{p_C} \left\{ \sum_{i \in W} \tilde{\Pi}_i^C(p_C) + p_C \cdot T_C Q \right\}.$$

The derivative of $v(W, Q)$ in the direction of c_k is positive means that $c_k' \nabla v(W, Q) \geq 0$. When this is true for each $c_k \in C$ and for all W , then $T_C^{-1}\nabla v(W, Q) \geq 0$ since the c_k form the columns of T_C^{-1} . It follows that we can minimize the left-hand side of (7) over the positive orthant to get $v(W \cup Z, Q) + v(W \cap Z, Q) \leq v(Z, Q) + v(W, Q)$ and $v(\cdot)$ is bidder-submodular. The assertions follow by Lemma 2 and Corollary 2. \square

Proof of Lemma 5 (Eigenvector Condition for Substitution Symmetry): Let T_C be a polyhedral generator for the cone generated by \mathcal{C} . It follows from the chain rule that $D_{q_i^C}^2 u_i(T_C^{-1} q_i^C) = T_C^{-1} D_{q_i}^2 u_i(q_i) T_C^{-1}$. Since \mathbb{R} is a chain, $u_i(\cdot)$ is separable in q_i^C if, and only if, it is modular in q_i^C ; that is, if, and only if, $D_x^2 u_i(T_C^{-1} x)$ is diagonal. This is true if, and only if, we can write $D_x^2 u_i(x) = T_C' M_i(x) T_C$ for diagonal $M_i(x)$, which we can do if, and only if, scalar multiples of \mathcal{C} are an orthonormal eigenbasis for $D^2 u_i(x)$. \square

Proof of Theorem 3 (Core Selection: Sufficient Conditions on the Primitives): Matrices commute if, and only if, they have the same eigenspace; let \mathcal{C} be their orthonormal eigenvectors, oriented so that Q is in the cone C they generate. (Such orthogonal eigenvectors exist because the matrices are symmetric.) By Lemma 5, $u_i(\cdot)$ is separable in the \mathcal{C} for each i , implying they are substitutes for each bidder. Hence we can write $u_i(q_i) = \sum_{k=1}^K u_i^k(\{q_i^C\}_k)$ and optimize in each package separately when computing $v(\cdot)$ and q_i^* . For each k , if the initial endowment is efficiently allocated among N , we have $u_i^{k'}(0) = d_k$ for each $i \in N$ and some d_k . Given $W \subseteq N$, from the first-order conditions of the coalitional optimization we have $u_i^{k'}(\{q_i^{*C}(W, Q)\}_k) = \mu_k(W, Q)$ for each i and some $\mu_k(W, Q)$. Then for each $k \in K$, either $\mu_k(W) \leq d_k$ and thus strict concavity implies $\{q_i^{*C}(W, Q)\}_k \geq 0$ for each i , or $\mu_k(W, Q) > d_k$ and thus $\sum_{i \in W} \{q_i^{*C}(W, Q)\}_k < 0$ for each i . The latter is impossible since $\{q_i^{*C}(W, Q)\}_k = T_C Q \geq 0$. \square

Proof of Proposition 1 (No Core Selection: Heterogeneity in Initial Endowments I): Let agents 1, 2, and 3 participate; choose

$$t_1 = 0, t_2 = (\nabla u_2)^{-1}(\nabla u_1(Q)), t_3 = \begin{cases} 0, & \nabla u_1(Q) \neq \nabla u_3(0) \\ \mathbf{1}, & \nabla u_1(Q) = \nabla u_3(0) \end{cases}.$$

