1. The Flexible-Price Monetary Approach

Assume uncovered interest rate parity (UIP), which is implied by perfect capital substitutability 1.

\[ i_t - i^*_t = \Delta s^e_{t+1} \equiv E_{t} s^e_{t+1} - s_t \]  \hspace{1cm} (1)

The object on the right-hand side of the equation is "expected depreciation", which is typically modeled as the mathematical expectation of the log spot exchange rate at time t, based on time t information set \( (\Phi_t) \) minus the time t log-spot exchange rate.

\[ i_t - i^*_t = \Delta s^e_{t+1} \equiv E_{t} s^e_{t+1} - s_t \]  \hspace{1cm} (2)

The next relation is purchasing power parity (PPP) in log-levels.

\[ s_t = p_t - p^*_t \]  \hspace{1cm} (3)

Finally, assume stable money demand functions in the two countries:

\[ (m_t - p_t)^d = \phi y_t - \lambda i_t \]
\[ (m^*_t - p^*_t)^d = \phi^* y^*_t - \lambda^* i^*_t \]  \hspace{1cm} (4)

---

1 In this course, we will take perfect capital substitutability to be as Frankel (1983) defined it: government bonds issued in different currencies by different government are perfectly substitutable, such that uncovered interest rate parity (UIP) holds. Perfect capital mobility is defined as the condition where there are no actual or incipient government restrictions on movements of capital. Then covered interest parity holds, or alternatively, the covered interest differential equals zero.
where the d superscripts indicate "demand". Rearranging, assuming money supply equals money demand, and imposing PPP one obtains:

\[ s_t = p_t - p_t^* = (m_t - m_t^*) - \phi(y_t - y_t^*) + \lambda(i_t - i_t^*) \]  \hspace{1cm} (5)

for \( \phi = \phi^* \) and \( \lambda = \lambda^* \)

Note that this step requires that stock (as well as flow) equilibrium holds. That is, the trade balance is zero.

2. A Rational-Expectations/Present Value Formulation of the FPMA
2.1 The General Present Value Expression

Some additional insights can be garnered by re-expressing the monetary model in terms of current and expected future values of the "fundamentals". Note by UIP,

\[ i_t - i_t^* = \Delta s_{t+1}^e = E(s_{t+1} | \Phi_t) - s_t \]  \hspace{1cm} (2)

Assuming perfectly flexible prices, the Fisherian model of interest rates should hold:

\[ r_t = i_t - \pi_{t+1}^e \]
\[ r_t^* = i_t^* - \pi_{t+1}^{e*} \]  \hspace{1cm} (7)

Combining (2) and (7) yields:

\[ i_t - i_t^* = \Delta s_{t+1}^e = \pi_{t+1}^e - \pi_{t+1}^{e*} \]

Hence equation (5) can be re-expressed as:

\[ s = \tilde{M}_t + \lambda(s_{t+1}^e) - \lambda(s_t) \]

where \( \tilde{M}_t \equiv (m_t - m_t^*) - \phi(y_t - y_t^*) \)  \hspace{1cm} (9)

By manipulating this expression
Imposing rational expectations yields and expression for the future expected spot rate in period \( t+1 \):

\[
E_t(s_{t+1}) = \frac{1}{1+\lambda} E_t \tilde{M}_{t+1} + \frac{\lambda}{1+\lambda} E_t \tilde{s}_{t+2} \quad (10)
\]

substituting equation (10) into (9) yields:

\[
s_t = \frac{1}{1+\lambda} \tilde{M}_t + \frac{\lambda}{(1+\lambda)^2} E_t \tilde{M}_{t+1} + \frac{\lambda}{1+\lambda} \frac{\lambda}{1+\lambda} E_t \tilde{s}_{t+2} \quad (11)
\]

but consider:

\[
E_t(s_{t+2}) = \frac{1}{1+\lambda} E_t \tilde{M}_{t+2} + \frac{\lambda}{1+\lambda} E_t \tilde{s}_{t+3} \quad (12)
\]

so that by substituting iteratively, one obtains:

\[
s_t = \left( \frac{1}{1+\lambda} \right) \sum_{\tau=0}^{\infty} \left( \frac{\lambda}{1+\lambda} \right)^\tau E_t \tilde{M}_{t+\tau} \quad (13)
\]

