Economics 310
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University of Wisconsin-Madison

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Problem Set #3 Answer Key **(Ans. X1 rev'd)**

5.56 a. Using Table II, $P(x \le 11) = .345$
$\mu = np = 25(.5) = 12.5$, $\sigma = \sqrt{npq} = \sqrt{25(.5)(.5)} = 2.5$

Using the normal approximation,

$$P(x \le 11) \approx P\left(z \le \frac{(11+.5)-12.5}{2.5}\right) = P(z \le -.40) = .5 - .1554 = .3446$$

b. Using Table II, $P(x \ge 16) = 1 - P(x \le 15) = 1 - .885 = .115$

Using the normal approximation,

$$P(x \ge 16) \approx P\left(z \ge \frac{(16-.5)-12.5}{2.5}\right) = P(z \ge 1.2) = .5 - .3849 = .1151$$

(from Table IV, Appendix B)

c. Using Table II, $P(8 \le x \le 16) = P(x \le 16) - P(x \le 7) = .946 - .022 = .924$

Using the normal approximation,

$$P(8 \le x \le 16) \approx P\left(\frac{(8-.5)-12.5}{2.5} \le z \le \frac{(16+.5)-12.5}{2.5}\right)$$

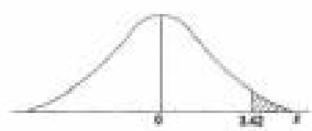
$$= P(-2.0 \le z \le 1.6) = .4772 + .4452 = .9224$$

(from Table IV, Appendix B)

5.62 Let x = number of white-collar employees in good shape who will develop stress related illnesses in a sample of 400. Then x is a binomial random variable with $n = 400$ and $p = .10$. To see if the normal approximation is appropriate for this problem:
$np \pm 3\sqrt{npq} \Rightarrow 400(.1) \pm 3\sqrt{400(.1)(.9)} \Rightarrow 40 \pm 18 \Rightarrow (22, 58)$

Since this interval is contained in the interval 0, $n = 400$, the normal approximation is appropriate.

$$P(x > 60) \approx P\left(z > \frac{(60+.5)-40}{6}\right)$$
$$= P(z > 3.42) \approx .5000 - .5000 = 0$$



5.68 a. If 80% of the passengers pass through without their luggage being inspected, then 20% will be detained for luggage inspection. The expected number of passengers detained will be:

$$E(x) = np = 1{,}500(.2) = 300$$

b. For $n = 4{,}000$, $E(x) = np = 4{,}000(.2) = 800$

c. $P(x > 600) \approx P\left(z > \frac{(600+.5)-800}{\sqrt{4000(.2)(.8)}}\right) = P(z > -7.89) = .5 + .5 = 1.0$

5.74 $f(x) = \lambda e^{-\lambda x} = e^{-x} \quad (x > 0)$

$$\mu = \frac{1}{\lambda} = \frac{1}{1} = 1, \sigma = \frac{1}{\lambda} = \frac{1}{1} = 1$$

a. $\mu \pm 3\sigma \Rightarrow 1 \pm 3(1) \Rightarrow (-2, 4)$

Since $\mu - 3\sigma$ lies below 0, find the probability that x is more than $\mu + 3\sigma = 4$.

$P(x > 4) = e^{-1(4)} = e^{-4} = .018316$ (using Table V in Appendix B)

b. $\mu \pm 2\sigma \Rightarrow 1 \pm 2(1) \Rightarrow (-1, 3)$

Since $\mu - 2\sigma$ lies below 0, find the probability that x is between 0 and 3.

$P(x < 3) = 1 - P(x \geq 3) = 1 - e^{-1(3)} = 1 - e^{-3} = 1 - .049787 = .950213$
(using Table V in Appendix B)

c. $\mu \pm .5\sigma \Rightarrow 1 \pm .5(1) \Rightarrow (.5, 1.5)$

$$\begin{aligned} P(.5 < x < 1.5) &= P(x > .5) - P(x > 1.5) \\ &= e^{-.5} - e^{-1.5} \\ &= .606531 - .223130 \\ &= .383401 \quad \text{(using Table V in Appendix B)} \end{aligned}$$

5.84 a. $P(x) = e^{-\lambda x} = e^{-.5x}$

b. $P(x \geq 4) = e^{-.5(4)} = e^{-2} = .135335$ (Table V, Appendix B)

c. $\mu = \frac{1}{\lambda} = \frac{1}{.5} = 2$

$P(x > \mu) = P(x > 2) = e^{-.5(2)} = e^{-1} = .367879$ (Table V, Appendix B)

