Stochastic Integration and Ito's Lemma: Definitions and Results

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Acknowledgement 1 *These notes contain some mathematical results taken from several different references (missing)*

1 Riemann vs. Lebesgue Integrals

Let $[a,b]$ be an interval of the real numbers. Let f be bounded on $[a,b]$. Let

$$\Pi^k = \{a = x_0^k < x_1^k < ... < x_n^k = b\}$$

be a partition. Let

$$\mid \Pi^k \mid = \max_{0 \leq i \leq n-1} \{x_{i+1}^k - x_i^k\}$$

be the mesh of the partition. We can now construct the upper and lower sums of f relative to Π as follows

$$\begin{aligned} M_i &= \sup\{f(x) : x_{i-1}^k < x \leq x_i^k\}, \qquad i = 1, 2, ..., n \quad \Leftrightarrow f_k^U(x) \\ m_i &= \inf\{f(x) : x_{i-1}^k < x \leq x_i^k\}, \qquad i = 1, 2, ..., n \quad \Leftrightarrow f_k^L(x). \end{aligned}$$

Define

$$\begin{aligned} U(\Pi^k) &\equiv \sum_{i=1}^{n} M_i(x_i^k - x_{i-1}^k), \\ L(\Pi^k) &\equiv \sum_{i=1}^{n} m_i(x_i^k - x_{i-1}^k). \end{aligned}$$

Note that, if we denote the Lebesgue measure by λ, then we have

$$\begin{aligned} U(\Pi^k) &= \int f_k^U(x)d\lambda, \\ L(\Pi^k) &= \int f_k^L(x)d\lambda. \end{aligned}$$

By the dominated convergence theorem,

$$\begin{aligned} \lim_{k\to\infty} U(\Pi^k) &= \int_a^b f^U(x)d\lambda, \\ \lim_{k\to\infty} L(\Pi^k) &= \int_a^b f^L(x)d\lambda. \end{aligned}$$

Then f is said to Riemann integrable iff

$$r = \int_a^b f^U(x)d\lambda = \int_a^b f^L(x)d\lambda.$$

Theorem 2 (Riemann Integrable [Ash p.55]) *Let f be bounded and real valued on $[a, b]$.*

1. *The function f is Riemann integrable on $[a, b]$ iff f is continuous $\lambda - as$.*
2. *If f is Riemann integrable on $[a, b]$ then f is Lebesgue integrable and both integrals coincide.*

What is the Lebesgue integral? First, define it for simple functions. Let

$$h(x) = \sum_{i=1}^{n} h_i \mathcal{X}_{A_i},$$

where the A_i are disjoint sets in some $\sigma-$algebra $\mathcal{F}$. Then the Lebesgue integral is

$$\int_\Omega h(x)d\lambda(x) = \sum_{i=1}^{n} h_i \lambda(A_i).$$

To extend this definition to a non-negative function h we define

$$\int_\Omega h(x)d\lambda(x) \equiv \sup\{\int_\Omega s(x)d\lambda(x) : \ s(x) \leq h(x) \ is \ simple\}.$$

Finally for an arbitrary Borel measurable h let

$$\begin{aligned} h^+(x) &= \max\{0, h(x)\}, \\ h^-(x) &= \max\{0, -h(x)\}, \end{aligned}$$

then we define

$$\int_\Omega h(x)d\lambda(x) = \int_\Omega h^+(x)d\lambda(x) - \int_\Omega h^-(x)d\lambda(x),$$

provided these terms are not of the form $+\infty - \infty$.

2 Stochastic Integral

Throughout this section we assume that the filtration $\{\mathcal{F}_t\}$ satisfies the usual conditions. Let M be a martingale and assume that $M_0 = 0$, $P - as$.

Let $\langle M \rangle_t$ be the unique adapted continuous and increasing process such that

$$\{M_t^2 - \langle M \rangle_t \ , \{\mathcal{F}_t\}, 0 \leq t < \infty\}$$

is a martingale.