Hence, the log spot exchange rate is equal to the present discounted value of the fundamentals from now to the infinite future, where the discount factor \([\lambda/(1+\lambda)]\) is a function of the interest semi-elasticity of money demand. This is a common form for rational expectations solutions to take; the current value of an asset depends upon all expected future values of that asset; but that in turn depends upon the values of the fundamentals expected from now to the infinite future (of course with declining weights).
2.2. Special Cases

Obviously, the expression in (13) is of limited usefulness, since one does not directly observe expected values for the future fundamentals. However, the expression for the current spot exchange rate can be determined under certain circumstances.

2.2.1. The fundamentals follow an AR(1) process. Let

\[ \tilde{M}_t = \rho \tilde{M}_{t-1} + \epsilon_t \Rightarrow E_t \tilde{M}_{t+1} = \rho \tilde{M}_t \]  

(14)

Then substituting this into (13), one obtains:

\[
s_t = \left( \frac{1}{1+\lambda} \right) \left[ \tilde{M}_t + \left( \frac{\lambda}{1+\lambda} \right) \rho \tilde{M}_t + \left( \frac{\lambda^2}{1+\lambda} \right) \rho^2 \tilde{M}_t + \ldots \right]
\]

\[
= \left( \frac{1}{1+\lambda} \right) \left[ \tilde{M}_t + \left( \frac{\lambda^2}{1+\lambda} \right) \rho \tilde{M}_t + \ldots \right]
\]

\[
= \left( \frac{1}{1+\lambda} \right) \left[ \frac{1}{1-(\lambda \rho)(1+\lambda)} \tilde{M}_t \right]
\]

\[
s_t = \frac{1}{1+\lambda - \lambda \rho} \tilde{M}_t
\]  

(15)

Equation (16) expresses the current exchange rate as a function of the current fundamentals, because all the expected future fundamentals can be expressed as a simple function of current fundamentals.

Notice in the special case of no autocorrelation, such that the fundamentals follow a white noise process, \( \rho = 0 \). The expression for the current exchange rate degenerates to a function of the current money supply if the conditional mean of the driving process is zero:
If the fundamentals are a white noise process around a mean, say $\mu$, then one obtains:

$$s_t = \mu + \left( \frac{1}{1+\lambda} \right) \varepsilon_t$$  \hfill (17)

2.2.2. The fundamentals follow an simple random walk. Let

$$\tilde{M}_t = \tilde{M}_{t-1} + \varepsilon_t \Rightarrow E_t\tilde{M}_{t+1} = \tilde{M}_t$$  \hfill (18)

One way to see the end result is by substituting $\rho=1$ into equation (16) (since a simple random walk is an AR(1) process with an autoregressive coefficient equal to unity); this yields:

$$s_t = \tilde{M}_t$$  \hfill (19)

That is, the current exchange rate equals the current fundamentals, because with a random walk process for the fundamentals, all future fundamentals are expected to equal the current value.

One can also for the solution explicitly setting all the $E_tM_{t+i}$'s equal to $M_t$ in equation (13), thus obtaining:

$$s_t = \left( \frac{1}{1+\lambda} \right) \sum_{\tau=0}^{\infty} \left( \frac{\lambda}{1+\lambda} \right)^\tau E_t\tilde{M}_{t+\tau}$$

$$= \left( \frac{1}{1+\lambda} \right) \left[ \frac{1}{1 - \lambda/(1+\lambda)} \right] \tilde{M}_t$$

$$= \left( \frac{1}{1+\lambda} \right) (1+\lambda)\tilde{M}_t$$  \hfill (20)

which ends up being the same as equation (19).

