Introduction to Dynamic Economics: Stopping Time Problems

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1 Introduction

In this note, we present some results on existence and characterization of solutions to some optimal stopping problems. The notes follow Brock, Rothschild and Stiglitz (1989), and Shiryayev (1978).

2 A Simple Optimal Stopping Problem

Consider the following problem: Given a stochastic process that satisfies the stochastic differential equation following SDE

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \tag{1}$$

where W_t is a standard Brownian Motion or its integral version

$$X_t = X_0 + \int_0^t \mu(X_s)ds + \int_0^t \sigma(X_s)dW_s, \qquad 0 \le t < \infty, \tag{2}$$

and given a probability space, $(\Omega, \mathcal{F}, P)$, and the filtration $\{\mathcal{F}_t\}$ (which we assume satisfies the usual conditions), choose a stopping time, T, to maximize

$$E[e^{-rT} X_T \mid X_0 = x].$$

Let's assume (for now) that we search for the class of stopping times defined by

$$T = \inf\{t : X_t = y\}. \tag{3}$$

Let $H(x, y)$ be defined by

$$H(x, y) \equiv E[e^{-rT} y \mid X_0 = x]. \tag{4}$$

Given the structure of this problem, the Feynman-Kac theorem implies that $H(x, y)$ is a C^2 function of x, and it satisfies

$$rH(x, y) = 0 + \mu(x)H_x(x, y) + \frac{\sigma^2(x)}{2} H_{xx}(x, y). \tag{5a}$$

$$H(y, y) = y \tag{5b}$$

Here we are assuming that the pair of functions $[\mu(x), \sigma(x)]$ satisfies the conditions of the Feynman-Kac theorem, which, in this case, amount to assuming that $\{X_t\}$ is a canonical diffusion process. The formal definition is

Definition 1 (Canonical Diffusion Process) *A process X is a canonical diffusion if it satisfies*

$$X_t(\omega) = X_0(\omega) + \int_0^t \mu(X_s(\omega))ds + \int_0^t \sigma(X_s(\omega))dW_s$$

where X_t is a d-dimensional process, and W is a r-dimensional Brownian motion. Let the matrix

$$a(x) \equiv \sigma(x)\sigma^T(x)$$

and $\mu(x)$ satisfy the following Hölder condition

$$\begin{aligned} \| \quad \mu(x) - \mu(y) \| &\leq K \| x - y \|^\lambda, \\ \| \quad \sigma(x) - \sigma(y) \| &\leq K \| x - y \|^\lambda \end{aligned}$$

where if $x \in \mathbb{R}^d$, $\| x \| \equiv (\sum_{i=1}^d x_i^2)^{1/2}$, while if a is a $d \times d$ matrix, $\| a \| \equiv (\sum_{i=1}^d \sum_{j=1}^d a_{ij}^2)^{1/2}$. Assume that $\exists \gamma > 0$ such that for all $x \in E$ (the feasible state space) and for a d-tuples λ

$$\sum_{i=1}^d \sum_{j=1}^d a_{ij}(x)\lambda_i\lambda_j \geq \gamma \sum_{i=1}^d \lambda_i^2 = \gamma \| \lambda \|^2,$$

or

$$\hat{\lambda}^T a(x)\hat{\lambda} \geq \gamma; \qquad \| \hat{\lambda} \| = 1.$$

We can now define

$$V(x) = \sup_y H(x, y). \tag{6}$$

Before we proceed, it is useful to think of an example that fits this framework. Let X_t be the 'intrinsic'value of a forest at time t —the value of the forest if it was harvested today—, and let T be the time at which the trees are cut down. In this interpretation $V(x)$ is the market value of the forest before it is cut down. Thus, $V(x) \geq x$, while at the time of harvesting $V(x) = x$, which corresponds to the boundary condition (5b).

We are now ready to prove that the market value $V(x)$ —as opposed to the 'intrinsic' value X— grows at the rate r. For this we need to assume that $V(x)$ is a C^2 function, which we do for now (without proof).

