

Lecture Notes 3: Socially Optimal and Private Provision of Public Goods

Optimal Public Expenditure (ref: Samuelson 1954, 1955 REStat)

- (i) Private consumption good X_1 such that $X_1 = X_1^1 + X_1^2$,
 X_1^i , $i = 1, 2$.
→ bread
- (ii) Public consumption good X_2 such that $X_2 = X_2^1 = X_2^2$,
 X_2^i , $i = 1, 2$.
→ national defense, outdoor circus

Graphs:

The following page contains three graphs depicting the consumption of the private and the public good by persons 1 and 2, and the productive capabilities of the 2 person society.

Charts 1 and 2 show the indifference curves of persons 1 & 2 in public-private good space. The charts are aligned vertically because 1 & 2 must consume the same level of the public good.

Curve AB in Chart 3 depicts the feasible total productions of the private and public goods for the society, and clearly represents diminishing returns in production. (This could be relaxed--)

Conditions for Pareto optima in the production & allocation of the public and private goods:

What is the best or ideal state of the system? This is a *normative* question, and we need to specify a *social welfare function* in order to answer it.

Samuelson on SWF: “Use of the word ‘best’ indicates we are in the ascientific area of ‘welfare economics’ and must be provided with a set of norms. Economic science cannot deduce a social welfare function; what it can do is neutrally interpret any arbitrarily specified welfare function.”

Suppose that the SWF has the following (weak) properties:

- (a) leaving each i on the same indifference curve u^i will leave social welfare unchanged
- (b) at any point, a move of each i to a higher indifference level can be found that increases SW.

Given (a) & (b) we can determine tangency conditions necessary, though not sufficient, for the determination of the social optimum.

Return to graphs: Hold person 2’s indifference level fixed at u^2 , as on 2’s indifference curve CD. Find the allocation such that person 1 reaches her highest indifference level given person 2 receives u^2 .

- copy curve CD onto Chart 3 as C'D', still in (X_1, X_2) space.
- the vertical distance from curve CD (representing 2’s private good consumption requirement for u^2 for a given level of public good) to curve AB (representing the social private good production capacity for a fixed amount of the public good) is the private consumption available to 1 when 2 enjoys u^2 and public good amount X_2 .

- graph the CD to AB vertical distance in Chart 1 as curve cd , the consumption sets for person 1 that guarantee u^2 to person 2.
- Clearly the tangency point E_1 in Chart 1 represents the Pareto optimal consumption for 1 GIVEN 2 receives utility u^2 . At this point, 2 consumes the public and private goods represented by point E_2 in Chart 2 and society produces the private & public good levels described by point E in Chart 3.

Note that there are an infinite number of such optimal points, one for each possible level of agent 2's utility.

Chart 4 presents the feasible combinations of utilities u^1 & u^2 for agents 1 & 2 given the social production possibilities meeting condition $F(X_1^1 + X_1^2, X_2) = 0$.

Contours U' , U'' , and U''' represent level sets for a social welfare function that provides the required 'ordinal scoring of every state of the world', or state of indifference levels for the 2 agents. A SWF that can be represented by curves U' , U'' , and U''' determines the unique socially best obtainable point to be g .

The E_1 point of tangency with the E_2 consumption for agent 2 and the point E production in Charts 1-3 effectively describe the optimality condition for a public good

$$MRS^1 + MRS^2 = MC.$$

The socially optimal point g in Chart 4 adds the property that the social welfare significance of a unit of any private good allocated to any individual must at the margin be the same for each person.

PS 1 may require that you derive more technical expressions for the above conditions on MRS's & marginal social welfare for the 2 person society from the conditions for the social optimum in Samuelson's "The Pure Theory of Public Expenditure," REStat 1954.

Impossibility of the decentralized determination of the social optimum

Consider the attainment of the social optimum for the case of n goods in which all goods are private & all markets competitive.

Each agent i satisfies $p_1X_1^i + p_2X_2^i + \dots + p_nX_n^i = L^i$, $i=1, \dots, s$, where L^i is agent i 's purchasing power.

