Stochastic Processes and Brownian Motion: Basic Results

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1 Stochastic Processes

Let Ω be a space of events, or sample space, and P a probability measure. Let $\mathcal{F}$ be σ-algebra of subsets. Let $S \subseteq \Re^d$ be the state space, and $\mathcal{S}$ the Borel σ-algebra. A stochastic process is a function

$$X : [0, \infty) \times \Omega \to S$$

such that, for each t, $X(t, \bullet)$ is $\mathcal{F}$-measurable. The mapping

$$X(\bullet, \omega) : [0, \infty) \to S$$

is a sample path.

Definition 2 *The process X is measurable if for any $A \in \mathbf{S}$, $X^{-1}(A) \in \mathcal{B}([0, \infty)) \otimes \mathcal{F}$.*

Definition 3 (Filtration) *A filtration is a non-decreasing family of sub σ-fields of $\mathcal{F}$, $\mathcal{F}_s \subseteq \mathcal{F}_t$, for $0 \leq s \leq t < \infty$. Let $\mathcal{F}_\infty = \sigma(\cup_{t \geq 0} \mathcal{F}_t)$*

$\mathcal{F}_{t-} \equiv \sigma(\cup_{s<t} \mathcal{F}_s)$,

$\mathcal{F}_{t+} \equiv \cap_{\epsilon>0} \mathcal{F}_{t+\epsilon}$

A filtration is right continuous if $\mathcal{F}_t = \mathcal{F}_{t+}$.

Definition 4 (Adapted) *A process X is adapted to $\{\mathcal{F}_t\}$ if X_t is $\mathcal{F}_t$-measurable*

2 Stopping Times

Definition 5 (Stopping Time) *Let $(\Omega, \mathcal{F})$ be a measurable space equipped with the filtration $\{\mathcal{F}_t\}$. A random time T is a stopping time of the filtration if*

$$\{\omega : T(\omega) \leq t\} \in \mathcal{F}_t.$$

It is called ***optional time if***

$$\{\omega : T(\omega) < t\} \in \mathcal{F}_t.$$

It follows that all constants are stopping times and if $\{\mathcal{F}_t\}$ is right continuous every stopping time is optional (exercise).

There is a special class of stopping times denoted *hitting times*. Let $\Gamma \in \mathcal{B}(R^d)$ and let

$$H_\Gamma(\omega) = \inf\{t \geq 0; X_t(\omega) \in \Gamma\}$$

Results:

- If Γ is open then H_Γ is optional.
- If Γ is closed and the sample paths of X are continuous, then H_Γ is a stopping time.

Claim 6 (Properties of Stopping Times) *1. T optional and $\theta > 0$ implies that $T + \theta$ is a stopping time.*

2. T and S stopping times $\rightarrow$

$$T \wedge S$$
$$T \vee S$$
$$T + S$$

are stopping times

3. T and S optional times $\rightarrow T+S$ optional. $T+S$ is a stopping time if either

$$\begin{aligned} a) \qquad T &> 0,\ S > 0, \\ b) \qquad T &> 0,\ T \textit{ is a stopping time} \end{aligned}$$

4. $\{T_n\}$ sequence of stopping times so is $\sup_{n\geq 1} T_n$.

Definition 7 *Let T be a stopping time of the filtration $\{\mathcal{F}_t\}$. The σ-field $\mathcal{F}_T$ of events determined prior to the stopping time T consists of those events $A \in \mathcal{F}$ for which*

$$for\ all\ t, \qquad A \cap \{T \leq t\} \in \mathcal{F}_t$$

Claim 8 *If $T = t$ then $\mathcal{F}_T = \mathcal{F}_t$.*

Claim 9 *Let T be a stopping time and S a random time such that $S \geq T$. If S is $\mathcal{F}_T$-measurable it is also a stopping time.*

Claim 10 *Let T and S be stopping times. For any $A \in \mathcal{F}_S$*

$$A \cap \{S \leq T\} \in \mathcal{F}_T.$$

In particular, if $S \leq T \rightarrow \mathcal{F}_S \subseteq \mathcal{F}_T$.