Formally, we show

1. **Proposition 2** *Let $X_0 = x$, $x < y$. Then, for $t > 0$ (t small) we have*

$$\log E[V(X_t) \mid X_0 = x] - \log V(x) = rt.$$

Proof. Use Ito's lemma on $V(x)$ to get

$$dV(X_t) = [V'(X_t)\mu(X_t) + \frac{\sigma^2(X_t)}{2}V''(X_t)]dt + \sigma(X_t)V'(X_t)dW_t. \quad (7)$$

However, using (5a) on (7) we get

$$dV(X_t) = rV(X_t)dt + \sigma(X_t)V'(X_t)dW_t,$$

or

$$V(X_t) - V(x) = \int_0^t rV(X_s)ds + \int_0^t \sigma(X_s)V'(X_s)dW_s.$$

Taking expectations of this expression, we get

$$E[V(X_t) \mid X_0 = x] = V(x) + \int_0^t rE[V(X_s) \mid X_0 = x]ds.$$

It follows that the function $\phi(t)$ defined by $\phi(t) \equiv E[V(X_t) \mid X_0 = x]$, satisfies the integral equation

$$\phi(t) = V(x) + \int_0^t r\phi(s)ds.$$

The solution is given by

$$E[V(X_t) \mid X_0 = x] = V(x)e^{rt},$$

and this proves the proposition. ■

Thus, the expected market value of the forest grows at the interest rate, and it is independent of the growth rate of the intrinsic value (provided that the forest is harvested). The reason for this is simple: an investor will only hold an asset (in this case the forest) if its rate of return is at least as high as the rate of return on the safe bond. If the investor is risk neutral —as our investor is— then both rates of return are equal.

Remark 3 *Note that if*

$$\frac{\sigma(X_t)V'(X_t)}{V(X_t)} = \lambda$$

*for some λ then the value of the forest satisfies (see (**??**))*

$$dV(X_t) = rV(X_t)dt + \lambda V(X_t)dW_t,$$

i.e. $V(X_t)$ is a geometric Brownian motion.

Remark 4 *If $\mu(x) = \mu x$, then we need $\mu \le r$ for the process to be stopped. To see this, consider the case in which $\mu > r$. It follows that*

$$X_t = x + \int_0^t \mu X_s ds + \int_0^t \sigma(X_s)dW_s,$$

and, using standard arguments,

$$E[X_t \mid X_0 = x] = xe^{\mu t}.$$

It follows that the value of cutting the tree at time t —for an arbitrary t— is just

$$E[e^{-rt}X_t \mid X_0 = x] = xe^{(\mu-r)t},$$

which goes to ∞ as $t \to \infty$. Thus, it is optimal to never stop.

3 A Proof of the Smooth Pasting Condition

In this section, we present a proof of the smooth pasting condition. Given a probability space, $(\Omega, \mathcal{F}, P)$, and a a filtration of sub σ-fields of $\mathcal{F}$, $\{\mathcal{F}_t\}$, satisfying the usual conditions, let

$$V(x) \equiv \max_{T \in T} E\{\int_0^T e^{-\int_0^t k(X_s(\omega))ds} g(X_t(\omega))dt + e^{-\int_0^T k(X_s(\omega))ds}\varphi(X_T(\omega)) \mid X_0(\omega) = x\}, \tag{8}$$

where the set $\mathcal{T}$ is the set of all stopping times of $\{\mathcal{F}_t\}$. Note that this is a Feynman-Kac type of problem except that if the stopping time can be defined as

$$T = \inf\{t : X_t \in G\}$$

for some G closed, then G is chosen optimally. We assume that X_t is an element of $\mathbb{R}$. [Extension to the multidimensional case is missing!]

Remark 5 *In general, it is always possible —subject ot some technical conditions— to restrict the Feynman-Kac problem to a problem of choosing the stopping time T to be of the form above; i.e. of the first passage time variety. To see this, let T be any given stopping time for the Feynman-Kac problem. Then, define G as*

$$\tilde{G} \equiv \{x : V(x) = \varphi(x)\}.$$

Then defining $\tilde{T} = \inf\{t : X_t \in \tilde{G}\}$, and as $\tilde{\mathcal{T}}$ as the class of stopping times defined in this way (i.e. the class of first passage times), it follows that $V(x) = \tilde{V}(x)$, where $\tilde{V}(x)$ is the value of the problem (8), restricted to $\tilde{G}$.