We want to define integrals of the form

$$I_T(X) = \int_0^T X_t(\omega)dM_t(\omega).$$

Remark 3 *In the case of a (μ, σ) Brownian Motion $\langle M \rangle_t = \sigma t$. The existence of $\langle M \rangle_t$ as the unique adapted continuous and increasing process such that $M_t^2 - \langle M \rangle_t$ is a martingale is guaranteed by the Doob-Meyer decomposition.*

Why don't we do it path-wise, i.e. for each ω, take the function

$$G(t)(\omega) \equiv M_t(\omega),$$

and integrate —in the Lebesgue-Stieltjes sense— with respect to the measure induced by G. This will not work because, as we have already argued, if M is a BM then its sample paths are of unbounded variation. Thus, we need to try something else.

Assume that $M \in \mathcal{M}_2^c$ (i.e square integrable martingale). First, let's define a measure on $[0,\infty)\times\mathcal{F}$ by

$$\mu_M(A) = E[\int_0^\infty \mathcal{X}_A(t,\omega)d\langle M\rangle_t(\omega)].$$

Note that this is a well defined measure.

Let X be measurable and $\{\mathcal{F}_t\}$-adapted. Define

$$[X]_T^2 \equiv E[\int_0^T X_t^2(\omega)d\langle M\rangle_t(\omega)],$$

provided that the right hand side is finite. Note that this is an L^2 norm restricted to $[0,T]$.

Remark 4 *In the case in which the integrand is Brownian Motion, then $d\langle M\rangle_t(\omega) = dt$, and this is a standard L^2 norm.*

Definition 5 *Let $\mathcal{L}$ be the space of all measurable, $\{\mathcal{F}_t\}$-adapted processes X for which $[X]_T < \infty$. Define a metric on $\mathcal{L}$ by*

$$[X-Y] = \sum_{n=1}^\infty \left(\frac{1}{2}\right)^n \min\{1,[X-Y]_n\}$$

If $\mathcal{L}^*$ is the set of progressively measurable processes satisfying $[X]_T^2 < \infty$, define a metric on $\mathcal{L}^*$ in the same way we did it on $\mathcal{L}$.

Note that if a process X is in $\mathcal{L}^*$, then, for any $T<\infty$, it must be the case that

$$E[\int_0^T X_t^2(\omega)d\langle M\rangle_t(\omega)] < \infty$$

Let $\mathcal{L}_T^*$ denote the class of processes such that

$$\begin{aligned} X &\in \mathcal{L}^*, \\ X_t(\omega) &= 0, \qquad t>T,\ \omega\in\Omega. \end{aligned}$$

For $T = \infty$, $\mathcal{L}^*_T$ is the class of processes for which

$$E[\int_0^\infty X_t^2(\omega) d\langle M\rangle_t(\omega)] < \infty.$$

Note that since $X \in \mathcal{L}^*$ to begin with, this last definition (i.e. the definition of $\mathcal{L}^*_\infty$) simply extends the upper limit of integration to include the value ∞.

With this definition, $\mathcal{L}^*_T$ is a closed subspace of an L^2 space, the Hilbert space given by

$$\mathcal{H}_T \equiv L^2\left([0,T] \times \Omega, \mathcal{B}([0,T]) \otimes \mathcal{F}_T, \mu_M\right).$$

Note that $\mathcal{L}^*_T$ is complete under the norm

$$[X]_T^2 \equiv E[\int_0^T X_t^2(\omega) d\langle M\rangle_t(\omega)].$$

Definition 6 (Simple Processes) *A process X is called simple if there exists a strictly increasing sequence of real numbers $\{t_n\}_{n=0}^\infty$ with $t_0 = 0$ and $\lim_{n\to\infty} t_n = \infty$, as well as a sequence of random variables $\{\zeta_n\}$ and a (non-random) constant $C < \infty$ with*

$$\sup_n \mid \zeta_n(\omega) \mid \leq C, \qquad \omega \in \Omega,$$

such that $\zeta_n(\omega)$ is $\mathcal{F}_{t_n}$-measurable and

$$X_t(\omega) = \zeta_0(\omega) X_{\{0\}}(t) + \sum_{i=0}^\infty \zeta_i(\omega) X_{(t_i, t_{i+1}]}(t).$$

This class is denoted by $\mathcal{L}_0$.