Claim 11 *Let T and S be stopping times. Then,*

$$\mathcal{F}_{T\wedge S} = \mathcal{F}_T \cap \mathcal{F}_S,$$

and each of the following events

$$\begin{aligned}
\{T &< S\},\\
\{S &< T\},\\
\{T &\leq S\},\\
\{T &= S\},
\end{aligned}$$

belongs to $\mathcal{F}_{T\wedge S}= \mathcal{F}_T \cap \mathcal{F}_S$.

Definition 12 (Usual Conditions) *A Filtration* $\{\mathcal{F}_t\}$ *satisfies the usual conditions if it is right continuous and* $\mathcal{F}_0$ *contains all the* $P-null$ *sets in* $\mathcal{F}$.

Proposition 13 *If the process* X *has RCLL paths and is adapted to the filtration* $\{\mathcal{F}_t\}$ *which satisfies the usual conditions, then there exists a sequence of stopping times* $\{T_n\}$ *which exhausts the jumps of* X, *i.e.*

$$\{(t,\omega) : X_t(\omega) \neq X_{t^-}(\omega)\} \subseteq \cup_{n=1}^{\infty}\{(t,\omega) : T_n(\omega) = t\}$$

3 Continuous Time Martingales

Throughout this section we assume

1. $(\Omega, \mathcal{F}, P)$ is a probability space.
2. $\{\mathcal{F}_t\}$ is a filtration (to which X_t is adapted)
3. $E[| X_t |] < \infty$, for all $t \geq 0$.

Definition 14 (Submartingale-Supermatingale-Martingale) X *is a submartingale (supermartingale) is for all* $0 \leq s < t < \infty$ *we have* $P-as$

$$\begin{aligned}
X_s &\leq E[X_t \mid \mathcal{F}_s]\\
(X_s &\geq E[X_t \mid \mathcal{F}_s]).
\end{aligned}$$

If X *is both a sub and a supermartingale then it is a martingale*

3.1 Inequalities

Let X be a submartingale and let $\mathcal{F}_\infty = \sigma(\cup_{t\geq 0}\mathcal{F}_t)$, then it follows that if there exists an F_∞-measurable random variable, X_∞, such that

$$X_t \leq E[X_\infty \mid \mathcal{F}_t], \qquad t \geq 0,\ P - as,$$

then we say that $\{X_t, \{\mathcal{F}_t\}, 0 \leq t \leq \infty\}$ is a submartingale with *last element* X_∞.

Claim 15 *$\{X_t\}$ is a (sub)martingale and $\phi : R \to R$ convex (convex non-decreasing) such that*

$$E[\mid \phi(X_t) \mid] < \infty,\ t \geq 0$$

then $\{\phi(X_t)\}$ is a submartingale.

3.2 Convergence Results

In this section assume that all processes are *right continuous*.

Theorem 16 (Submartingale Convergence) *Let $\{X_t, \{\mathcal{F}_t\}, 0 \leq t < \infty\}$ be a right continuous submartingale. If*

$$C \equiv \sup_{t\geq 0} E[X_t^+] < \infty,$$

then

$$\begin{aligned} X_\infty(\omega) &\equiv \lim_{t\to\infty} X_t(\omega) \qquad \textit{exists } P - as, \\ E[&\mid X_\infty \mid] < \infty. \end{aligned}$$

where

$$X_t^+ = \max\{X_t, 0\}$$

Remark 17 *If the submartingale is **non-negative** then the existence part is guaranteed without any further assumptions and*

$$\{X_t, \{\mathcal{F}_t\}, 0 \leq t \leq \infty\}$$

is a submartingale.

Remark 18 *If instead of a submartingale we have a supermartingale then the analogous conditions/results are*

$$C \equiv \sup_{t \geq 0} E[X_t^-] < \infty,$$

then

$$\begin{aligned} X_\infty(\omega) &\equiv \lim_{t\to\infty} X_t(\omega) \qquad \textit{exists } P - as, \\ E[&\mid X_\infty \mid] < \infty. \end{aligned}$$

where

$$X_t^- = \max\{-X_t, 0\}$$

Definition 19 (Uniform Integrability) *A collection of random variables* $\{X_t\}$ *defined on* $(\Omega, \mathcal{F}, P)$ *is uniformly integrable if*

$$\int_{\{|X_t| \geq c\}} \mid X_t(\omega) \mid dP(\omega)$$

converges to 0, uniformly in t, *as* $c \to \infty$.

