Problem Set 3
Solutions

Economics 310 – Professor Menzie Chinn
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University of Wisconsin-Madison

5.8 a. For this problem, $c = 0$ and $d = 1$.

$$f(x) = \begin{cases} \frac{1}{d - c} = \frac{1}{1 - 0} & (0 \leq x \leq 1) \\ 0 & \text{otherwise} \end{cases}$$

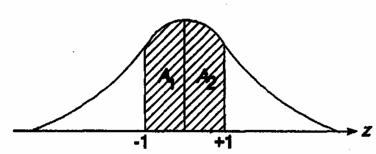
$$\mu = \frac{c + d}{2} = \frac{0 + 1}{2} = .5$$

$$\sigma^2 = \frac{(d - c)^2}{12} = \frac{(1 - 0)^2}{12} = \frac{1}{12} = .0833$$

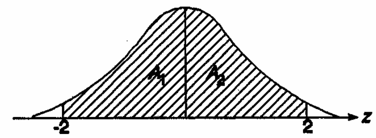
b. $P(.2 < x < .4) = (.4 - .2)(1) = .2$

c. $P(x > .995) = (1 - .995)(1) = .005$. Since the probability of observing a trajectory greater than .995 is so small, we would not expect to see a trajectory exceeding .995.

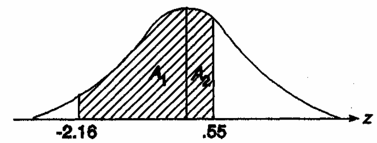
5.14 a.
$$\begin{aligned} P(-1 \leq z \leq 1) &= A_1 + A_2 \\ &= .3413 + .3413 \\ &= .6826 \end{aligned}$$



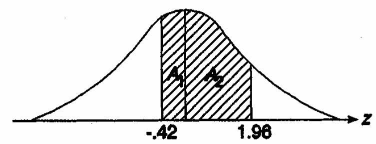
b. $P(-2 \le z \le 2) = A_1 + A_2$
$= .4772 + .4772$
$= .9544$



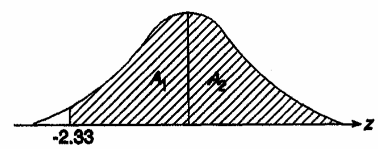
c. $P(-2.16 < z \le 0.55) = A_1 + A_2$
$= .4846 + .2088$
$= .6934$



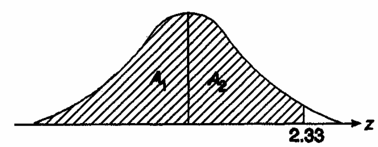
d. $P(-.42 < z < 1.96)$
$= P(-.42 \le z \le 0) + P(0 \le z \le 1.96)$
$= A_1 + A_2$
$= .1628 + .4750$
$= .6378$



e. $P(z \ge -2.33) = P(-2.33 \le z \le 0) + P(z \ge 0)$
$= A_1 + A_2$
$= .4901 + .5000$
$= .9901$



f. $P(z < 2.33) = P(z \le 0) + P(0 \le z \le 2.33)$
$= A_1 + A_2$
$= .5000 + .4901$
$= .9901$

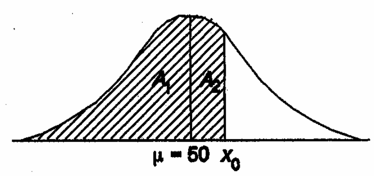


5.24 The random variable x has a normal distribution with $\mu = 50$ and $\sigma = 3$.

a. $P(x \le x_0) = .8413$

So, $A_1 + A_2 = .8413$

Since $A_1 = .5$, $A_2 = .8413 - .5 = .3413$.
Look up the area .3413 in the body of Table IV, Appendix B; $z_0 = 1.0$.

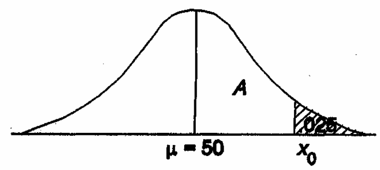


To find x_0, substitute all the values into the z-score formula:

$$z = \frac{x - \mu}{\sigma}$$
$$1.0 = \frac{x_0 - 50}{3}$$
$$x_0 = 50 + 3(1.0) = 53$$

b. $P(x > x_0) = .025$
So, $A = .5000 - .025 = .4750$

Look up the area .4750 in the body of Table IV, Appendix B; $z_0 = 1.96$.