For now we fix G, and we let ∂G to be the boundary of G. Recall that it follows that

$$V(z) = \varphi(z), \qquad z \in \partial G.$$

Let $z \in G$ and $\varepsilon > 0$ (arbitrarily small). Define the stopping time $\tau_\varepsilon(z) \equiv \inf\{t : X_t = z - \varepsilon,\ or\ X_t = z + \varepsilon\}$. Assume that

Condition 6 (Finiteness) $P[\tau_\varepsilon(z) < 1] = 1.$

Associated with the stopping time $\tau_\varepsilon(z)$, we define an operator $M_{\tau_\varepsilon(z)}$ by

$$M_{\tau_\varepsilon(z)}f(x) \equiv E[e^{-\int_0^{\tau_\varepsilon(z)} k(X_s(\omega))ds} f(X_{\tau_\varepsilon(z)}) \mid X_0 = x]. \tag{9}$$

Condition 7 *We assume that the function φ is differentiable and that it satisfies*

$$\varphi(z) = M_{\tau_\varepsilon(z)}\varphi(z) + o(\varepsilon)) \tag{10}$$

This last assumption amounts to a regularity condition on φ, and it is satisfied if φ is C^1. To see this, note that

$$\begin{aligned} M_{\tau_\varepsilon(z)}\varphi(z) - \varphi(z) &= E[e^{-\int_0^{\tau_\varepsilon(z)} k(X_s(\omega))ds}[\varphi(X_{\tau_\varepsilon(z)}) - \varphi(z)] \mid X_0 = z] \leq \\ E[e^{-\int_0^{\tau_\varepsilon(z)} k(X_s(\omega))ds} &\mid X_0 = z] \max[\varphi(z+\varepsilon) - \varphi(z), \varphi(z-\varepsilon) - \varphi(z)] \leq \\ &\| \varphi'(x) \|_{[z-\varepsilon, z+\varepsilon]} E[e^{-\int_0^{\tau_\varepsilon(z)} k(X_s(\omega))ds} \mid X_0 = z]\varepsilon \end{aligned}$$

where $\| \varphi'(x) \|_{[z-\varepsilon,z+\varepsilon]} = \sup_{z-\varepsilon \leq x \leq z+\varepsilon} \mid \varphi'(x) \mid$. By continuity of $\varphi'(x)$, it follows that $\lim_{\varepsilon \to 0} \| \varphi'(x) \|_{[z-\varepsilon,z+\varepsilon]} = 0$. With this assumption, $M_{\tau_\varepsilon(z)}\varphi(z) - \varphi(z)$ is $o(\varepsilon)$.

We will assume for simplicity that the function g is non-negative. The arguments, easily extend to any g that is bounded below. For functions g that are unbounded below, I suspect it can be done but one has to be careful!

Note that the principle of optimality implies that for an arbitrary stopping time $\tau_\varepsilon(z)$

$$\begin{aligned} V(x) &\geq E\{\int_0^{\tau_\varepsilon(z)} e^{-\int_0^t k(X_s(\omega))ds} g(X_t(\omega))dt + e^{-\int_0^{\tau_\varepsilon(z)} k(X_s(\omega))ds}\varphi(X_{\tau_\varepsilon(z)}(\omega)) \mid X_0(\omega) = z\} \\ &= E\{\int_0^{\tau_\varepsilon(z)} e^{-\int_0^t k(X_s(\omega))ds} g(X_t(\omega))dt + M_{\tau_\varepsilon(z)}V(z). \end{aligned} \tag{11}$$

We are now ready to state and prove the smooth pasting condition. Without loss of generality let $z \in \partial G$, and assume that, for a small enough ε, $z + \varepsilon \in G$ as well. Of course, $z - \varepsilon \notin G$.

Proposition 8 (Smooth Pasting) *Assume that $g \geq 0$, that the finitness condition hold, and that (10) is satisfied. If V and φ are differentiable and the problem is such that*

$$\lim_{\varepsilon \to 0} \int_{\{\omega : X_{\tau_\varepsilon(z)} = z - \varepsilon\}} e^{-\int_0^{\tau_\varepsilon(z)} k(X_s(\omega))ds} P_z(d\omega) \geq c > 0,$$

then $V'(z) = \varphi'(z)$, for $z \in \partial G$.