Definition 20 (L^1 norm) $\parallel X_t \parallel_1 \equiv \int \mid X_t(\omega) \mid dP(\omega)$, *provided it is finite.*

Theorem 21 ***Corollary 22*** *If* $\{X_t, \{\mathcal{F}_t\}, 0 \leq t < \infty\}$ *is a* ***uniformly integrable*** *supermartingale then,*

$$\sup_t \parallel X_t \parallel_1 < \infty,$$

and since $E[X_t^-] \leq \parallel X_t \parallel_1$ *the supermartingale version of the theorem implies*

$$X_\infty(\omega) \equiv \lim_{t\to\infty} X_t(\omega) \qquad \textit{exists } P - as.$$

By uniform integrability convergence is also in $L^1 \to \{X_t, \{\mathcal{F}_t\}, 0 \leq t \leq \infty\}$ *is also a supermartingale!*

Claim 23 *The following three conditions are equivalent for a* ***right continuous martingale***

1. *It is a uniformly integrable family of random variables*

2. *It converges in L^1 as $t \to \infty$.*

3. *It converges $P-as$ to an integrable random variable X_∞ such that $\{X_t, \{\mathcal{F}_t\}, 0 \leq t \leq \infty\}$ is a martingale.*

4. *There exists a random variable Y such that*

$$X_t = E[Y \mid \mathcal{F}_t], \qquad t \geq 0,\ P - as.$$

5. *If (4) holds, and X_∞ is the random variable in (3)*

$$X_\infty = E[Y \mid \mathcal{F}_\infty], \qquad P - as.$$

Theorem 24 (Optional Sampling) *Let $\{X_t, \{\mathcal{F}_t\}, 0 \leq t \leq \infty\}$ be a right continuous submartingale with last element X_∞ and let $S \leq T$ be two optional times of the filtration $\{\mathcal{F}_t\}$. Then,*

1. $$X_S \leq E[X_T \mid \mathcal{F}_{S^+}], \qquad t \geq 0,\ P - as,$$

2. *If S is a stopping time then*

$$X_S \leq E[X_T \mid \mathcal{F}_S], \qquad t \geq 0,\ P - as,$$

and, in particular,

$$E[X_0] \leq E[X_T].$$

3. *For a martingale with last element X_∞ the previous inequalities are equalities.*

Remark 25 *Note that the theorem assumes that the process has a last element X_∞. This is a strong assumption. See before to check the iff conditions for this!*

Remark 26 (Extensions of the OST Theorem) *Is it necessary for the process to have a last element? Here are alternative conditions under which the theorem holds*

1. *Let* $\{X_t, \{\mathcal{F}_t\}, 0 \leq t < \infty\}$ *be a right continuous submartingale and let the* ***optional*** *times satisfy* $S \leq T$, *then OST holds if either of these two conditions is satisfied*

   (a) T *is bounded (i.e.* $\exists a$ *such that* $T(\omega) \leq a$ $P - as$*)*

   (b) *There exists an integrable random variable* Y *such that*

   $$X_t \leq E[Y \mid \mathcal{F}_t], \qquad t \geq 0, \; P - as.$$

2. *Let* $\{X_t, \{\mathcal{F}_t\}, 0 \leq t < \infty\}$ *be a right continuous submartingale and let the* ***stopping*** *times satisfy* $S \leq T$, *then*

   (a) $\{X_{T \wedge t}, \{\mathcal{F}_t\}, 0 \leq t < \infty\}$ *is a submartingale,*

   (b)

   $$X_{S \wedge t} \leq E[X_{T \wedge t} \mid \mathcal{F}_{S \wedge t}], \qquad t \geq 0, \; P - as.$$

4 Brownian Motion

Definition 27 *A standard one dimensional Brownian Motion (BM) is a continuous adapted process* $W = \{W_t, \{\mathcal{F}_t\}, 0 \leq t < \infty\}$ *defined on some probability space* $(\Omega, \mathcal{F}, P)$ *with the properties that*

1. $W_0 = 0$

2. *For* $0 \leq s < t$, $W_t - W_s$ *is independent of* $\mathcal{F}_s$.