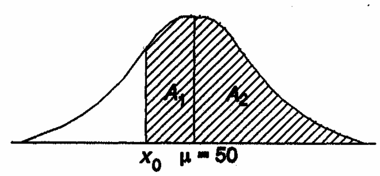


To find x_0, substitute all the values into the z-score formula:

$$z = \frac{x - \mu}{\sigma}$$
$$1.96 = \frac{x_0 - 50}{3}$$
$$x_0 = 50 + 3(1.96) = 55.88$$

c. $P(x > x_0) = .95$

So, $A_1 + A_2 = .95$. Since $A_2 = .5$, $A_1 = .95 - .5 = .4500$. Look up the area .4500 in the body of Table IV, Appendix B; (since it is exactly between two values, average the z-scores). $z_0 \approx -1.645$.



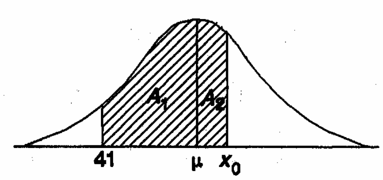
To find x_0, substitute into the z-score formula:

$$z = \frac{x - \mu}{\sigma}$$
$$-1.645 = \frac{x_0 - 50}{3}$$
$$x_0 = 50 - 3(1.645) = 45.065$$

d. $P(41 \le x < x_0) = .8630$

$$z = \frac{x - \mu}{\sigma} = \frac{41 - 50}{3} = -3$$

$$A_1 = P(41 \le x \le \mu) = P(-3 \le z \le 0)$$
$$= P(0 \le z \le 3)$$
$$= .4987$$



$A_1 + A_2 = .8630$, since $A_1 = .4987$, $A_2 = .8630 - .4987 = .3643$. Look up .3643 in the body of Table IV, Appendix B; $z_0 = 1.1$.

To find x_0, substitute into the z-score formula:

$$z = \frac{x - \mu}{\sigma}$$
$$1.1 = \frac{x_0 - 50}{3}$$
$$x_0 = 50 + 3(1.1) = 53.3$$

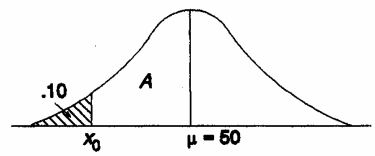
e. $P(x < x_0) = .10$

So $A = .5000 - .10 = .4000$

Look up area .4000 in the body of Table IV, Appendix B; $z_0 = 1.28$. Since z_0 is to the left of 0, $z_0 = -1.28$.

To find x_0, substitute all the values into the z-score formula:

$$z = \frac{x - \mu}{\sigma}$$
$$-1.28 = \frac{x_0 - 50}{3}$$
$$x_0 = 50 - 1.28(3) = 46.16$$



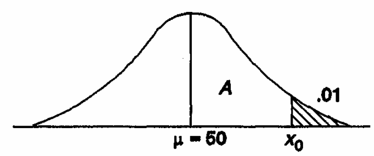
f. $P(x > x_0) = .01$

So $A = .5000 - .01 = .4900$

Look up area .4900 in the body of Table IV, Appendix B; $z_0 = 2.33$.

To find x_0, substitute all the values into the z-score formula:

$$z = \frac{x - \mu}{\sigma}$$
$$2.33 = \frac{x_0 - 50}{3}$$
$$x_0 = 50 + 2.33(3) = 56.99$$



5.30 Let x = wage rates. Then x is normally distributed with $\mu = 14$ and $\sigma = 1.25$.

a. $P(x > 15.30) = P\left[z > \frac{15.30 - 14}{1.25}\right] = P(z > 1.04)$
$= .5 - .3508 = .1492$
(Using Table IV, Appendix B)

b. $P(x > 15.30) = P\left[z > \frac{15.30 - 14}{1.25}\right] = P(z > 1.04)$
$= .5 - .3508 = .1492$
(Using Table IV, Appendix B)

c. $P(x \geq \eta) = P(x \leq \eta) = .5$

Therefore, $\eta = \mu = 14$

(Recall from Section 2.5 that in a symmetric distribution, the mean equals the median.)