Note that if $X_t \in \mathcal{L}_0$ then it is progressively measurable and bounded and hence,

$$\mathcal{L}_0(M) \subseteq \mathcal{L}^*(M) \subseteq \mathcal{L}(M)$$

If $X_t \in \mathcal{L}_0$ define the integral

$$I_t(X) \equiv \sum_{i=0}^{n-1} \zeta_i(\omega) \left(M_{t_{i+1}}(\omega) - M_{t_i}(\omega)\right) + \zeta_n(\omega)(M_t(\omega) - M_{t_n}(\omega)), \quad t_n \leq t < t_{n+1},$$

or, equivalently,

$$I_t(X) \equiv \sum_{i=0}^{\infty} \zeta_i(\omega) \left(M_{t \wedge t_{i+1}}(\omega) - M_{t \wedge t_i}(\omega) \right).$$

The basic idea is that the integral is well defined for these simple random variables and then it can be extended to $\mathcal{L}^*(M)$ and $\mathcal{L}(M)$.

For a given martingale M, it follows that

$$\mathcal{L}_0(M) \subseteq \mathcal{L}^*(M) \subseteq \mathcal{L}(M).$$

Thus the set of simple processes that are square integrable is a subset of the progressively measurable processes that are square integrable and this, in turn, is a subset of the measurable processes that are square integrable with respect to the quadratic variation of a given martingale, $\langle M \rangle$.

What is the difference between $\mathcal{L}^*(M)$ and $\mathcal{L}(M)$? There are two important results.

Proposition 7 (Meyer (1968)) *If the stochastic process X is measurable and adapted to the filtration $\{\mathcal{F}_t\}$, then it has a progressively measurable modification.*

Recall that Y is a *modification* of X if, for all t, we have

$$P[X_t = Y_t] = 1.$$

Thus, if we can 'live' with the idea of modifications (not clear that we can, as it depends on the application) we can ignore the difference between $\mathcal{L}^*(M)$ and $\mathcal{L}(M)$.

However, for the applications that are relevant for this course, the following result is more useful

Proposition 8 *If the stochastic process X is adapted to the filtration $\{\mathcal{F}_t\}$ and every sample path is right-continuous or else every sample path is left continuous, then X is also progressively measurable with respect to the filtration $\{\mathcal{F}_t\}$.*

In this course we will be dealing (mostly? exclusively?) with either continuous or right continuous processes (need to check when we choose a process; i.e. when we endogenously define a process to see if this is too restrictive) this proposition shows that we can ignore the differences between $\mathcal{L}^*(M)$ and $\mathcal{L}(M)$.

2.1 Construction and Elementary Properties of Stochastic Integrals

Recall that we defined the stochastic integral of $X \in \mathcal{L}_0(M)$ with respect to the martingale M, $I_t(X)$, as the stochastic process given by the martingale transform, i.e.

$$I_t(X) \equiv \sum_{i=0}^{n-1} \zeta_i(\omega) \left(M_{t_{i+1}}(\omega) - M_{t_i}(\omega) \right) + \zeta_n(\omega)(M_t(\omega) - M_{t_n}(\omega)), \quad t_n \leq t < t_{n+1},$$

or, equivalently,

$$I_t(X) \equiv \sum_{i=0}^{\infty} \zeta_i(\omega) \left(M_{t\wedge t_{i+1}}(\omega) - M_{t\wedge t_i}(\omega) \right).$$

The basic idea of the stochastic integral is that it is easily defined for $X \in \mathcal{L}_0(M)$ and then it can be extended to $\mathcal{L}^*(M)$ and $\mathcal{L}(M)$ (which are the same spaces for right or left continuous processes).

We now prove some properties of the stochastic integral for simple processes.

Proposition 9 (Properties of the Stochastic Integral) *Let $X, Y \in \mathcal{L}_0(M)$ and $0 \leq s < t < \infty$ then*

1. $I_0(X) = 0, \qquad P - as.$

2. $E[I_t(X) \mid \mathcal{F}_s] = I_s(X) \qquad P - as.$

3. $E[I_t(X)^2] = E[\int_0^t X_u^2(\omega) d\langle M\rangle_u(\omega)] = [X]_t^2,$ *or* $\| I(X) \|_t = [X]_t$,

   where $\| I(X) \|_t \equiv (E[I_t(X)^2])^{1/2}$ *is the L^2 norm in the space of martingales previously defined.*