3. $W_t - W_s \sim N(0, t - s)$

Is this definition restrictive? Maybe. Consider the following

Claim 28 (Levy) *If $X \in \mathcal{M}^c$ (continuous path martingale) and X is such that its quadratic variation $\langle X \rangle_t = t$ as, for all interval $[0, t]$, then X is a Brownian Motion.*

Claim 29 (Breiman/Harrison) *If X is continuous (i.e. has continuous sample paths) and X has identical independent increments, then X is a Brownian Motion.*

Does a Brownian Motion exist? Yes. There are several ways to construct a Brownian Motion. We will not go into the details, but the interested reader can consult Karatzas and Shreve or Harrison.

Here is one simple construct that gives a random walk interpretation of a BM. Consider a random walk Z_t such that

$$\Delta Z = Z_{j+1}^h - Z_j^h = \begin{cases} \Delta h & \textit{with probability } p \\ -\Delta h & \textit{with probability } 1-p. \end{cases}$$

It follows that

$$\begin{aligned} E[Z_{j+1}^h - Z_j^h] &= \Delta h(2p-1), \\ Var[Z_{j+1}^h - Z_j^h] &= 4p(1-p)(\Delta h)^2. \end{aligned}$$

Consider a time interval of length T, and let $T = n\Delta t$. Then the cumulative change in $Z_{j+1}^h - Z_j^h$ over the whole interval is

$$\Delta_T Z = \underbrace{\Delta Z_1 + ... + \Delta Z_n}_{n \textit{ times}}.$$

It follows that

$$\begin{aligned} E[\Delta_T Z] &= n\Delta h(2p-1) = T\frac{\Delta h}{\Delta t}(2p-1), \\ Var[\Delta_T Z] &= n4(1-p)\Delta h^2 = T\frac{\Delta h^2}{\Delta t}4p(1-p). \end{aligned}$$

Hence if we wanted to pick the random walk (which in this context means picking p and Δh) to match the first two moments of a Brownian Motion we would end up with a system of equations given by

$$\begin{aligned} T\frac{\Delta h}{\Delta t}(2p-1) &= \mu T \\ T\frac{\Delta h^2}{\Delta t}4p(1-p) &= \sigma^2 T. \end{aligned}$$

This system is then,

$$\begin{aligned} \Delta h &= \frac{\mu \Delta t}{2p-1} \\ (\Delta h)^2 &= \left(\frac{\sigma^2 \Delta t}{4p(1-p)}\right)^{1/2}. \end{aligned}$$

The solution is

$$\begin{aligned} \Delta h &= \sigma\sqrt{\Delta t}\left[\sqrt{1+\left(\frac{\mu}{\sigma}\right)\Delta t}\right], \\ p_i &= \frac{1}{2}\{1 \pm \left(\frac{\mu}{\sigma}\right)\frac{\sqrt{\Delta t}}{\sqrt{1+\left(\frac{\mu}{\sigma}\right)\Delta t}}\}. \end{aligned}$$

Since this is an approximation that is valied when $\Delta t \to 0$, the solution can be described as

$$\begin{aligned} \Delta h &= \sigma\sqrt{\Delta t}, \\ p_i &= \frac{1}{2}\{1 \pm \left(\frac{\mu}{\sigma}\right)\frac{\sqrt{\Delta t}}{\sqrt{1+\left(\frac{\mu}{\sigma}\right)\Delta t}}\}. \end{aligned}$$

In the special case of a Weiner process or standard Brownian motion, we have $\mu = 0$ and $\sigma = 1$. Thus, in that case,

$$\begin{aligned} \Delta h &= \sigma\sqrt{\Delta t}, \\ p_i &= \frac{1}{2}. \end{aligned}$$

Thus, one way of thinking about Brownian Motion is as the sum of a sequence of Bernoulli *i.i.d.* random variables, with the size of the jump of the order of the square root of dt

What is the *quadratic variation* of this process?