Therefore each agent consumes such that $MRS_{r,j}^i = \frac{p_j}{p_r}$, competitive markets equate price with marginal cost, and the social optimum for private goods is obtained in a decentralized economy as long as L^i , $i=1, \dots, s$, is set to achieve the 'ethical observer's optimum'.

No such price mechanism emerges to produce the social optimum $MRS^1 + MRS^2 = MC$ given the appropriate allocation of property rights/purchasing power in the presence of public goods.

Alternative solutions: each agent in the society with public goods 'votes' or 'signals' so as to communicate her or his MRS profile, and a social calculator aggregates this information to determine the production & allocation that max the SWF.

Problem:

It is difficult to design a mechanism under which the individual's optimal report or vote reflects her true valuation of the public good.

References on revelation mechanisms & tax structures to overcome this problem: Lindahl, E. (1919), "Just taxation—a positive solution," in Musgrave & Peacock, *Classics in the Theory of Public Finance*. London: Macmillan. Clarke, E. (1971), "Multipart pricing of public goods," *Public Choice* 11, 17-33. Groves, T. (1973), "Incentives in Teams," *EMA* 21, 63-80.

Voluntary Contributions to the Public Good (ref: Bergstrom, Blume, & Varian (1985), JPubE)

Quick overview:

→ Neutrality result: A redistribution of income among contributors that does not alter the contributing set does not change the supply of the public good.

→ Since only a small subset of consumers contribute to the public good in most voluntary contributions cases of interest, assuming interior solutions may be misleading & the participation margin is likely to be of interest.

→ BBV analyze the extents to which government contributions to the public good crowd out private contributions & demonstrate that equalizing income redistributions tend to reduce voluntary contributions to the PG.

Model

n consumers, indexed $i=1, \dots, n$.

i consumes x_i of the private good & contributes g_i to the public good.

$G = \sum_i g_i$ is the total public good. i 's utility is $u(x_i, G)$.

Agent i is endowed with wealth w_i , so that $x_i + g_i = w_i$. G_{-i} denotes the sum of all contributions from agents other than i ,

$$G_{-i} = \sum_{j \neq i} g_j.$$

Definition A Nash equilibrium is a gift vector (g_i^*) , $i=1, \dots, n$, such that for each i , (x_i^*, g_i^*) solves

$$\begin{aligned} \max_{x_i, g_i} u_i(x_i, g_i + G_{-i}^*) \\ \text{s.t. } x_i + g_i = w_i, g_i \geq 0. \end{aligned}$$

Consider the problem of the individual consumer. Each consumer effectively chooses the equilibrium level of G itself.

An equivalent specification of the consumer's problem is

$$\begin{aligned} \max_{x_i, G} u_i(x_i, G) \\ \text{s.t. } x_i + G = w_i + G_{-i}^*, G \geq G_{-i}^*. \end{aligned}$$

See Figure 1. The unconstrained consumer optimizes subject to AD, while the constraint requiring $G \geq +G_{-i}^*$ restricts the consumer to segment AB of the budget constraint.

Figure 3 shows 3 consumers with identical preferences and differing wealth levels $w_1 > w_2 > w_3$. The first two consumers contribute while the 3rd does not, accepting $G = G_{-3}$.

Theorem 1. (see fuller statement in BBV paper) If consumers have convex prefs & contributions are originally Nash, then after a redistribution of income among contributors such that no contributor loses more than her original contribution there exists a new Nash equilibrium in which each previous contributor changes the amount of her gift by the change in her income with the redistribution. Each consumer consumes the same amount of the

private and public goods in both the pre- and post-redistribution equilibria.

Proof. Let G^* be the original equilibrium sum of contributions, g_i^* consumer i 's original contribution, & Δw_i i 's change in wealth with the redistribution.

Sps after the redistribution every other consumer ($-i$) changes her contribution by exactly Δw_j , $j \neq i$.

