Algebra for the Growth Accounting Formula

Suppose that each time period $A, N, K$ grow by $\Delta A/A, \Delta N/N, \Delta K/K$. Further assume:

$$Y = F(N,K,A) = Af(N,K)$$

Take the total differential with respect to time $t$, set $\Delta t = 1$, and manipulating yields:

$$\Delta Y / Y = \Delta A / A + \frac{\Delta f(N,K)}{f(N,K)}$$

(5.6)

The last term in (5.6) is a differential with respect to the two arguments of the function, $N$ and $K$. The differential of $f(.)$ with respect to $N$ is the marginal productivity of labor; the differential of $f(.)$ with respect to $K$ is the marginal productivity of capital. Let these be denoted by $M_N$ and $M_K$ respectively, and substitute these expressions into (5.6):

$$\frac{\Delta f(N,K)}{f(N,K)} = M_N \frac{\Delta N}{Y} + M_K \frac{\Delta K}{Y}$$

(5.7)

(When there are constant returns to scale due to the assumption of a Cobb-Douglas production function.) Substituting (5.7) into (5.6) yields:

$$\Delta Y / Y = \Delta A / A + M_N \frac{\Delta N}{Y} + M_K \frac{\Delta K}{Y}$$

(5.8)

Suppose (as we assumed in Chapter 11 under perfect competition), that firms hire factors of production up to the point where the marginal product of a factor equals the real factor price (e.g., capital is hired up to the point where rental cost of capital equals the marginal product of capital), viz.

$$M_N = \frac{W}{P}, M_K = \frac{R^K}{P}$$

(5.9)

Substituting into (5.8) yields:

$$\Delta Y / Y = \Delta A / A + \left(\frac{W}{P}\right) \frac{\Delta N}{Y} + \left(\frac{R^K}{P}\right) \frac{\Delta K}{Y}$$

(5.10)

Which can be rewritten as:

$$\Delta Y / Y = \Delta A / A + \left(\frac{W}{PY}\right) \frac{\Delta N}{N} + \left(\frac{R^K}{PY}\right) \frac{\Delta K}{K}$$

(5.11)
Note that \((WN/PY)\) is the labor share of income, and \((R^KPY)\) is the capital share of output. These are measurable; in the United States, the former is 0.7, and the latter is 0.3, on average.

Substituting these values into (5.11) leads to equation (5.1) in the text, (5.12) in the appendix.

\[
\frac{\Delta Y}{Y} = \frac{\Delta A}{A} + 0.7 \left[ \frac{\Delta N}{N} \right] + 0.3 \left[ \frac{\Delta K}{K} \right] \tag{5.12}
\]

One can re-express (5.12) in terms of output per unit labor \((Y/N)\) by subtracting the growth rate of labor from both sides:

\[
\left( \frac{\Delta Y}{Y} \right) - \left( \frac{\Delta N}{N} \right) = \frac{\Delta A}{A} + 0.7 \times \left[ \frac{\Delta N}{N} \right] + 0.3 \times \left[ \frac{\Delta K}{K} \right] - \left( \frac{\Delta N}{N} \right)
\]

Which leads to:

\[
\left( \frac{\Delta Y}{Y} \right) - \left( \frac{\Delta N}{N} \right) = \frac{\Delta A}{A} + 0.3 \times \left[ \frac{\Delta K}{K} - \frac{\Delta N}{N} \right]
\]

Where the left hand side is the growth rate of labor productivity and the term in the [square brackets] on the right hand side is the growth rate of capital per unit labor.

This expression indicates that output per hour can rise because of improving technology (the A term) or because of capital deepening.

**Figure 1:** Average long-term labor productivity growth
Nonfarm business sector

<table>
<thead>
<tr>
<th>Year</th>
<th>Percent</th>
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</thead>
<tbody>
<tr>
<td>1947-1971</td>
<td>3.0</td>
</tr>
<tr>
<td>1972-1995</td>
<td>2.5</td>
</tr>
<tr>
<td>1996-2001</td>
<td>3.0</td>
</tr>
<tr>
<td>2002-2008</td>
<td>2.5</td>
</tr>
</tbody>
</table>


**Figure 2:** Labor productivity
Nonfarm business sector, year-over-year percent change

<table>
<thead>
<tr>
<th>Year</th>
<th>Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>2002-2004</td>
<td>3.5%</td>
</tr>
<tr>
<td>2005-2007</td>
<td>1.4%</td>
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<tr>
<td>2008</td>
<td>2.9%</td>
</tr>
</tbody>
</table>