4. $\| I(X) \| = [X]$ *where*

$$\| I(X) \| = \| X \| \equiv \sum_{n=1}^{\infty} \frac{1 \wedge \| I(X) \|_n}{2^n},$$

with

$$\| X \|_t \equiv \sqrt{E[X_t^2]},$$

and

$$[X] = \sum_{n=1}^{\infty} \frac{1 \wedge [X]_n}{2^n},$$

with

$$[X]_t = \left(E[\int_0^t X_u^2(\omega) d\langle M \rangle_t(\omega)] \right)^{1/2}.$$

Thus, this result says that the distance from 0 to $I_t(X)$ (in the L^2 sense) coincides with the distance from 0 of X in the square integrable sense with respect to $\langle M \rangle$.

5. $E[(I_t(X) - I_s(X))^2 \mid \mathcal{F}_s] = E[\int_s^t X_u^2(\omega) d\langle M \rangle_t(\omega) \mid \mathcal{F}_s]$.

6. *For any $\alpha, \beta \in \mathcal{R}$ $I(\alpha X + \beta Y) = \alpha I(X) + \beta I(Y)$.*

Corollary 10 *$I_t(X) \in M_2^c$ (i.e. the martingale $I_t(X)$ has* ***continuous*** *sample paths) and*

$$\langle I(X) \rangle_t = \int_0^t X_u^2 d\langle M \rangle_u$$

Remark 11 *If M_t is a Brownian Motion or a Poisson process $\langle M \rangle_t = ct$ for some constant c. In this case we have that*

$$\langle I(X) \rangle_t = c \int_0^t X_u^2 du$$

Now we are almost ready to extend the definition of the integral to $\mathcal{L}(M)$. The notion is not standard. It is a probabilistic construct. The strategy consists of showing for $X \in \mathcal{L}(M)$, there exists a square integrable martingale that is a "natural" candidate to be $I(X)$.

It follows from part 4 of the previous Proposition that if there exists a sequence $X^{(n)} \in \mathcal{L}_0(M)$ such that, for any $X \in \mathcal{L}(M)$, $[X^{(n)} - X] \to 0$ as $n \to \infty$, then

$$\| I(X^{(n)}) - I(X^{(m)}) \| = \| I(X^{(n)} - X^{(m)}) \| = [X^{(n)} - X^{(m)}] \to 0$$

as $n, m \to \infty$.

In other words, $\{I(X^{(n)})\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathcal{M}_2^c$. Since this space is complete, then there exists a process $I(X) = \{I_t(X); 0 \leq t < \infty\}$ in $\mathcal{M}_2^c$ such that

$$\| I(X^{(n)}) - I(X) \| \to 0.$$

Note that, for $A \in \mathcal{F}_s$ the Proposition implies

$$\begin{aligned}
E[\mathcal{X}_A(I_t(X) - I_s(X))^2] &= \lim_{n\to\infty} E[\mathcal{X}_A(I_t(X^{(n)}) - I_s(X^{(n)}))^2] \\
&= \lim_{n\to\infty} E[\mathcal{X}_A \int_s^t (X_u^{(n)})^2 d\langle M\rangle_u] \\
&= E[\mathcal{X}_A \int_s^t X_u^2 d\langle M\rangle_u],
\end{aligned}$$

where the last inequality follows from $[X^{(n)} - X] \to 0$. Thus, the limiting process satisfies part 5 of the proposition. The other parts are verified as well. Thus we have a definition

Definition 12 *For $X \in \mathcal{L}^*(M)$, the stochastic integral of X with respect to the martingale $M \in \mathcal{M}_2^c$ is the unique square integrable martingale $I(X) = \{I_t(X); 0 \leq t < \infty\}$ which satisfies $\lim_{n\to\infty} \| I(X^{(n)}) - I(X) \| = 0$ for every sequence $\{X^{(n)}\}_{n=1}^{\infty} \subseteq \mathcal{L}_0(M)$ with $\lim_{n\to\infty}[X^{(n)} - X] = 0$. We write*

$$I_t(X) = \int_0^t X_s dM_s$$

Proposition 13 *The stochastic integral defined above satisfies parts 1-5 of the previous proposition and the corollary as well. Furthermore, for any two stopping times $S \leq T$ and any $t > 0$ we have*

$$E[I_{t\wedge T}(X) \mid \mathcal{F}_S] = I_{t\wedge S}(X), \qquad P - as,$$

and for $X, Y \in \mathcal{L}^*(M)$ we have

$$E[(I_{t\wedge T}(X) - I_{t\wedge S}(X))(I_{t\wedge T}(Y) - I_{t\wedge S}(Y)) \mid \mathcal{F}_S] = E[\int_{t\wedge S}^{t\wedge T} X_u Y_u d\langle M\rangle_u \mid \mathcal{F}_S].$$