So for coalitions $\{0, 1\}, \{0, 1, 2\}$, the price $\nabla u_1(Q)$ clears the market, hence allocating the entire quantity vector to bidder 1 is efficient and $v(\cdot, Q)$ is the same for each of them. Now $v(\{0, 1, 2, 3\}, Q) \geq v(\{0, 1, 3\}, Q)$ because we could always allocate nothing to bidder 2 in the grand coalition. The inequality is strict: for $q_2 = 0$ to be efficient in the grand coalition, the market would have to clear at $\nabla u_1(Q)$ and the entire quantity vector would have to again be allocated to bidder 1; but at such a price, bidder 3 would demand some amount different

than zero. Then, $\pi_0 + \pi_1 = v(\{0, 1, 2\}, Q) + v(\{0, 1, 3\}, Q) - v(\{0, 1, 2, 3\}, Q) < v(\{0, 1\}, Q)$, and the coalition $\{0, 1\}$ blocks the Vickrey payoff profile. \square

Proof of Proposition 2 (No Core Selection: Heterogeneity in Initial Endowments II): Let agents 1, j , and ℓ participate; choose $Q = (\nabla u_1)^{-1}(\nabla u_j(0))$. The rest of the proof follows exactly the same as that of Proposition 1, with agents 2 and 3 replaced by agents j and ℓ respectively. \square

Proof of Proposition 3 (No Core Selection: Heterogeneity in Substitution Patterns): We first need the following result, which is proved in the Supplementary Material.

Lemma A.2 (No Core Selection: Quadratic Utilities). *When agents have quadratic valuations $u_i(q_i) = \theta'_i q_i - \frac{1}{2} q'_i S_i q_i$, if $\theta_i = \theta$ for each i and $S_\ell^{-1} H(Z \setminus \ell + Z \setminus j) S_j^{-1}$ has a negative eigenvalue for some coalition $Z \subset N$ and agents $\ell, j \in Z$, then $v(\cdot, Q)$ is not bidder-submodular for some Q . In particular, this implies that $(H(Z) + H(Z \setminus \{\ell \cup j\})) - H(Z \setminus j) - H(Z \setminus \ell)$ has a negative eigenvalue, and that $v(\cdot, Q)$ is not bidder-submodular when Q is a scalar multiple of the eigenvector associated with the negative eigenvalue.*

Let A be the rotation matrix that maps Q to the eigenvector associated with the negative eigenvalue of $(H(Z) + H(Z \setminus \{\ell \cup j\})) - H(Z \setminus \ell) - H(Z \setminus j)$ that, by Lemma A.2, must exist. Since $(\sum_{i \in W} (A' S_i A)^{-1})^{-1} = A' H(W) A$, we then have

$$v(Z, Q) + v(Z \setminus \{\ell \cup j\}, Q) > v(Z \setminus \ell, Q) + v(Z \setminus j, Q)$$

and for some profile of participants the Vickrey outcome is not in the core by Lemma 2. \square

Hence, if the difference-in-difference of the harmonic mean has a negative eigenvalue, then there is some cone which gives no core selection when it contains Q .

Proof of Proposition 4 (Bidder-Submodularity and Quantity Auctioned): Let $p_C^*(W, Q)$ be the market-clearing package price vector when W participate in a Vickrey auction for Q . The envelope conditions for agents' primal (profit-maximization) problems yield $\nabla \Pi_i^C(p_C) = q_i^C \Rightarrow \sum_{i \in W} \nabla \Pi_i^C(p_C^*(W, Q)) = T_C Q$. The inverse function theorem yields

$$D_Q p_C^*(W, Q) = \left(\sum_{i \in W} D^2 \Pi_i^C(p_C^*(W, Q)) \right)^{-1} T_C = \left(\sum_{i \in W} T_C D^2 u_i(q_i^*(W, Q))^{-1} T_C' \right)^{-1}.$$

From the proof of Theorem 1, $D q_i^C(p_C) = T_C D^2 u_i(q_i^C(p_C))^{-1} T_C'$, so applying the chain rule,

$$\begin{aligned} D_Q T_C q_i^*(W, Q) &= T_C D^2 u_i(q_i^*(W, Q))^{-1} T_C' \left(\sum_{j \in W} T_C D^2 u_j(q_j^*(W, Q))^{-1} T_C' \right)^{-1} T_C \\ &= T_C D^2 u_i(q_i^*(W, Q))^{-1} \left(\sum_{j \in W} D^2 u_j(q_j^*(W, Q))^{-1} \right)^{-1}. \end{aligned}$$