5.36 Let x = monthly rate of return to stock ABC and y = monthly rate of return to stock XYZ. The random variable x is normally distributed with $\mu = .05$ and $\sigma = .03$ and y is normally distributed with $\mu = .07$ and $\sigma = .05$. You have $100 invested in each stock.

a. The average monthly rate of return for ABC stock is $\mu = .05$.
The average monthly rate of return for XYZ stock is $\mu = .07$.
Therefore, stock XYZ has the higher average monthly rate of return.

b. $E(x) = .05$ for each $1.

Since we have $100 invested in stock ABC, the monthly rate of return would be $100(.05) = \$5$.

Therefore, the expected value of the investment in stock ABC at the end of 1 month is $100 + 5 = \$105$.

$E(y) = .07$ for each $1.

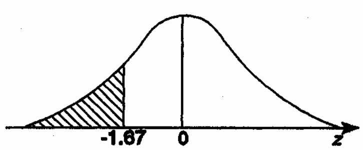
Since we have $100 invested in stock XYZ, the monthly rate of return would be $100(.07) = \$7$.

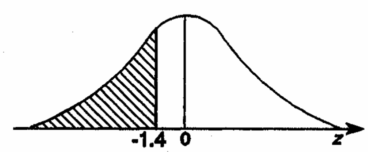
Therefore, the expected value of the investment in stock XYZ at the end of 1 month is $100 + 7 = \$107$.

c. We need to find the probability of incurring a loss for each stock and compare them.

P(incurring a loss on stock ABC)
= P(monthly rate of return is negative on stock ABC)
= $P(x < 0)$

P(incurring a loss on stock XYZ)
= P(monthly rate of return is negative on stock XYZ)
= $P(y < 0)$





$$P(x < 0) = P\left[z < \frac{0 - .05}{.03}\right] = P(z < -1.67) = .5000 - .4525 = .0475 \text{ (Table IV, Appendix B)}$$

$$P(y < 0) = P\left[z < \frac{0 - .07}{.05}\right] = P(z < -1.4) = .5000 - .4192 = .0818 \text{ (Table IV, Appendix B)}$$

Since the probability of incurring a loss is smaller for stock ABC, stock ABC would have a greater protection against occurring a loss next month.

5.44 a. If the data are normal, then $IQR/s \approx 1.3$. For this data, $IQR = Q_U - Q_L = 51.25 - 45 = 6.25$. $IQR/s = 6.25/6.015 = 1.04$. Since this value is fairly close to 1.3, the data are approximately normal.

b. From Exercise 2.38d, the relative frequency histogram looks fairly mound-shaped. This indicates that the data are approximately normal.

Problem 1.X: DY = $\log(GDP_t) - \log(GDP_{t-1})$; DY~N(.0074, $.008593^2$); X~ N(0, 1).

a. We want to find the probability that $\log(GDP_t) - \log(GDP_{t-1})$ is less than 0, which is equivalent to the probability that DY<0.

P[DY<0] = P[(DY-.0074)/.008593 < (0-.0074)/.008593]
= P[X<-.86]
=.1949

b. The quarterly growth rate corresponding to an annual 6% rate is 1.5%. Hence, we want to find the probability that $\log(GDP_t) - \log(GDP_{t-1})$ is greater than 1.5%, which is equivalent to the probability that DY > 0.015.

P[DY>.015] = P[(DY-.0074)/.008593 > (.015-.0074)/.008593]
= P[X>.88]
= .1894

Note: If you used the exact formula for converting the exact formula for growth rates into logs, you would follow the following steps:
The quarterly growth rate is 1.5%. We want to find the probability that $(GDP_t - GDP_{t-1})/ GDP_{t-1}$ is greater than 1.5%, which is equivalent to the probability that DY>log 1.015.
P[DY>log1.015] = P[(DY-.0074)/.008593 > (log1.015-.0074)/.008593]
= P[X>.87]
= .1922