Note that this holds for any pair of numbers $t > s$ as well.

3 Ito's Lemma

In this section we state (and provide a proof of) Ito's lemma.

Definition 14 (Continuous Semimartingale) *A continuous semimartingale $X = \{X_t, \{\mathcal{F}_t\}; 0 \le t < \infty\}$ is an adapted process which has decomposition*

$$X_t = X_0 + M_t + B_t, \qquad 0 \le t < \infty$$

where $M = \{M_t, \{\mathcal{F}_t\}; 0 \le t < \infty\} \in \mathcal{M}^{c,loc}$, and $B = \{B_t, \{\mathcal{F}_t\}; 0 \le t < \infty\}$ is the difference of continuous, nondecreasing, adapted processes $\{A_t^{\pm}, \{\mathcal{F}_t\}; 0 \le t < \infty\}$:

$$B_t = A_t^+ - A_t^-$$

Remark 15 *In this definition we assume that A is the minimal decomposition of B, i.e. A_t^+ is the positive variation of B and A_t^- is the negative variation of B.*

Ito's formula states that a "smooth function" of a continuous semimartingale is a continuous semimartingale and provides its decomposition.

Theorem 16 (Ito (1944)) *Let $f : \Re \to \Re$ be of class C^2 and let $X = \{X_t, \{\mathcal{F}_t\}; 0 \le t < \infty\}$ be a continuous semimartingale with decomposition as in the definition. Then, $P - as$*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) dM_s + \int_0^t f'(X_s) dB_s + \frac{1}{2}\int_0^t f''(X_s) d\langle M\rangle_s$$

Proof (from Karatzas and Shreve). *There are several parts to this proof. First, there is a localization argument and then a second order Taylor expansion.*

1. Localization Argument. *First, note that since the process B_t is of finite variation, then it can be written as*

$$B_t = A_t^+ - A_t^-.$$

Define $\check{B}_t$ by

$$\check{B}_t = A_t^+ + A_t^-.$$

Let the stopping time T_n be given by

$$T_n = \begin{cases} 0 & if \quad \mid X_0 \mid \geq n \\ \inf\{t : \mid M_t \mid \geq n, \ or \ \mid \check{B}_t \mid \geq n, \ or \ \langle M \rangle_t \geq n & if \quad \mid X_0 \mid < n \\ \infty & if \quad \mid X_0 \mid < n \ and \ \{...\} = \varnothing \end{cases}$$

The resulting sequence of stopping times is non-decreasing with $\lim_{n\to\infty} T_n = \infty$. Thus, if we prove the theorem for the stopped process $X_{t\wedge T_n}$, $M_{t\wedge T_n}$ then we have the desired result upon letting $n \to \infty$. Thus, we may assume that all functions are bounded by some constant K. It follows that $\mid X_t(\omega) \mid \leq 3K$. The values of f outside $[-3K, 3K]$ are irrelevant. Thus we assume that f has compact support and this means that f' and f'' are bounded.

2. Taylor Expansion. *Fix $t > 0$ and let $\Pi = \{0 = t_0 < t_1 < ... < t_m = t\}$ be a partition. A Taylor expansion yields*

$$\begin{aligned} f(X_t) - f(X_0) &= \sum_{k=1}^{m} \{f(X_{t_k}) - f(X_{t_{k-1}})\}, \\ &= \sum_{k=1}^{m} f'(X_{t_{k-1}})[X_{t_k} - X_{t_{k-1}}] + \frac{1}{2}\sum_{k=1}^{m} f''(\eta_k)[X_{t_k} - X_{t_{k-1}}]^2 \end{aligned}$$

