Debt and Deficits

Mathematics (Blanchard & Johnston)

\[-BuS_t \equiv BuD_t = rB_{t-1} + G_t - T_t\]

Where for simplicity \( P=1 \), \( r \) is the real interest rate.

\[B_t - B_{t-1} = BuD_t\]

Divide by \( Y \):

\[\frac{B_t}{Y_t} = (1 + r) \left( \frac{B_{t-1}}{Y_{t-1}} \right) + \frac{G_t - T_t}{Y_t}\]

Sometimes this is written as:

\[\frac{\Delta S}{Y_t} \equiv \frac{\Delta S_t}{Y_{t-1}} = (r - g) \frac{B_{t-1}}{Y_{t-1}} + \frac{G_t - T_t}{Y_t}\]

Where the primary surplus is defined as:

\[d_t \equiv \frac{G_t - T_t}{Y_t}\]

The foregoing is in real terms (as \( P=1 \)). Allowing for inflation, and time-variation in interest rates, inflation rates, and real growth (IMF, WEO 2012):

\[b_t = (1 + r - g) b_{t-1} + d_t \approx b_t = \frac{(1 + i_t)}{(1 + \pi_t)(1 + r_t)} b_{t-1} + d_t\]
Now B, Y, S in nominal terms:

\[
B_t + (1 + R_t)(B_t - S_t)
\]

\[
\frac{B_{t+1}}{Y_{t+1}} = \frac{Y_t \left(1 + R_t\right)}{Y_{t+1}} \left(1 + R_t\right)(B_t - S_t)
\]

\[
b_{t+1} = \left\{ \frac{Y_t}{Y_{t+1}} \left(1 + R_t\right) \right\} (b_t - s_t)
\]

\[
b_{t+1} = \{(1 + r'_t)\} (b_t - s_t)
\]

\[
Y_{t+1} = (1 + g'_t)Y_t
\]

Suppose r* constant net borrowing cost, and b* constant debt-to-GDP ratio, satisfied by s*.

\[
b^* = (1 + r''^*) (b^* - s^*)
\]

\[
s^* = \frac{r''^* b^*}{1 + r''^*}
\]

Note if \(R_t > g'_t\) then debt dynamics are explosive. If \(R_t < g'_t\) then dynamics are stable. What if the interest rate depends on debt-to-GDP ratios? Then increased debt-to-GDP ratios can make an economy more susceptible to tipping into unsustainable debt dynamics.

\[
R_u = \hat{\alpha} + \hat{\gamma} + 0.0313 b_{t-1} + 0.0142 b^2_{t-1} - 0.184 c_{t-1} + e_u
\]

\[
R^2 = 0.69 \quad \text{log likelihood} = -288.32.
\]

\[
R_u = \hat{\alpha} + \hat{\gamma} + 0.0029 b_{t-1} + 0.245 c_{t-1} + 0.000203 b^2_{t-1} + 0.00793 c_{t-1} - 0.00636 c_{t-1} b_{t-1} + e_u
\]

\[
R^2 = 0.82 \quad \text{log likelihood} = -224.28.
\]

\[
R_u = \hat{\alpha} + \hat{\gamma} + 0.0370 b^n_{t-1} - 0.157 c_{t-1} + 0.000365 (b^n_{t-1})^2 + 0.0101 c^2_{t-1} - 0.00124 c_{t-1} b^n_{t-1} + e_u
\]

\[
R^2 = 0.76 \quad \text{log likelihood} = -259.74.
\]