5.52 a. For $n = 100$ and $p = .01$:

$$\mu \pm 3\sigma \Rightarrow np \pm 3\sqrt{npq} \Rightarrow 100(.01) \pm 3\sqrt{100(.01)(.99)}$$
$$\Rightarrow 1 \pm 3(.995) \Rightarrow 1 \pm 2.985 \Rightarrow (-1.985, 3.985)$$

Since the interval does not lie in the range 0 to 100, we cannot use the normal approximation to approximate the probabilities.

b. For $n = 100$ and $p = .5$:

$$\mu \pm 3\sigma \Rightarrow np \pm 3\sqrt{npq} \Rightarrow 100(.5) \pm 3\sqrt{100(.5)(.5)}$$
$$\Rightarrow 50 \pm 3(5) \Rightarrow 50 \pm 15 \Rightarrow (35, 65)$$

Since the interval lies in the range 0 to 100, we can use the normal approximation to approximate the probabilities.

c. For $n = 100$ and $p = .9$:

$$\mu \pm 3\sigma \Rightarrow np \pm 3\sqrt{npq} \Rightarrow 100(.9) \pm 3\sqrt{100(.9)(.1)}$$
$$\Rightarrow 90 \pm 3(3) \Rightarrow 90 \pm 9 \Rightarrow (81, 99)$$

Since the interval lies in the range 0 to 100, we can use the normal approximation to approximate the probabilities.

5.72 Let x = the shelf-life of bread. The mean of an exponential distribution is $\mu = 1/\lambda$. We know that $\mu = 2$; therefore, $\lambda = .5$.

We want to find:

$$P(x > 3) = e^{-.5(3)} = e^{-1.5} = .223130 \text{ (using Table V in Appendix B)}$$

6.4 $$E(x) = \mu = \sum xp(x) = 1(.2) + 2(.3) + 3(.2) + 4(.2) + 5(.1)$$
$$= .2 + .6 + .6 + .8 + .5 = 2.7$$

$$E(\bar{x}) = \sum \bar{x}p(\bar{x}) = 1.0(.04) + 1.5(.12) + 2.0(.17) + 2.5(.20) + 3.0(.20) + 3.5(.14) + 4.0(.08) + 4.5(.04) + 5.0(.01)$$
$$= .04 + .18 + .34 + .50 + .60 + .49 + .32 + .18 + .05 = 2.7$$

6.8. a. $$\mu = \sum xp(x) = 0\left(\frac{1}{3}\right) + 1\left(\frac{1}{3}\right) + 4\left(\frac{1}{3}\right) = \frac{5}{3} = 1.667$$

$$\sigma^2 = \sum (x - \mu)^2 p(x) = \left(0 - \frac{5}{3}\right)^2\left(\frac{1}{3}\right) + \left(1 - \frac{5}{3}\right)^2\left(\frac{1}{3}\right) + \left(4 - \frac{5}{3}\right)^2\left(\frac{1}{3}\right)$$
$$= \frac{78}{27} = 2.889$$

b.

Sample	$\bar{x}$	Probability
0, 0	0	1/9
0, 1	0.5	1/9
0, 4	2	1/9
1, 0	0.5	1/9
1, 1	1	1/9
1, 4	2.5	1/9
4, 0	2	1/9
4, 1	2.5	1/9
4, 4	4	1/9

$\bar{x}$	Probability
0	1/9
0.5	2/9
1	1/9
2	2/9
2.5	2/9
4	1/9

c. $$E(\bar{x}) = \sum \bar{x}p(\bar{x}) = 0\left(\frac{1}{9}\right) + 0.5\left(\frac{2}{9}\right) + 1\left(\frac{1}{9}\right) + 2\left(\frac{2}{9}\right) + 2.5\left(\frac{2}{9}\right) + 4\left(\frac{1}{9}\right)$$
$$= \frac{15}{9} = \frac{5}{3} = 1.667$$

Since $E(\bar{x}) = \mu$, $\bar{x}$ is an unbiased estimator for μ.

d.