$$\langle \Delta Z \rangle = \sum_{j=1}^{n} [\underbrace{\Delta Z_j}_{\Delta h}]^2 = \sum_{j=1}^{n} [\sqrt{\Delta t}]^2 = n\Delta t = T.$$

Thus the random walk has the property that its quadratic variation is t in any interval of length t

What do we know about Brownian Motion? There are a number of results that hold for all martingales (and some for martingales with continuous paths) that directly apply to a Brownian Motion

Claim 30 *Let $W = \{W_t, \{\mathcal{F}_t\}, 0 \leq t < \infty\}$ be a Brownian Motion then,*

1. *W is a Markov process.*
2. *W is a martingale.*
3. $\langle W \rangle_t = t$

   Proof. *Using the alternative definition of quadratic variation we have that*

   $$\langle W \rangle_t \equiv \lim_{n\to\infty} \sum_{i=1}^{2^n-1} [W_{t^n_{i+1}} - W_{t^n_i}]^2,$$

   where

   $$t^n_i = \frac{it}{2^n}, \qquad i = 0, 1, ..., 2^n.$$

   Since $W_{t^n_{i+1}} - W_{t^n_i} \sim N(0, t^n_{i+1} - t^n_i) = N(0, \frac{t}{2^n})$, it follows that

   $$Z_i \equiv \sqrt{2^n}[W_{t^n_{i+1}} - W_{t^n_i}] \sim N(0, t).$$

   Thus,

   $$\sum_{i=1}^{2^n-1} [W_{t^n_{i+1}} - W_{t^n_i}]^2 = \langle W \rangle_t \equiv \sum_{i=1}^{2^n-1} \frac{1}{2^n} Z_i{}^2.$$

   By the law of large numbers this term converges (a.s.) to $E[Z_i^2] = t$ ■

4. *$W^2 - \langle W \rangle_t$ is a martingale*

Claim 31 *If a function is differentiable (with bounded derivative) then its quadratic variation is zero*

Proof. *Let $f(t)$ be a function. Then, given a partition $\Pi = \{0 = t_0 < t_1 < ... < t_n = t\}$, the mean Value Theorem implies that*

$$f(t_{i+1}) - f(t_i) = f'(\tau_i)(t_{i+1} - t_i).$$

The quadratic variation satisfies

$$\begin{aligned}\left|\sum_{i=1}^{n}[f(t_{i+1}) - f(t_i)]^2\right| &= \left|\sum_{i=1}^{n}[f'(\tau_i)(t_{i+1} - t_i)]^2\right| \leq \max|f'(\tau)| \sum_{i=1}^{n}(t_{i+1} - t_i)^2 \\ &\leq \max_i |f'(\tau_i)| \max_i (t_{i+1} - t_i)t.\end{aligned}$$

However, as the mesh of the partition goes to 0, $\max(t_{i+1} - t_i) \to 0$ and, hence, the quadratic variation is zero. ■

Claim 32 *Let $W = \{W_t, \{\mathcal{F}_t\}, 0 \leq t < \infty\}$ be a Brownian Motion then its total variation is ∞*

Proof. *Let $\Pi = \{0 = t_0 < t_1 < ... < t_n = t\}$ be a partition of the interval $[0, t]$. Then,*

$$\sum_{i=1}^{n}[W_{t_{i+1}} - W_{t_i}]^2 \leq \max_i (W_{t_{i+1}} - W_{t_i}) \sum_{i=1}^{n} \left|W_{t_{i+1}} - W_{t_i}\right|$$

but the first term on the right hand side converges to zero (as) as the mesh of the partition goes to 0 (this follows from continuity of the sample paths). Since $\langle W \rangle_t = \sum_{i=1}^{n}[W_{t_{i+1}} - W_{t_i}]^2 = t$ is positive, it follows that the term $\sum_{i=1}^{n} \left|W_{t_{i+1}} - W_{t_i}\right|$ must converge to ∞ ■

Starting from the standard BM we can create what we will denote a (μ, σ)-BM. This is a process X_t given by

$$X_t(\omega) = X_0 + \mu t + \sigma W_t(\omega).$$

In the following claims the X_t process is the one given above.