Since $\sum_i \Delta w_i = 0$, i 's budget constraints become

$$x_i + G = w_i + \Delta w_i + G^*_{-i} - \Delta w_i, \quad G \geq G^*_{-i} - \Delta w_i.$$

Clearly it is only the nonnegativity constraint on her gift that changes.

Since by assumption $g_i + \Delta w_i \geq 0$, the consumer can achieve her pre-redistribution amount of both the public and the private good by contributing $g_i^* + \Delta w_i$ to the public good.

Considering Figure 1, we see that if $\Delta w_i < 0$ then i 's budget set moves back from AB to AC, and is smaller than before. Because i can still afford G^* & x_i^* , by the logic of revealed preference she prefers this point and contributes $g_i^* + \Delta w_i$.

Turning to Figure 2, if $\Delta w_i > 0$, the consumer's budget constraint has expanded from AC to AB. Here we need the convexity of preferences & interiority of the original contribution. The individual still prefers G^* & x_i^* because if there were a better

choice for her on CB then there would exist a convex combination of this new preferred choice and the old choice G^* & x_i^* which would make i better off than the old choice and would have been attainable under budget constraint AC.

Existence & Uniqueness when the redistribution may be large enough to change the set of contributors

Note that the above pf demonstrates the existence of equilibrium given a fixed set of initial contributors and a redistribution of wealth among the contributors such that no contributor loses more than her contribution.

Please see BBV 1985 pp 32-35 for pfs of existence & uniqueness.

Comparative Statics on Wealth

Some definitions will help us evaluate comparative statics:

Def 1 Let $f_i(w_i + G_{-i})$ be consumer i's demand for the public good given income & others' contributions $w_i + G_{-i}$ and ignoring the inequality constraint requiring that i contributes a positive amount.

Then for contributors,

$$G = f_i(w_i + G_{-i}). \tag{2}$$

Then including the inequality constraint, we have

$$G = \max\{f_i(w_i + G_{-i}), G_{-i}\}, \text{ or}$$

$$g_i = \max\{f_i(w_i + G_{-i}) - G_{-i}, 0\}.$$

The proofs of existence and uniqueness of equilibrium required one further assumption on preferences.

Assumption. There is a single-valued demand function for the public good, $f_i(w_i + G_{-i})$, which is a differentiable function of wealth. The MPC the public good is >0 and <1 , so $0 < f_i'(w_i + G_{-i}) < 1$ for $i = 1, \dots, n$.

Fact 2 There exists a real valued function $F(G, C)$, differentiable & increasing in G , such that in a Nash equilibrium

$$F(G, C) = \sum_{i \in C} w_i.$$

Where does this function come from? Start by noting that since $f_i(w_i + G_{-i})$ is strictly increasing in its argument for each i it has an inverse for each i . Call that inverse ϕ_i .

Applying ϕ_i to each side of (2), we find

$$\phi_i(G) = w_i + G_{-i} \text{ for } i \in C.$$

Summing over i ,

$$\sum_{i \in C} \phi_i(G) + (1-c)G = \sum_{i \in C} w_i.$$

where c is the number of contributors & C is the set of contributors. Now call the LHS $F(G, C)$ and we have our increasing, differentiable function of G . If you're not convinced on the increasing point, that's because we haven't covered it yet. p 34 of the paper shows $\partial F / \partial G > 1$.

Fact 3 BBV also demonstrate that for Nash equilibria (g_i) & (g_i') , $i=1,\dots,n$, given wealth distributions w_i and w_i' , with corresponding sets of contributors C and C' , it must be true that

$$F(G',C) - F(G,C) \geq \sum_{i \in C} (w_i' - w_i) \quad \forall i \in C.$$

Theorem 4. In a Nash equilibrium:

- (i) Any change in the w distribution that leaves the aggregate wealth of contributors unchanged will either increase or leave unchanged the eq'm supply of G .
- (ii) Any change in the wealth dist'n that increases the aggregate wealth of current contributors must necessarily increase the eq'm supply of G .
- (iii) If a redistribution of income among current contributors increases the equilibrium supply of G , then the set of contributing consumers after the redistribution must be a proper subset of the original set of contributors.
- (iv) Any simple transfer of income from another consumer to a currently contributing consumer will either increase or leave constant the eq'm supply of G .