3.2.3. The fundamentals follow a random walk with drift. Let
\[ \tilde{M}_t = \delta + \tilde{M}_{t-1} + \epsilon_t \Rightarrow E_t \tilde{M}_{t+\tau} = \tilde{M}_t + \delta \tau \]  \hspace{1cm} (21)

where \( \delta \) is the drift term. Let

\[ \phi = \frac{\lambda}{1+\lambda} \]

for notational purposes. Substituting (21) into (13) yields:

\[
s_t = \left( \frac{1}{1+\lambda} \right) \left[ \tilde{M}_t + \phi E_t \tilde{M}_{t+1} + \phi^2 E_t \tilde{M}_{t+2} + \phi^3 E_t \tilde{M}_{t+3} + \ldots \right] \\
= \left( \frac{1}{1+\lambda} \right) \left[ \tilde{M}_t + \phi (\tilde{M}_t + \delta) + \phi^2 (\tilde{M}_t + 2\delta) + \phi^3 (\tilde{M}_t + 3\delta) + \ldots \right]
\]

re-arranging:

\[
s_t = \left( \frac{1}{1+\lambda} \right) \left[ \tilde{M}_t + \phi E_t \tilde{M}_t + \phi^2 E_t \tilde{M}_t + \phi^3 E_t \tilde{M}_t + \ldots \right] \\
+ \left( \frac{1}{1+\lambda} \right) \left[ \phi \delta + \phi^2 \delta 2 + \phi^3 \delta 3 + \ldots \right] \hspace{1cm} (24)
\]

Let

\[ Z = [\phi \delta + \phi^2 \delta 2 + \phi^3 \delta 3 + \ldots \]  \hspace{1cm} (25)

\[ \phi Z = \phi [\phi \delta + \phi^2 \delta 2 + \phi^3 \delta 3 + \ldots ] \hspace{1cm} (26)
\]

subtracting equation (26) from equation (25) yields:

\[ Z(1-\phi) = [\phi \delta + \phi^2 \delta 2 + \phi^3 \delta 3 + \ldots] = \delta [\phi + \phi^2 + \phi^3 + \ldots] = \delta [\hspace{1cm} (27)\]

This implies that

\[ Z = \left( \frac{1}{1+\lambda} \right) \delta \lambda = (1+\lambda) \delta \lambda \hspace{1cm} (28)\]
This model has the following interesting property, which is called the "magnification effect". Taking the total differential of equation (30),

\[ ds_t = d\ddot{M}_t + \lambda d\delta \]  

holding \( M_t \) constant, and allowing only the growth rate (the drift \( \delta \)) to change yields:

\[ ds_t = \lambda d\delta \]  

which graphically is depicted as follows:
The magnification effect means that an increase in the growth rate of the money stock, holding constant the actual level of the current fundamentals, causes an immediate and discontinuous depreciation, and then a more rapid rate of depreciation thereafter.

4. Sticky-Price Monetary Approach to Exchange Rates

4.1. Overview

The flexible price monetary approach (which some termed monetarist, in earlier, more ideologically charged days) yields some very strong predictions. One of the most controversial is that increasing interest differential will be associated with weakening currencies. In the context of a model with purchasing power parity holding in both the long run and short run, this result makes sense; positive interest differentials arise from positive inflation differentials (via the Fisher relation). The more rapid a currency loses value against a basket of real goods, the more rapid a currency loses value against another currency, given that PPP links prices of home and foreign real goods.

The positive relationship between the interest differential and the exchange rate runs counter to casual empiricism, at least as far as the developed economies are concerned (the high-inflation LDCs such as Argentina and Brazil are another matter). Hence we consider allowing the PPP condition to hold only in the long run.

4.2. Derivation of the Frankel Model

The assumption of long run PPP is denoted as follows, where "overbars" indicate long run variables:

\[ \bar{s}_t = \bar{p}_t - \bar{p}^*_t \]  

Hence, rewrite the flexible price monetary model equation for the exchange rate (1) is re-written:

\[ s_t = (m_t - m^*_t) - \phi(y_t - y^*_t) + \lambda(i_t - i^*_t) \]
where the inflation rates stand in for long run interest rates, given the Fisher relation holds in the long run.