Claim 33 *The process* $X_t(\omega) = X_0 + \mu t + \sigma W_t(\omega)$ *is a submartingale (supermartingale) if* $\mu \geq (\leq) 0$.

Claim 34 $Y_t \equiv X_t - \mu t$ *is a martingale,*

Claim 35 $\langle Y \rangle_t = \sigma^2 t$.

Claim 36 $Y_t^2 - \sigma^2 t$ *is a martingale.*

Claim 37 $\left(\frac{Y_t}{\sigma}\right)^2 - t$ *is a martingale.*

Claim 38 (Wald Martingale) *Let* $Z_t \equiv e^{[\beta X_t - q(\beta)t]}$, *with* $q(\beta) \equiv \beta\mu + \frac{1}{2}\beta^2\sigma^2$. *Then,* Z_t *is a martingale.*

Proof. *To see this, note that*

$$e^{[\beta X_t - q(\beta)t]} = e^{[\beta(X_t - X_s) - q(\beta)(t-s)]} e^{[\beta X_s - q(\beta)s]}.$$

Taking expectations conditional on $\{\mathcal{F}_s\}$ *on both sides we get*

$$E[Z_t \mid \mathcal{F}_s] = Z_s E[e^{[\beta(X_t - X_s) - q(\beta)(t-s)]} \mid \mathcal{F}_s].$$

Thus, it suffices to show that

$$E[e^{[\beta(X_t - X_s) - q(\beta)(t-s)]} \mid \mathcal{F}_s] = 1.$$

However, $\beta(X_t - X_s) \sim N(\beta\mu(t-s), \beta^2\sigma^2(t-s))$, *and the expectation of any such log-normal random variable is*

$$E[e^{\beta(X_t - X_s)} \mid \mathcal{F}_s] = e^{\beta\mu(t-s) + \frac{1}{2}\beta^2\sigma^2(t-s)} = e^{q(\beta)(t-s)},$$

which completes the proof ■

Claim 39 $(X_t - \mu t)^2 - \sigma^2 t$ *is a martingale*

An important property of Brownian Motion is that it is a Strong Markov process.

Definition 40 (Strong Markov) *A process X with initial distribution μ (and associated probability measure P^μ) is said to be a strong Markov process if*

1. $P^\mu[X_0 \in \Gamma] = \mu(\Gamma)$, *for all* $\Gamma \in \mathcal{B}(\Re^d)$

2. *For any optional time S, $t \geq 0$ and* $\Gamma \in \mathcal{B}(\Re^d)$

$$P^\mu[X_{S+t} \in \Gamma \mid \mathcal{F}_{S^+}] = P^\mu[X_{S+t} \in \Gamma \mid X_S],$$

   $P^\mu - as$ *on* $\{S < \infty\}$

Claim 41 *A Brownian Motion is a strong Markov process*

There is another way of describing the strong Markov property which reveals more clearly its implications. Let $W = \{W_t, \{\mathcal{F}_t\}, 0 \leq t < \infty\}$ be a Brownian Motion and let S be a stopping time of the filtration $\{\mathcal{F}_t\}$. Let

$$W_t^* \equiv W_{S+t} - W_S.$$

Then

1. W_t^* is a Brownian motion.

2. W_t^* is independent of S.

Here is an *implication* of the strong Markov property that will be useful later. Let T be an arbitrary stopping time. Then we can define the random Z_t^* by

$$Z_t^*(w) \equiv W_{T(\omega)+t}(\omega), \qquad t \geq 0, \qquad on\ \{\omega : T(\omega) < \infty\}.$$

Note that the process Z_t^* need not be defined at all on the set $\{\omega : T(\omega) = \infty\}$. Now let the function F be a measurable mapping such that

$$E^x[\mid F(x) \mid] < \infty, \qquad \forall x \in \Re.$$

Then, the strong Markov property implies that

$$E^x[F(Z_t^*) \mid \mathcal{F}_T] = F(Z_0^*) = F(W_T), \qquad on\ \{\omega : T(\omega) < \infty\}.$$

This result comes in handy when one needs to compute functions of a stopped BM.