6.20 In Exercise 6.19, it was determined that the mean and standard deviation of the sampling distribution of the sample mean are 20 and 2 respectively. Using Table IV, Appendix B:

a. $P(\bar{x} < 16) = P\left(z < \frac{16-20}{2}\right) = P(z < -2) = .5 - .4772 = .0228$

b. $P(\bar{x} > 23) = P\left(z > \frac{23-20}{2}\right) = P(z > 1.50) = .5 - .4332 = .0668$

c. $P(\bar{x} > 25) = P\left(z > \frac{25-20}{2}\right) = P(z > 2.5) = .5 - .4938 = .0062$

d. $P(16 < \bar{x} < 22) = P\left(\frac{16-20}{2} < z < \frac{22-20}{2}\right) = P(-2 < z < 1)$
$= .4772 + .3413 = .8185$

e. $P(\bar{x} < 14) = P\left(z < \frac{14-20}{2}\right) = P(z < -3) = .5 - .4987 = .0013$

6.28 a. $\mu_{\bar{x}} = \mu = 3.5 \quad \sigma_{\bar{x}} = \dfrac{\sigma}{\sqrt{n}} = \dfrac{.5}{\sqrt{100}} = .05$

b. $P(3.40 < \bar{x} < 3.60) = P\left(\dfrac{3.40 - 3.5}{.05} < z < \dfrac{3.60 - 3.5}{.05}\right)$

$= P(-2 < z < 2) = .4772 + .4772 = .9544$

(using Table IV, Appendix B)

c. $P(\bar{x} > 3.62) = P\left(z > \dfrac{3.62 - 3.5}{.05}\right) = P(z > 2.40) = .5 - .4918 = .0082$

(using Table IV, Appendix B)

d. $\mu_{\bar{x}} = \mu = 3.5 \quad \sigma_{\bar{x}} = \dfrac{\sigma}{\sqrt{n}} = \dfrac{.5}{\sqrt{200}} = .03536$

The mean of the sampling distribution of would stay the same, but the standard deviation would decrease.

$P(3.40 < \bar{x} < 3.60) = P\left(\dfrac{3.40 - 3.5}{.03536} < z < \dfrac{3.60 - 3.5}{.03536}\right)$

$= P(-2.83 < z < 2.83) = .4977 + .4977 = .9954$

(using Table IV, Appendix B)

This probability is larger than when the sample size was 100.

$P(\bar{x} > 3.62) = P\left(z > \dfrac{3.62 - 3.5}{.03536}\right) = P(z > 3.39) \approx .5 - .5 = 0$

(using Table IV, Appendix B)

This probability is smaller than when the sample size was 100.

6.46 By the Central Limit Theorem, the sampling distribution of $\bar{x}$ is approximately normal with $\mu_{\bar{x}} = \mu = .501$ and $\sigma_{\bar{x}} = \dfrac{\sigma}{\sqrt{n}} = \dfrac{.001}{\sqrt{36}} = .000167$

$$P(\bar{x} < .1994 \text{ } or \text{ } \bar{x} > .5006)$$

$$= P(\bar{x} < .4994) + P(\bar{x} > .5006) = P\left(z < \frac{.4994 - .501}{.000167}\right) + P\left(z > \frac{.5006 - .501}{.000167}\right)$$

$$= P(z < -9.58) + P(z > -2.40) = (.5 - .5) + (.5 + .4918) = .9918$$

Thus, if the true mean is .501, the test will almost surely imply the process is out of control.

7.4 a. For confidence coefficient .95, $\alpha = .05$ and $\alpha/2 = .05/2 = .025$. From Table IV, Appendix B, $z_{.025} = 1.96$. The confidence interval is:

$$\bar{x} \pm z_{.025}\frac{s}{\sqrt{n}} \Rightarrow 25.9 \pm 1.96\frac{2.7}{\sqrt{90}} \Rightarrow 25.9 \pm .56 \Rightarrow (25.34,\ 26.46)$$

b. For confidence coefficient .90, $\alpha = .10$ and $\alpha/2 = .10/2 = .05$. From Table IV, Appendix B, $z_{.05} = 1.645$. The confidence interval is:

$$\bar{x} \pm z_{.05}\frac{s}{\sqrt{n}} \Rightarrow 25.9 \pm 1.645\ \frac{2.7}{\sqrt{90}} \Rightarrow 25.9 \pm .47 \Rightarrow (25.43,\ 26.37)$$

c. For confidence coefficient .99, $\alpha = .01$ and $\alpha/2 = .01/2 = .005$. From Table IV, Appendix B, $z_{.005} = 2.58$. The confidence interval is:

$$\bar{x} \pm z_{.005}\frac{s}{\sqrt{n}} \Rightarrow 25.9 \pm 2.58\frac{2.7}{\sqrt{90}} \Rightarrow 25.9 \pm .73 \Rightarrow (25.17,\ 26.63)$$

7.6 If we were to repeatedly draw samples from the population and form the interval $\bar{x} \pm 1.96\sigma_{\bar{x}}$ each time, approximately 95% of the intervals would contain μ. We have no way of knowing whether our interval estimate is one of the 95% that contain μ or one of the 5% that do not.