It follows that $T_C q_i^*(W, Q)$ are each increasing in exactly the directions in the polyhedral cone $\bigcap_{i \in W} \left(\sum_{j \in W} D^2 u_j(q_j^*(W, Q))^{-1} \right) D^2 u_i(q_i^*(W, Q)) T_C^{-1} \mathbb{R}_+^K$. \square

Lemma 6 (Dominant Strategy Property under Separability). *Suppose that all agents' utilities are additively separable in the packages \mathcal{C} . When the \mathcal{C} are sold in separate Vickrey auctions, submitting the demand curves $b_i^k(p_k) = (u_i^{k'})^{-1}(p_k)$ for each package $c_k \in \mathcal{C}$ is a dominant strategy and therefore optimal ex post.*

Proof of Lemma 6 (Dominant Strategy Property under Separability): Each agent i 's utility is separable in the \mathcal{C} , and by design, payments are separable in the \mathcal{C} . Hence, the optimality conditions in each auction are unaffected by the outcome of the others. By the logic of Section 2.3, then, submitting $b_i^k(p_k) = (u_i^{k'})^{-1}(p_k)$ is a dominant strategy. \square

Proof of Theorem 4 (Equivalent Implementation): (*Equivalent prices and identical allocation*) Market clearing in each of the separate Vickrey auctions is given by

$$c'_k Q = \sum_{i=1}^N \{q_i^C\}_k \quad \forall k \in K \text{ and } b_i^k(p_k) = \{q_i^C\}_k \quad \forall i \in N, k \in K.$$

Since utilities $u_i(\cdot)$ are separable, for equilibrium bids this is equivalent to

$$T_C Q = \sum_{i=1}^N q_i^C \text{ and } p_C = D_{q_i^C} u_i(T_C' q_i^C) \quad \forall i \in N,$$

by the previous lemma. Applying the chain rule yields $p_C = T_C \nabla u_i(T_C' q_i^C) \quad \forall i \in N$. Since $p_C = T_C p$ and $q_i^C = T_C q_i$, this gives us $Q = \sum_{i=1}^N q_i$ and $p = \nabla u_i(q_i) \quad \forall i \in N$, which are just the market clearing conditions for the K -good Vickrey auction, given equilibrium bids.

(*Equal payments*) In the K -good Vickrey auction, payments are given by $\int_0^{q_i} R_i(\mathbf{r}) \cdot d\mathbf{r}$. This integral is path-independent since DR_i is symmetric, and so can alternatively be written

$$\sum_{k=1}^K \int_0^{c'_k q_i} c'_k R_i(rc_k + \sum_{\ell < k} (c'_\ell q_i) c_\ell) dr,$$

since $q_i = T_C' T_C q_i = \sum_{k \in K} (c'_k q_i) c_k$. We want to show that $c'_k R_i(q_i + \epsilon c_\ell) = c'_k R_i(q_i)$ for all $k, q_i, \epsilon, \ell \neq k$ so that we can get rid of the sum inside R_i . The easiest way to do this is to show that $D_t T_C R_i(T_C' t)$ is diagonal. The inverse function theorem and the chain rule gives

$$\begin{aligned} D_t T_C R_i(T_C' t) &= T_C D R_i(T_C' t) T_C' \\ &= -T_C \left(\sum_{j \neq i} D u_j(t_j)^{-1} \right)^{-1} T_C' \text{ for some } t_j \\ &= - \left(\sum_{j \neq i} M_j(t_j)^{-1} \right)^{-1} \text{ for some } t_j, \end{aligned}$$

for some diagonal $M_j(t_j)$, by Lemma 5. Thus, we can write payments as $\sum_{k=1}^K \int_0^{c_k^{q_i}} c'_k R_i(rc_k) dr$, which is equivalent to the total payment $\sum_{k=1}^K \int_0^{\{q_i^C\}^k} R_i^k(r) dr$ when packages are sold separately.⁵⁰ \square