Now introduce "overshooting": exchange rates tend to revert back towards the long run value at some rate $\theta$. That is, if exchange rates are too high, relative to some long run value, they will then tend to fall toward the long run value. This suggests the following mechanism:

$$\Delta s_t \equiv s_{t+1} - s_t = -\theta(s_t - \bar{s}) + \pi_{t+1} - \pi_{t+1}^*$$  \hspace{1cm} (36)$$

In words, if the exchange rate is undervalued, the exchange rate will appreciate. The $\theta$ parameter is the rate of reversion. If $\theta = 0.5$, then a 0.10 (10%) undervaluation induces a 0.05 (5%) exchange rate appreciation in the subsequent period, holding everything else constant. The inflation rates are added because the more rapid the inflation rate, the faster the exchange rate is losing value against the other currency, everything else held constant.

Assuming "rational expectations", that is on average people's expectations match what actually happens, then

$$s_{t+1}^e - s = -\theta(s_t - \bar{s}) + \pi_{t+1}^e - \pi_{t+1}^e$$  \hspace{1cm} (37)$$

However, by uncovered interest parity, the left-hand side of equation (5) is also equal the interest differential:

$$i_t - i_t^* = -\theta(s_t - \bar{s}) + \pi_{t+1}^e - \pi_{t+1}^e$$  \hspace{1cm} (38)$$

Rearranging and solving for $s$:

$$s_t = \bar{s}_t - (1/\theta)\left[(i_t - \pi_{t+1}^e) - (i_t^* - \pi_{t+1}^e)\right]$$  \hspace{1cm} (39)$$
we have an expression for the long run $s$; substituting that in:

$$ s_t = (m_t - m_t^*) - \phi(y_t - y_t^*) - \frac{1}{\theta}(i_t - i_t^*) + (\lambda + \frac{1}{\theta})(\pi_{t+1}^e - \pi_{t+1}^{e*}) $$  

This expression can be rewritten as:

$$ s_t = (m_t - m_t^*) - \phi(y_t - y_t^*) - \frac{1}{\theta}(r_t - r_t^*) + \lambda(\pi_{t+1}^e - \pi_{t+1}^{e*}) \quad (41) $$

where  

$$ r_t = i_t - \pi_t^{e*} $$

Since the real interest rate shows up in this expression, this model is sometimes called the "real interest differential" model.

**Interpretation:** The current exchange rate depends positively on current money stocks, and inflation rates, and negatively on income levels and interest rates. This result regarding interest rates differs from the flex-price monetary model because in the short run, inflation rate differentials can differ from interest rate differentials.

A common error in using this model is to trace out the following logic: higher real interest rates in the US induce investors to shift their capital to the US, resulting in a capital inflow. The capital inflow causes a greater demand for US dollars, thereby appreciating the currency. This interpretation cannot literally be correct since, as noted above, the trade balance is always zero so that the capital account is also always zero. Recall also that uncovered interest parity always holds, so investors are always indifferent between holding US versus foreign assets.

3. **The Dornbusch Overshooting Model in Detail**

This is the derivation from Dornbusch (1976). Using the numbering from the article, first consider UIP:

\[ \text{---} \]

\[ 2 \text{ For a more complete model which allows for adjustment to changing long run price levels, etc., see Mussa (1982). Frankel (1979) is this model, allowing for secular inflation.} \]
$r = r^* + x$

where $x = E\bar{e}$

(1)

and the following adjustment mechanism:

$x = \theta(\bar{e} - e)$

(2)

Money demand and money market equilibrium are given by:

$m - p = \phi y - \lambda r$

(3)

Solving for $(p-m)$ and substituting in (1):

$p - m = -\phi y + \lambda(r^* + x)$

substituting in for $x$ from equation (2):

$p - m = -\phi y + \lambda(r^* + \theta(\bar{e} - e))$

rearranging yields (4);

$p - m = -\phi y + \lambda r^* + \lambda\theta(\bar{e} - e)$

(4)

In order to solve for $p - \bar{p}$, note the long run version of (4) is:

$\bar{p} = m + (\lambda r^* - \phi y)$

(5)

which can be solved for $\lambda r^*$

$\lambda r^* = \bar{p} - m + \phi y$

substitute this in for $r^*$ in (4):

$p - m = -\phi y + (\bar{p} - m + \phi y) + \lambda\theta(\bar{e} - e)$

$p - \bar{p} = \lambda\theta(\bar{e} - e)$

solving for $e$ yields:
The domestic macroeconomy is described as:

$$\ln(D/Y) = u + \delta(e-p) + \gamma y - \sigma r$$  \hspace{1cm} \textit{aggregate demand}  \hspace{1cm} (7)$$

$$\dot{p} = \pi [\ln(D/Y)] = \pi [u + \delta(e-p) + (\gamma-1)y - \sigma r]$$ \hspace{1cm} \textit{price adjustment}  \hspace{1cm} (8)$$

where $\ln(D/Y)$ is log excess demand, $(e-p)$ is the real exchange rate (since $p^* = 1$), and $u$ is a fiscal shift parameter. One can solve for long run values of $e$ and $p$ by setting $p=0$, $r = r^*$, in equation (8):

$$0 = \pi [u + \delta(e-p) + (\gamma-1)y - \sigma r^*]$$

$$\bar{e} = \bar{p} + \frac{1}{\delta} [\sigma r^* + (1-\gamma)y - u]$$ \hspace{1cm} (9)$$

Notice that (1) and (2) imply:

$$e - \bar{e} = -\frac{1}{\theta}(r - r^*)$$

but since (6) implies a relation between $(e-\bar{e})$ and $(p-\bar{p})$:

$$e = \bar{e} + \frac{1}{\theta}(r - r^*) = \bar{e} - \frac{1}{\lambda \theta}(p - \bar{p})$$

substituting into (8):

$$\dot{p} = \pi [u + \delta(e - \frac{1}{\lambda \theta}(p - \bar{p})) - p] + (\gamma-1)y - \sigma r$$

but equation (9) is an expression for $\bar{e}$,
\[ \dot{p} = \pi[u + \delta \bar{p} + \frac{1}{\delta}(\alpha r + (1-\gamma)y - u) - \frac{1}{\lambda \theta}(p - \bar{p}) - \gamma - 1)] \]

\[ = \pi[\delta \bar{p} + \alpha r - \frac{\delta}{\lambda \theta}(p - \bar{p}) - \delta p - \gamma - 1] \]

\[ = \pi\left[-\frac{\delta}{\lambda \theta} + \delta(p - \bar{p}) - \delta r \right] \]

\[ = \pi\left[-\frac{\delta}{\lambda \theta} + \delta(p - \bar{p}) - \frac{\sigma}{\lambda}(p - \bar{p}) \right] \]

where the last line is obtained by recalling from (1), (2) and (6) that:

\[ (r - r^*) = \frac{1}{\lambda}(p - \bar{p}) \]

Then one obtains (10):

\[ \dot{p} = -\pi\left[\frac{\delta}{\lambda \theta} + \delta + \frac{\sigma}{\lambda}(p - \bar{p}) \right] \]

\[ = -\pi\left[\frac{\delta}{\lambda \theta} + \delta + \frac{\sigma}{\lambda}(p - \bar{p}) \right] \]

for the following expression (11) holding true:

\[ v = \pi\left[\frac{\delta}{\lambda \theta} + \delta + \frac{\sigma}{\lambda} \right] \]  \hspace{1cm} (11)

Solving (10) yields:

\[ p(t) = \bar{p} + (p_0 - \bar{p})\exp(-vt) \]  \hspace{1cm} (12)

Substituting (12) into (6) yields:

\[ e(t) = \bar{e} - \left(\frac{1}{\lambda \theta}\right)(p_0 - \bar{p})\exp(-vt) \]

but since

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3 See Alpha Chiang, *Fundamental Methods of Mathematical Economics*, chapter on nonhomogeneous first order differential equations.
and

\[ e(t) = \bar{e} + (e_0 - \bar{e})\exp(-\nu t) \]  \quad \text{(13)}

This result implies (14):

\[ \theta = \nu = \pi \left[ \frac{(\delta + \sigma \theta)}{\theta \lambda} + \delta \right] \]  \quad \text{(14)}