7.20 $\bar{x} = \dfrac{11{,}298}{5{,}000} = 2.26$

For confidence coefficient, .95, $\alpha = .05$ and $\alpha/2 = .025$. From Table IV, Appendix B, $z_{.025} = 1.96$. The confidence interval is:

$$\bar{x} \pm z_{\alpha/2}\frac{s}{\sqrt{n}} \Rightarrow 2.26 \pm 1.96\frac{1.5}{\sqrt{5000}} \Rightarrow 2.26 \pm .04 \Rightarrow (2.22,\ 2.30)$$

We are 95% confident the mean number of roaches produced per roach per week is between 2.22 and 2.30.

7.42 a. The point estimate of p is $\hat{p} = \dfrac{x}{n} = \dfrac{67}{105} = .638$.

b. To see if the sample size is sufficiently large:

$$\hat{p} \pm 3\sigma_{\hat{p}} \approx \hat{p} \pm 3\sqrt{\frac{\hat{p}\hat{q}}{n}} \Rightarrow .638 \pm 3\sqrt{\frac{.638(.362)}{105}} \Rightarrow .638 \pm .141 \Rightarrow (.497,\ \ .779)$$

Since the interval is wholly contained in the interval (0, 1), we may conclude that the normal approximation is reasonable.

For confidence coefficient .95, $\alpha = .05$ and $\alpha/2 = .05/2 = .025$. From Table IV, Appendix B, $z_{.025} = 1.96$. The confidence interval is:

$$\hat{p} \pm z_{.025}\sqrt{\frac{\hat{p}\hat{q}}{n}} \Rightarrow .638 \pm 1.96\sqrt{\frac{.638(.362)}{105}} \Rightarrow .638 \pm .092 \Rightarrow (.546,\ \ .730)$$

c. We are 95% confident that the true proportion of on-the-job homicide cases that occurred at night is between .546 and .730.

7.46 a. The population of interest is the set of all debit cardholders in the U.S.

c. Of the 1252 observations, 180 had used the debit card to purchase a product or service on the Internet $\Rightarrow$

$$\hat{p} = \frac{180}{1252} = .144$$

To see if the sample size is sufficiently large:

$$\hat{p} \pm 3\sigma_{\hat{p}} \approx \hat{p} \pm 3\sqrt{\frac{\hat{p}\hat{q}}{n}} \Rightarrow .144 \pm 3\sqrt{\frac{.144(.856)}{1252}} \Rightarrow .144 \pm .030 \Rightarrow (.114, .174)$$

Since this interval is wholly contained in the interval (0, 1), we may conclude that the normal approximation is reasonable.

d. For confidence coefficient .98, $\alpha = 1 - .98 = .02$ and $\alpha/2 = .02/2 = .01$. From Table IV, Appendix B, $z_{.01} = 2.33$. The confidence interval is:

$$\hat{p} \pm z_{.01}\sqrt{\frac{\hat{p}\hat{q}}{n}} \Rightarrow .144 \pm 2.33\sqrt{\frac{.144(.856)}{1252}} \Rightarrow .144 \pm .023 \Rightarrow (.121, .167)$$

We are 98% confident that the proportion of debit cardholders who have used their card in making purchases over the Internet is between .121 and .167.

e. Since we would have less confidence with a 90% confidence interval than with a 98% confidence interval, the 90% interval would be narrower.

7.66 To compute the necessary sample size, use

$$n = \frac{(z_{\alpha/2})^2\sigma^2}{SE^2} \text{ where } \alpha = 1 - .90 = .10 \text{ and } \alpha/2 = .05.$$

From Table IV, Appendix B, $z_{.05} = 1.645$. Thus,

$$n = \frac{(1.645)^2(10)^2}{1^2} = 270.6 \approx 271$$

7.68 a. To compute the needed sample size, use

$$n = \frac{(z_{\alpha/2})^2\sigma^2}{SE^2} \text{ where } \alpha = 1 - .90 = .10 \text{ and } \alpha/2 = .05.$$

From Table IV, Appendix B, $z_{.10} = 1.645$. Thus,

$$n = \frac{(1.645)^2(2)^2}{.1^2} = 1{,}082.41 \approx 1{,}083$$

b. As the sample size decreases, the width of the confidence interval increases. Therefore, if we sample 100 parts instead of 1,083, the confidence interval would be wider.

c. To compute the maximum confidence level that could be attained meeting the management's specifications,

$$n = \frac{(z_{\alpha/2})^2\sigma^2}{SE^2} \Rightarrow 100 = \frac{(z_{\alpha/2})(2)^2}{.1^2} \Rightarrow (z_{\alpha/2})^2 = \frac{100(.01)}{4} = .25 \Rightarrow z_{\alpha/2} = .5$$

Using Table IV, Appendix B, $P(0 \le z \le .5) = .1915$. Thus, $\alpha/2 = .5000 - .1915 = .3085$, $\alpha = 2(.3085) = .617$, and $1 - \alpha = 1 - .617 = .383$.