Sample	s^2	Probability
0, 0	0	1/9
0, 1	0.5	1/9
0, 4	8	1/9
1, 0	0.5	1/9
1, 1	0	1/9
1, 4	4.5	1/9
4, 0	8	1/9
4, 1	4.5	1/9
4, 4	0	1/9

s^2	Probability
0	3/9
0.5	2/9
4.5	2/9
8	2/9

e. $E(s^2) = \sum s^2 p(s^2) = 0\left[\frac{3}{9}\right] + 0\left[\frac{2}{9}\right] + 4.5\left[\frac{2}{9}\right] + 8\left[\frac{2}{9}\right] = \frac{26}{9} = 2.889$

Since $E(s^2) = \sigma^2$, s^2 is an unbiased estimator for σ^2.

6.16 a. $\mu_{\bar{x}} = \mu = 10,\ \sigma_{\bar{x}} = \sigma/\sqrt{n} = 3/\sqrt{25} = 0.6$

b. $\mu_{\bar{x}} = \mu = 100,\ \sigma_{\bar{x}} = \sigma/\sqrt{n} = 25/\sqrt{25} = 5$

c. $\mu_{\bar{x}} = \mu = 20,\ \sigma_{\bar{x}} = \sigma/\sqrt{n} = 40/\sqrt{25} = 8$

d. $\mu_{\bar{x}} = \mu = 10,\ \sigma_{\bar{x}} = \sigma/\sqrt{n} = 100/\sqrt{25} = 20$

6.20 In Exercise 6.19, it was determined that the mean and standard deviation of the sampling distribution of the sample mean are 20 and 2 respectively. Using Table IV, Appendix B:

a. $P(\bar{x} < 16) = P\left[z < \frac{16 - 20}{2}\right] = P(z < -2) = .5 - .4772 = .0228$

b. $P(\bar{x} > 23) = P\left[z > \frac{23 - 20}{2}\right] = P(z > 1.50) = .5 - .4332 = .0668$

c. $P(\bar{x} > 25) = P\left[z > \frac{25 - 20}{2}\right] = P(z > 2.5) = .5 - .4938 = .0062$

d. $P(16 < \bar{x} < 22) = P\left[\frac{16 - 20}{2} < z < \frac{22 - 20}{2}\right] = P(-2 < z < 1)$

$= .4772 + .3413 = .8185$

e. $P(\bar{x} < 14) = P\left[z < \frac{14 - 20}{2}\right] = P(z < -3) = .5 - .4987 = .0013$

6.24 a. By the Central Limit Theorem, the sampling distribution of $\bar{x}$ is approximately normal with

$\mu_{\bar{x}} = \mu = 213$ and $\sigma_{\bar{x}} = \sigma/\sqrt{n} = 15/\sqrt{49} = 2.143$

b. $P(\bar{x} > 213) = P\left(z > \dfrac{213 - 213}{2.143}\right) = P(z > 0) = .5$

$P(\bar{x} > 217) = P\left(z > \dfrac{217 - 213}{2.143}\right) = P(z > 1.87) = .5 - .4693 = .0307$

$$P(209 < \bar{x} < 217) = P\left(\frac{209 - 213}{2.143} < z < \frac{217 - 213}{2.143}\right) = P(-1.87 < z < 1.87)$$
$$= .4693 + .4693 = .9386$$

6.30 a. By the Central Limit Theorem, the sampling distribution of $\bar{x}$ is approximately normal with $\mu_{\bar{x}} = \mu$ and $\sigma_{\bar{x}} = \sigma/\sqrt{n} = \sigma/\sqrt{100}$.

b. The mean of the $\bar{x}$ distribution is equal to the mean of the distribution of the fleet or the fleet mean score.

c. $\mu_{\bar{x}} = \mu = 30$ and $\sigma_{\bar{x}} = \sigma/\sqrt{100} = 60/\sqrt{100} = 6.$

$P(\bar{x} \geq 45) = P\left(z \geq \dfrac{45 - 30}{6}\right) = P(z \geq 2.5) = .5 - .4938 = .0062$

(using Table IV, Appendix B)

d. The sample mean of 45 tends to refute the claim. If the true fleet mean was as high as 30, observing a sample mean of 45 or higher would be extremely unlikely (probability = .0062). Thus, we would infer that the true mean is actually not 30 but something higher. Thus, we would refute the company's claim that the mean "couldn't possibly be as large as 30."