5 The Reflection Principle

Let $W = \{W_t, \{\mathcal{F}_t\}, 0 \leq t < \infty\}$ be a standard BM, and let

$$T_b(\omega) \equiv \inf\{t \geq 0 : W_t(\omega) = b\}.$$

It follows that T_b is a stopping time of the filtration $\{\mathcal{F}_t\}$.

Claim 42 $P^0[T_b < t] = 2P^0[W_t > b] = \frac{2}{\sqrt{2\pi}} \int_{bt^{-1/2}}^{\infty} e^{-\frac{x^2}{2}} dx$

Proof. We first claim that $P^0[T_b < t] = P^0[T_b < t, W_t > b] + P^0[T_b < t, W_t < b]$. This is obvious as $W_t = b$ —the missing part— has zero probability. Next, note that the set $\{T_b < t, W_t > b\} = \{W_t > b\}$. Why? If $W_t > b$ it must be the case that $T_b < t$, as $W_0 = 0$. Thus, the extra conditioning does not change the set. What about the other set, $\{T_b < t, W_t < b\}$? On this set, the Brownian motion hit b at some point, and then drifted to a point $c < b$. Given that the distribution of a Brownian motion is symmetric (because the Normal is a symmetric distribution), the probability of traveling *down* a distance $b - c$ is the same as the probability of traveling *up* the same distance. Thus, we just argued that

$$P^0[T_b < t, W_t < b] = P^0[T_b < t, W_t > b].$$

Thus,

$$\begin{aligned} P^0[T_b &< t] = 2P^0[T_b < t, W_t > b] = 2P^0[W_t > b] \\ &= \frac{2}{\sqrt{2\pi}} \int_{bt^{-1/2}}^{\infty} e^{-\frac{x^2}{2}} dx \end{aligned}$$

■

The above proof is somewhat heuristic (but close to being right) but it relies heavily on the idea that a Brownian motion has the strong Markov property. Where was this used? We basically used to argue that the process

$$W_{T_b+t} - W_{T_b}$$

is a Brownian motion that is independent of $\mathcal{F}_{T_b}$, the σ-field of events determined prior to T_b. Of course, to claim the strong Markov property, T_b has to be a stopping time as we showed (it is a hitting time).

Here is another application of the strong Markov property

Claim 43 *Let $W = \{W_t, \{\mathcal{F}_t\}, 0 \le t < \infty\}$ be a standard BM, with $X_0 = 0$. Let*

$$M_t \equiv \sup\{W_s, 0 \le s \le t\},$$

then

$$P^0[W_t \le x, M_t \le y] = \Phi(\frac{x}{\sqrt{t}}) - \Phi(\frac{x-2y}{\sqrt{t}})$$

where Φ is the cdf of a standard normal random variable, i.e. a $N(0,1)$

Proof. *Note that*

$$P^0[W_t \le x, M_t \le y] = P^0[W_t \le x] - P^0[W_t \le x, M_t > y].$$

The first term is

$$P^0[W_t \le x] = \Phi(\frac{x}{\sqrt{t}})$$

given that W_t is an $N(0,t)$ random variable. What about $P^0[W_t \le x, M_t > y]$? This event corresponds to the case in which the process W_t hit the value y, and then traveled to a point below x. However, conditional on being at y, traveling down *below x (a distance $y - x$) has the same probability (given symmetry of the Normal distribution) of traveling* up *a distance $y - x$. Let T be the first time that the process W_t hits y. Then, define $W^*_{t-T} \equiv W_t - W_T = W_t - y$. Using this notation, we need to compute the probability of W^*_{t-T} increasing by at least $(y - x)$. This is the probability of the event $\{W^*_{t-T} \ge y - x\} = \{W_t \ge 2y - x\} = \{W_t \le x - 2y\} = \Phi(\frac{x-2y}{\sqrt{t}})$. Thus, it follows that*

$$\begin{aligned} P^0[W_t &\le x, M_t > y] = P^0[T < t, W^*_{t-T} > y - x] \\ &= P^0[T < t, W^*_{t-T} > y - x] = P^0[W_t \le x - 2y] = \Phi(\frac{x-2y}{\sqrt{t}}). \end{aligned}$$

Thus, putting together these two pieces we get the desired result. ∎