Proof of Proposition 5 (Revenue in Split vs. Two-Sided Vickrey Auction): In the split auction, if $i \in N_+$, i receives payoff $v(N_+, Q_+) - v(N_+ - i, Q_+)$; if $i \in N_-$, i receives payoff $v(N_-, Q_-) - v(N_- - i, Q_-)$. By price equalization, marginal utilities equalize across the two submechanisms and the allocation is efficient; hence $v(N, Q) = v(N_+, Q_+) + v(N_-, Q_-)$. We also have, by the definition of $v(\cdot)$ and disjoint participation, $v(W_+, Q_+) + v(W_-, Q_-) \leq v(W_+ \cup W_-, Q)$. So, if $i \in N_+$, i receives payoff $\pi_i = v(N_+, Q_+) - v(N_+ - i, Q_+) = v(N_+, Q_+) + v(N_-, Q_-) - (v(N_+ - i, Q_+) + v(N_-, Q_-)) \geq v(N, Q) - v((N \setminus i), Q)$; and if $i \in N_-$, i receives payoff $\pi_i = v(N_-, Q_-) - v(N_- - i, Q_-) = v(N_+, Q_+) + v(N_-, Q_-) - (v(N_- - i, Q_-) + v(N_+, Q_-)) \geq v(N, Q) - v((N \setminus i), Q)$. Since in either the split or two-sided auction, the auctioneer receives revenue $\pi_0 = v(N, Q) - \sum_{i=1}^N \pi_i$, weakly higher payoffs for bidders implies weakly lower revenue for the auctioneer. \square

Proof of Proposition 6 (Revenue in Vickrey vs. Uniform-Price Auctions): The expected revenue in the uniform price auction is computed in the main text. For the expected Vickrey revenue, $E(\pi_0^{VA}) = E(v(N)) - E(\sum_{i \in N} \pi_i)$, from Example 6 (derived in the Supplementary Material) we have

$$E\left(\sum_{i \in N} \pi_i\right) = \frac{N}{2} \sum_{k \in K} \text{Var}(\lambda'_k \theta_i) + \frac{1}{2(N-1)} Q' S Q.$$

Computing $E(v(N))$:

$$\begin{aligned} v(N) &= \frac{1}{2} \sum_{i \in N} \theta'_i S^{-1} \theta_i - \frac{1}{2N} \left(\sum_{i \in N} \theta_i - S Q \right)' S^{-1} \left(\sum_{i \in N} \theta_i - S Q \right) \\ &= \frac{N-1}{2N} \sum_{i \in N} \theta'_i S^{-1} \left(\theta_i - \frac{1}{N-1} \sum_{j \neq i} \theta_j \right) - \frac{1}{2N} Q' S Q + \frac{1}{N} \left(\sum_{i \in N} \theta_i \right)' Q \\ \Rightarrow E(v(N)) &= \frac{N-1}{2} \sum_{k \in K} \text{Var}(\lambda'_k \theta_i) - \frac{1}{2N} Q' S Q + E(\theta_i)' Q \end{aligned}$$

and so expected revenue in the Vickrey auction is

$$E(\pi_0^{VA}) = -\frac{1}{2} \sum_{k \in K} \text{Var}(\lambda'_k \theta_i) - \frac{2N-1}{2N(N-1)} Q' S Q + E(\theta_i)' Q. \quad (8)$$

The ranking then follows from (2) and (8). \square

⁵⁰ That $c'_k R_i(rc_k) = R_i^k(r)$ is implied by the equivalence result of part I of the proof.