The maximum confidence level would be 38.3%.

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7.7. A point estimator is a function of some data which gives us a single number as an estimate for μ, ignoring any information we have about how accurate this estimator may be. An interval estimator takes into account how accurate the estimator is; a wider interval means that our point estimator is less accurate, for example.

7.9. Of course. This is an immediate consequence of the central limit theorem and the fact that $s \to \sigma^2$ in probability.

X1. We are asked to find a number n s.t. $P(n\bar{X} > 3500) \leq \frac{3}{10{,}000} = .0003$. $X \sim N(155, 625)$. It is elementary to show that $nX \sim N(155n, 625n)$. The answer then follows directly. To apply the tools we've used in class,

$$P(n\bar{X} > 3500) = P(n\bar{X} - 155n > 3500 - 155n)$$

$$= P(\frac{n\bar{X} - 155n}{\sqrt{625n}} > \frac{3500 - 155n}{\sqrt{625n}})$$

where the expression on the left is exactly distributed $N(0,1)$. Thus, the quantity above is equal to:

$$P(N(0,1) > \frac{3500 - 155n}{\sqrt{625n}})$$

We want this probability to be .0003. A check of the standard normal probability tables gives us that $P(N(0,1) > 3.43) = .0003$. Thus we require that

$$\frac{3500 - 155n}{\sqrt{625n}} \leq 3.43$$

There a variety of ways to solve for the largest n such that this holds. The easiest is probably to use a spreadsheet. We see that $n = 20$ is the largest capacity that will give us the desired probability. That is, if 20 people ride the elevator, they fall to their deaths with probability less than $\frac{3}{10{,}000}$ and live with complimentary probability. Another acceptable answer would be to use 3000 instead of 3500 in the problem setup; this would give us a capacity such that exceeding the listed capacity would be less than $\frac{3}{10{,}000}$.

X2. We are asked to find the probability that a sample of 10 students has a mean of at least 112. We cannot solve this problem without being told that IQ is normally distributed. Taking this as given, however, $\frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}} = N(0,1)$ exactly. Thus,

$$P(\bar{x} > 112) = P(\bar{x} - 110 > 112 - 110)$$

$$= P(\frac{\bar{x} - 110}{\frac{4}{\sqrt{10}}} > \frac{112 - 110}{\frac{4}{\sqrt{10}}})$$

$$= P(N(0,1) > 1.58) = .0571$$

X3. A 100% confidence interval would include entire range of possible values the parameter of interest can take on. For example, if we are interesting in writing down a 100% confidence for μ, what we can do is

$$P(-\infty < N(0,1) < \infty) = 1$$

$$P(-\infty < \frac{\bar{x} - \mu}{\frac{s}{\sqrt{n}}} < \infty) = 1$$

$$P(\bar{x} - \infty\frac{s}{\sqrt{n}} < \mu < \bar{x} + \infty\frac{s}{\sqrt{n}}) = 1$$

$$P(-\infty < \mu < \infty) = 1$$

So we are saying that we know the mean is a number. We could have said this without any confidence interval language; the idea of estimating parameters through confidence intervals is to narrow the range of values in which we think the parameter could fall.

X4. As seen numerous times in class,

$$P(\bar{x} - z_{\frac{\alpha}{2}}\frac{s}{\sqrt{n}} < \mu < \bar{x} + z_{\frac{\alpha}{2}}\frac{s}{\sqrt{n}}) = .95$$

Let's make up data. Suppose we are interested in knowing the proportion of Bush supporters who think gay marriage is more important than the Iraq war. Suppose an exit poll measured a sample of 1025 people and found that 60% of Bush supporters in the sample believed this. Then, $\bar{x} = .6$ and $s = \sqrt{\bar{x}(1-\bar{x})} = .49$. Then, a 95% confidence interval for this proportion is

$$[.6 - .03, .6 + .3]$$

or, as pollsters would say, the proportion is .6, with a margin or error of $\pm 3\%$.

X5. $s = \sqrt{3.39} = 1.84$. $\bar{x} = 6$. Now,

$$P(-1.96 < N(0,1) < 1.96) = .95 \implies$$

$$P(-1.96 < \frac{\bar{x} - \mu}{\frac{s}{\sqrt{n}}} < 1.96) = .95 \implies$$

$$P(\bar{x} - 1.96\frac{s}{\sqrt{n}} < \mu < \bar{x} + 1.96\frac{s}{\sqrt{n}}) = .95$$

And so a 95% confidence interval for μ is $[5.76, 6.24]$.