where

$$\eta_k(\omega) = X_{t_{k-1}}(\omega) + \theta_k(\omega)[X_{t_k}(\omega) - X_{t_{k-1}}(\omega)]$$

can be chosen so that $f''(\eta_k)$ is measurable. Thus, we conclude that

$$f(X_t) - f(X_0) = J_1(\Pi) + J_2(\Pi) + \frac{1}{2}J_3(\Pi),$$

where

$$\begin{aligned}
J_1(\Pi) &\equiv \sum_{k=1}^{m} f'(X_{t_{k-1}})[B_{t_k} - B_{t_{k-1}}], \\
J_2(\Pi) &\equiv \sum_{k=1}^{m} f'(X_{t_{k-1}})[M_{t_k} - M_{t_{k-1}}], \\
J_3(\Pi) &\equiv \sum_{k=1}^{m} f''(\eta_k)[X_{t_k} - X_{t_{k-1}}]^2.
\end{aligned}$$

What do we know about these integrals? First, it is clear that $J_1(\Pi)$ converges to the Lebesgue integral $\int_0^t f'(X_s)dB_s$ as $\| \Pi \| \to 0$, where $\| \Pi \| = \max_{1\leq k\leq m} \mid t_k - t_{k-1} \mid$.

Since $Y_s(\omega) \equiv f'(X_s(\omega))$ is in $\mathcal{L}(M)$ we can approximate Y_s by the following simple process

$$Y_s^{\Pi}(\omega) = f'(X_0(\omega))\mathcal{X}_{\{0\}}(s) + \sum_{k=1}^{m} f'(X_{t_{k-1}}(\omega))\mathcal{X}_{\{(t_k - t_{k-1}]\}}(s).$$

Given the properties of the stochastic integral,

$$E[I_t^2(Y^{\Pi} - Y)] = E[\int_0^t \mid Y_s^{\Pi} - Y_s \mid^2 d\langle M\rangle_s.$$

However, it is clear that the right hand side of the previous expression converges to 0 $P-as$ as $\| \Pi \| \to 0$. This follows because $\mid Y_s^{\Pi} - Y_s \mid^2$ converges $P-as$ to 0 and the bounded convergence theorem implies that it also converges in the L^2 norm. Thus,

$$J_2(\Pi) = \int_0^t Y_s^{\Pi} dM_s \underset{\|\Pi\|\to 0}{\longrightarrow} \int_0^t Y_s dM_s \ .$$

3. The Quadratic Variation. The last term can be written as

$$J_3(\Pi) = J_4(\Pi) + J_5(\Pi) + J_6(\Pi)$$

where

$$\begin{aligned}
J_4(\Pi) &\equiv \sum_{k=1}^{m} f''(\eta_k)[B_{t_k} - B_{t_{k-1}}]^2, \\
J_5(\Pi) &\equiv 2\sum_{k=1}^{m} f''(\eta_k)[B_{t_k} - B_{t_{k-1}}][M_{t_k} - M_{t_{k-1}}], \\
J_6(\Pi) &\equiv \sum_{k=1}^{m} f''(\eta_k)[M_{t_k} - M_{t_{k-1}}]^2.
\end{aligned}$$

Because B_t has total variation bounded by K we have

$$\begin{aligned} &| \quad J_4(\Pi) \mid + \mid J_5(\Pi) \mid \\ \leq \; & 2K \parallel f' \parallel_\infty \left(\max_{1\leq k\leq m} \mid B_{t_k} - B_{t_{k-1}} \mid + \max_{1\leq k\leq m} \mid M_{t_k} - M_{t_{k-1}} \mid \right), \end{aligned}$$

and since the processes B_t and M_t are continuous, the last term converges to zero $P-as$ as $\parallel \Pi \parallel \to 0$. (Given the bounded convergence theorem, it also converges in $L^1(\Omega, \mathcal{F}, P)$.

