Present Value, Future Value, and Bonds

**Present Value and Future Value**

For one period:

\[ FV_t = PV_t + PV_t \times i = PV_t (1 + i) \]

\[ PV_t = \frac{FV_t}{(1 + i)} \]

In general:

\[ FV_{n,t} = PV_t (1 + i)^n \]

\[ PV_t = \frac{FV_{n,t}}{(1 + i)^n} \]

When the interest rate and payments are not constant:

\[ PV_t = \left[ \frac{X_{t+1}}{(1 + i_t)} + \frac{X_{t+2}}{(1 + i_t)(1 + i_{t+1})} + \ldots + \frac{X_{t+n}}{(1 + i_t)(1 + i_{t+1}) \ldots (1 + i_{t+n-1})} \right] \]

And when the future is not known:

\[ PV_t = \left[ \varepsilon_t \left( \frac{X_{t+1}}{(1 + i_t)} \right) + \varepsilon_t \left( \frac{X_{t+2}}{(1 + i_t)(1 + i_{t+1})} \right) + \ldots + \varepsilon_t \left( \frac{X_{t+n}}{(1 + i_t)(1 + i_{t+1}) \ldots (1 + i_{t+n-1})} \right) \right] \]

**Bond Prices and Yields**

Types of bonds:
- Zero-coupon or discount bonds
- Fixed payment loans
- Coupon bonds
- Consols

For one year discount bonds,\n
\[ 1 + i_t = \frac{FaceValue_t}{P_{DB,t}} \quad \text{or} \quad i_t = \frac{FaceValue_t}{P_{DB,t}} - 1 \]

For fixed payment loan, where fixed payments and interest rates are constant:

\[ P_{FPL,t} = \frac{FixedPaym't}{(1 + i)} + \frac{FixedPaym't}{(1 + i)^2} + \ldots + \frac{FixedPaym't}{(1 + i)^n} \]

For coupon bonds where the coupons and interest rates are constant,
\[ P_{CB,t} = \frac{\text{CouponPaym}'t}{(1 + i)} + \frac{\text{CouponPaym}'t}{(1 + i)^2} + \ldots + \frac{\text{CouponPaym}'t}{(1 + i)^n} + \frac{\text{FaceValue}}{(1 + i)^n} \]

For consols,
\[ P_{\text{Consol},t} = \frac{\text{CouponPaym}'t}{(1 + i)} + \frac{\text{CouponPaym}'t}{(1 + i)^2} + \ldots + \frac{\text{CouponPaym}'t}{(1 + i)^n} = \frac{\text{CouponPaym}'t}{(i)^n} \]

The value of a bond varies inversely with the interest rate used to calculate the present value of the promised payment.

Types of interest rates:
- Yield to maturity
- Current yield
- Holding period yield

Yield to maturity for coupon bond is the interest rate that solves:
\[ P_{CB,t} = \frac{\text{CouponPaym}'t}{(1 + i)} + \frac{\text{CouponPaym}'t}{(1 + i)^2} + \ldots + \frac{\text{CouponPaym}'t}{(1 + i)^n} + \frac{\text{FaceValue}}{(1 + i)^n} \]

Current yield on a coupon bond is:
\[ \text{Current yield} = \frac{\text{CouponPaym}'t}{P_{CB,t}} \]

Holding period yield on a coupon bond held for one period is:
\[ \frac{\text{CouponPaym}'t}{P_{CB,t}} + \frac{P_{CB,t+1} - P_{CB,t}}{P_{CB,t}} \]

Where the first term is the current yield, and the second is the capital gains.

**Real-World Data on Bonds**
http://online.wsj.com/mdc/public/page/2_3020-treasury.html
Real Interest Rates

The expected (or ex ante) real interest rate is given by the Fisher equation:

\[ i_t = r_t + \pi_{t+1}^e \]
\[ r_t = i_t - \pi_{t+1}^e \]

The ex post real interest rate is given by:

\[ r_{t,\text{ex post}} = i_t - \pi_{t+1} \]

A quasi-guaranteed real interest rate is provided Treasury Inflation Protected Securities (TIPS):

http://online.wsj.com/mdc/public/page/2_3020-tips.html
Determination of Bond Prices (and hence interest rates)