Proof. (i) Based on Fact 3,

$$\sum_{i \in C} (w_i' - w_i) = 0 \text{ implies } F(G',C) \geq F(G,C).$$

Since $F(G, C)$ is strictly increasing in G , this implies $G' \geq G$.

(ii) Similarly, $\sum_{i \in C} (w_i' - w_i) > 0$ implies $F(G', C) > F(G, C)$
& this implies $G' > G$.

(iii) see p 36 discussion

(iv) Since a simple transfer of income from a non-contributor or contributor to a contributor either increases or leaves constant the aggregate wealth of contributors, (iv) follows immediately from (i) & (ii).

Let's pause to consider the effects of more and less equalizing transfers on contributions to G in the context of Theorem 4 for a moment.

Equilibrium with identical tastes

Definition. A redistribution of wealth is said to be *equalizing* if it is equivalent to a series of bilateral transfers in which the absolute value of the wealth difference between the two parties to the transfer is reduced.

Fact 4. If all consumers have equal preferences & G^* is an equilibrium supply of the PG, then there is a critical wealth level $w^* = \phi(G^*) - G^*$ such that all consumers with wealth $w_i \leq w^*$ contribute nothing and every consumer with income $w_i > w^*$ contributes the amount $g_i^* = w_i - w^*$ to the public good.

Given Fact 4, the intuition behind the five points of Theorem 5 may be clear. For proofs of each point, see BBV p38.

Theorem 5. If preferences are identical, then in a Nash equilibrium,

- (i) All contributors have greater wealth than all non contributors.
- (ii) All contributors will consume the same amount of the private as well as the public good.
- (iii) An equalizing wealth redistribution will never increase the voluntary equilibrium supply of the public good.
- (iv) Equalizing wealth redistributions among current non-contributors or among current contributors will leave the equilibrium supply unchanged.
- (v) Equalizing income redistributions that involve any transfers from contributors to non-contributors will decrease the equilibrium supply of the public good.

On surprising result (ii), see Figure 3 (again) for proof by contradiction.

Example: Identical, homothetic preferences

Homothetic preferences imply that the demand function relating total wealth to the desired level of the public good are

$$f(w) = \alpha w.$$

Let's order the wealth of members of our society so that

$$w_1 \geq w_2 \geq \dots \geq w_n.$$

If $\alpha = 0.5$, when will there be only one contributor to G ?

We know by Fact 4 that if there is only 1 contributor then she is the wealthiest member of society. The wealthiest member of society's optimal contribution if she is the only contributor is

$$f(w_1) = \alpha w_1.$$

Then the second wealthiest person contributes 0 iff

$$G = \alpha w_1 \geq \alpha(w_2 + G) = \alpha(w_2 + \alpha w_1), \text{ or}$$

$$(1 - \alpha)w_1 \geq w_2.$$

Thus for $\alpha = 0.5$, there is only one contributor to G if the wealthiest person is twice as wealthy as the second wealthiest person in society.

In another interesting case with identical homothetic preferences, we can consider the move from a society in which the wealthiest individual has all of the society's wealth, W , to a case in which society's wealth W is divided between two individuals.

The wealthiest person holding all society's riches clearly chooses

$$G = \alpha W.$$

Now give $W/2$ to a second member of society, and let the two with positive wealth determine G . Theorem 5 for identical prefs says they must be equal contributors since they have the same wealth. Therefore,

$$G_2 = \alpha(W/2 + G_2/2), \text{ or}$$

$$G_2 = \alpha W / (2 - \alpha).$$

Observe that $G_2 = \alpha W / (2 - \alpha) < \alpha W = G_1$.

More generally, if wealth is divided among k consumers in the society, then the net voluntary contributions to the public good are