We are now left with $J_6(\Pi)$. Define

$$J_6^*(\Pi) \equiv \sum_{k=1}^{m} f''(X_{t_{k-1}})[M_{t_k} - M_{t_{k-1}}]^2.$$

It follows that

$$\mid J_6^*(\Pi) - J_6(\Pi) \mid \leq \underbrace{V_t^{(2)}(\Pi)}_{quadratic\ var\ of\ M} \max_{1\leq k\leq m} \mid f''(X_{t_{k-1}}) - f''(\eta_k) \mid .$$

Next recall that "If $X \in \mathcal{M}_2$, $\mid X_s \mid \leq K$ for $s \in [0,t]$ then $E[V_t^{(2)}(\Pi)] \leq 6K^2$." In addition, the Cauchy-Schwartz inequality is just

$$E[\mid XY \mid] \leq E[\mid X \mid\mid Y \mid] \leq \sqrt{E[\mid X \mid^2]}\sqrt{E[\mid Y \mid^2]}.$$

Using these two results we get

$$E[\mid J_6^*(\Pi) - J_6(\Pi) \mid] \leq \sqrt{6K^2}\sqrt{E[(\max_{1\leq k\leq m} \mid f''(X_{t_{k-1}}) - f''(\eta_k) \mid)^2]}.$$

However, given that X has continuous sample paths and the bounded convergence theorem, the term

$$E[(\max_{1\leq k\leq m} \mid f''(X_{t_{k-1}}) - f''(\eta_k) \mid)^2] \to 0, \qquad as \;\parallel \Pi \parallel \to 0.$$

Thus, to show that

$$J_3(\Pi) \to \int_0^t f''(X_s) d\langle M \rangle$$

it suffices to show that

$$J_6^*(\Pi) \to J_7(\Pi) \equiv \sum_{k=1}^{m} f''(X_{t_{k-1}})[\langle M \rangle_{t_k} - \langle M \rangle_{t_{k-1}}].$$

We now do this

$$
\begin{aligned}
E \quad | \quad J_6^*(\Pi) - J_7(\Pi) \,|^2 &= E \,\left|\, \sum_{k=1}^{m} f''(X_{t_{k-1}})[(M_{t_k} - M_{t_{k-1}})^2 - (\langle M\rangle_{t_k} - \langle M\rangle_{t_{k-1}})] \,\right|^2 \\
&\leq E\left[\sum_{k=1}^{m}(f''(X_{t_{k-1}}))^2[(M_{t_k} - M_{t_{k-1}})^2 - (\langle M\rangle_{t_k} - \langle M\rangle_{t_{k-1}})]^2\right] \\
&= E[\sum_{k=1}^{m}(f''(X_{t_{k-1}}))^2[(M_{t_k} - M_{t_{k-1}})^4 \\
&\quad -2(M_{t_k} - M_{t_{k-1}})^2(\langle M\rangle_{t_k} - \langle M\rangle_{t_{k-1}}) + (\langle M\rangle_{t_k} - \langle M\rangle_{t_{k-1}})^2] \\
&\leq \max_{1\leq k\leq n}(f''(X_{t_{k-1}}))\{E[V_t^{(4)}(\Pi) \\
&\quad +2\max_{1\leq k\leq n}(\langle M\rangle_{t_k} - \langle M\rangle_{t_{k-1}})\sum_{k=1}^{m}(M_{t_k} - M_{t_{k-1}})^2 + \sum_{k=1}^{m}(\langle M\rangle_{t_k} - \langle M\rangle_{t_{k-1}})^2\} \\
&\leq \| f''(X_{t_{k-1}}) \|_\infty \{E[V_t^{(4)}(\Pi) + 2\max_{1\leq k\leq n}(\langle M\rangle_{t_k} - \langle M\rangle_{t_{k-1}})[\sum_{k=1}^{m}(M_{t_k} - M_{t_{k-1}})^2 + \sum_{k=1}^{m} \mid \langle M\rangle_{t_k} - \langle M\rangle \\
&= \| f''(X_{t_{k-1}}) \|_\infty \{E[\underbrace{V_t^{(4)}(\Pi)}_{<\infty \ \to 0 \ as \ \|\Pi\|\to 0} + \underbrace{2\max_{1\leq k\leq n}(\langle M\rangle_{t_k} - \langle M\rangle_{t_{k-1}})}_{\to 0 \ as \ \|\Pi\|\to 0 \ (continuity)}\underbrace{[V_t^{(2)}(\Pi) + \langle M\rangle_t]}_{bounded}\}.
\end{aligned}
$$

Thus, this argument shows that

$$
J_3(\Pi) \underset{\|\Pi\|\to 0}{\longrightarrow} \int_0^t f''(X_s)d\langle M\rangle_s
$$

where convergence is in the L^2 norm (and hence in L^1).