Proof. Note that $z \in G$ implies that $V(z) = \varphi(z)$ by the value matching condition. Thus,

$$\begin{aligned} o(\varepsilon) &= \varphi(z) - M_{\tau_\varepsilon(z)}\varphi(z) = V(z) - M_{\tau_\varepsilon(z)}\varphi(z) \geq \\ &\quad E\{\int_0^{\tau_\varepsilon(z)} e^{-\int_0^t k(X_s(\omega))ds} g(X_t(\omega))dt + M_{\tau_\varepsilon(z)}V(z) - M_{\tau_\varepsilon(z)}\varphi(z) \\ &\geq M_{\tau_\varepsilon(z)}(V - \varphi)(z) \geq 0. \end{aligned}$$

It follows that $H(z) \equiv V(z) - \varphi(z)$ satisfies

$$o(\varepsilon) = M_{\tau_\varepsilon(z)} H(z).$$

Given that $H(x) = 0$, for $x \in [z, z+\varepsilon]$ by assumption, it follows that

$$M_{\tau_\varepsilon(z)} H(z) = \int_{\{\omega : X_{\tau_\varepsilon(z)} = z-\varepsilon\}} e^{-\int_0^{\tau_\varepsilon(z)} k(X_s(\omega))ds} f(X_{\tau_\varepsilon(z)}(\omega)) P_z(d\omega)]$$

Next, consider a Taylor expansion of $H(x)$ around $x = z$. It follows that

$$\begin{aligned} o(\varepsilon) &= \int_{\{\omega : X_{\tau_\varepsilon(z)} = z-\varepsilon\}} e^{-\int_0^{\tau_\varepsilon(z)} k(X_s(\omega))ds} [H(z) + H'(z)(X_{\tau_\varepsilon(z)}(\omega) - z) \\ &\quad + \frac{1}{2} H''(Y(\omega))(X_{\tau_\varepsilon(z)}(\omega) - z)^2] P_z(d\omega), \end{aligned}$$

or,

$$o(\varepsilon) = H'(z) \int_{\{\omega : X_{\tau_\varepsilon(z)} = z-\varepsilon\}} e^{-\int_0^{\tau_\varepsilon(z)} k(X_s(\omega))ds} (X_{\tau_\varepsilon(z)}(\omega) - z) P_z(d\omega) + R(\varepsilon),$$

where $R(\varepsilon)$ is $o(\varepsilon)$, and it is the residual of the Taylor expansion. This expression is equal to

$$o(\varepsilon) = -\varepsilon H'(z) \int_{\{\omega : X_{\tau_\varepsilon(z)} = z-\varepsilon\}} e^{-\int_0^{\tau_\varepsilon(z)} k(X_s(\omega))ds} P_z(d\omega) + R(\varepsilon).$$

Then dividing through byε and taking the limit as $\varepsilon \to 0$, we get

$$0 = H'(z) \lim_{\varepsilon \to 0} \int_{\{\omega : X_{\tau_\varepsilon(z)} = z-\varepsilon\}} e^{-\int_0^{\tau_\varepsilon(z)} k(X_s(\omega))ds} P_z(d\omega),$$

which implies the desired result. ■

Remark 9 *Note that the term inside the integral is bounded* above *by*

$$P_z[X_{\tau_\varepsilon(z)} = z - \varepsilon].$$

We need a lower bound. Thus if we could show/argue that

$$\int_{\{\omega : X_{\tau_\varepsilon(z)} = z-\varepsilon\}} e^{-\int_0^{\tau_\varepsilon(z)} k(X_s(\omega))ds} P_z(d\omega) \geq K P_z[X_{\tau_\varepsilon(z)} = z - \varepsilon],$$

and if, for small $\varepsilon > 0$, we have

$$P_z[X_{\tau_\varepsilon(z)} = z - \varepsilon] \geq c > 0,$$

then we are done.

References

[1] Brock, William A., M. Rothschild and J. E. Stiglitz, 1989, "Stochastic Capital Theory," in George R. Feiwel (ed) *Joan Robinson and Modern Economic Theory*, pp: 591-622.

[2] Shiryayev, A. N., 1978, *Optimal Stopping Rules*, Springer-Verlag, New York.