Hence the ad hoc expression for exchange rate depreciation in (2) is equivalent to the perfect foresight expression if:

\[
\begin{align*}
\theta &= \pi (\delta + \frac{\sigma}{\lambda}) + \frac{\pi \delta}{\theta \lambda} \\
0 &= -\theta + \pi (\delta + \frac{\sigma}{\lambda}) + \frac{\pi \delta}{\theta \lambda} \\
0 &= \theta^2 - \pi (\delta + \frac{\sigma}{\lambda}) \theta - \frac{\pi \delta}{\lambda}
\end{align*}
\]

Using the quadratic formula, dropping the negative root, one obtains:

\[ \theta(\lambda, \delta, \sigma, \pi) = \frac{\pi (\delta + \sigma / \lambda) + \sqrt{\pi^2 (\sigma / \lambda + \delta)^2 - 4 \pi \delta / \lambda}}{2} \]

This yields (15):

\[ \theta = \frac{\pi (\delta + \sigma / \lambda)}{2} + \sqrt{\frac{\pi^2 (\sigma / \lambda + \delta)^2}{4} + \frac{\pi \delta}{\lambda}} \]  \quad \text{(15)}

Expression (15) indicates that the rate of convergence of the exchange rate to the long-
run values is greater, the larger $\delta, \sigma, \pi$ are (where the first two parameters are the exchange rate- and interest sensitivity of log-aggregate demand, and the third parameter is the sensitivity of prices to excess demand), and the smaller $\lambda$ is (where this parameter is the interest semi-elasticity of money demand).

These relationships should be kept in mind when noting that (4):

$$p - m = -\phi y + \lambda r^* + \lambda \theta (e^- - e)$$

implies for $de = dm = dp$, that:

$$dp - dm = \lambda \theta (de - de)$$
$$dp - dm = \lambda \theta dm - \lambda \theta de$$
$$dp/dm - 1 = \lambda \theta - \lambda \theta (de/dm)$$

but $dp/dm = 0$ in the short run, due to sticky prices:

$$-(1 + \lambda \theta) = -\lambda \theta \frac{de}{dm} \Rightarrow \frac{de}{dm} = 1 + \frac{1}{\lambda \theta}$$ (16)

substituting (15) into (16) yields (17):

$$\frac{de}{dm} = 1 + \frac{1}{\lambda \theta} = 1 + \frac{1}{\pi (\sigma + \delta \lambda)} + \left[ \frac{\pi^2 (\sigma + \delta \lambda)^2}{4} + \pi \delta \lambda \right]^{1/2}$$ (17)

Hence the extent of overshooting depends on the magnitude of $\lambda$, and more importantly the rate of convergence to PPP. The more rapid the convergence rate, the less overshooting.
The \( p-dot=0 \) line, which represents combined goods and money market equilibrium, is given by:

\[
\dot{p} = \left[ \frac{\delta \lambda}{\delta \lambda + \sigma} \right] e + \left[ \frac{\sigma}{\delta \lambda + \sigma} \right] m + \left[ \frac{\lambda}{\delta \lambda + \sigma} \right] [u + (1 - \gamma)v - \phi \sigma y/\lambda]
\]

Notice in (17) as:

\[
\lambda \to 0, \quad de/dm \to 1 + \frac{1}{\pi \sigma}
\]

\[
\theta \to 0, \quad \text{overshooting increases}
\]

The basic intuition for the dynamics over time of key variables in the case of a monetary shock is given by the following diagram.
As indicated in the text and appendix, if output responds to aggregate demand/monetary shocks, then the overshotting phenomenon may be dampened. If $1-\phi\mu\delta < 0$ then the exchange rate will undershoot (where $\mu = 1/(1-\gamma)$, and $\gamma$ is the income elasticity of demand for domestic goods).