Final Touches. If $\{\Pi^{(n)}\}$ is a sequence of partitions of $[0,t]$ with $\| \Pi^{(n)} \| \to 0$, then for some subsequence

$$
\begin{aligned}
\lim_{k\to\infty} J_1(\Pi^{(n_k)}) &= \int_0^t f'(X_s)dB_s, \qquad P-as, \\
\lim_{k\to\infty} J_2(\Pi^{(n_k)}) &= \int_0^t f'(X_s)dM_s, \qquad P-as, \\
\lim_{k\to\infty} J_3(\Pi^{(n_k)}) &= \int_0^t f''(X_s)d\langle M\rangle_s, \qquad P-as.
\end{aligned}
$$

This completes the proof. ■

There are several comments that are relevant for this definition.

1. For fixed ω and $t > 0$ the function $X_s(\omega)$ is bounded for $0 \leq s \leq t$, so is $f'(X_s(\omega))$. Thus, it follows that $\int_0^t f'(X_s) dM_s$ is defined on this interval and the integral is a continuous local martingale.

2. The other two integrals are defined in the Lebesgue sense and so, as functions of the upper limits of integration, are of bounded variation.

3. From the previous two, it follows that $\{f(X_t), \{\mathcal{F}_t\}; 0 \leq t < \infty\}$ is a continuous semimartingale.

4. The statement in Ito's Theorem is often written as

$$df(X_t) = f'(X_t)dM_t + f'(X_t)dB_t + \frac{1}{2}f'(X_t)d\langle M\rangle_t.$$

Here is the multidimensional version of Ito's rule

Theorem 17 (Generalized Ito) *Let $\{M_t = (M_t^{(1)}, .., M_t^{(d)}), \{\mathcal{F}_t\}; 0 \leq t < \infty\}$ be a vector of local martingales in $\mathcal{M}^{c,loc}$, $\{B_t = (B_t^{(1)}, .., B_t^{(d)}), \{\mathcal{F}_t\}; 0 \leq t < \infty\}$ be a vector of adapted processes of bounded variation with $B_0 = 0$, and set $X_t = X_0 + M_t + B_t$ where X_0 is an $\mathcal{F}_0$-measurable vector in $\Re^d$. Let $f(t, X) : [0, \infty) \times \Re^d \to \Re$ be of class $C^{1,2}$. Then, $P - as$*

$$\begin{aligned} f(t, X_t) &= f(0, X_0) + \int_0^t \frac{\partial}{\partial t} f(s, X_s) d_s + \sum_{i=1}^d \int_0^t \frac{\partial}{\partial x_i} f(s, X_s) dB_s^{(i)} + \sum_{i=1}^d \int_0^t \frac{\partial}{\partial x_i} f(s, X_s) dM_s^{(i)} \\ &+ \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d \int_0^t \frac{\partial^2}{\partial x_i \partial x_j} f(s, X_s) d\langle M^{(i)}, M^{(j)}\rangle_s \end{aligned}$$

3.1 Examples

There are several interesting special cases of the above formula

1. If $d = 1$ then the previous formula just extends Ito's Theorem to the case in which the f function depends on time. In this case the (differential) version of the formula is

$$df(t, X_t) = \frac{\partial}{\partial t} f(t, X_t) dt + \frac{\partial}{\partial x} f(t, X_t) dM_t + \frac{\partial}{\partial x} f(t, X_t) dB_t + \frac{1}{2} \frac{\partial^2}{\partial x^2} f(t, X_t) d\langle M\rangle_t.$$

2. In the case in which the process M is a Brownian Motion, W, $d\langle M\rangle_t = dt$.The formula is

$$df(t,X_t) = [\frac{\partial}{\partial t}f(t,X_t) + \frac{1}{2}\frac{\partial^2}{\partial x^2}f(t,X_t)]dt + \frac{\partial}{\partial x}f(t,X_t)dW_t + \frac{\partial}{\partial x}f(t,X_t)dB_t.$$

3. If the semimartingales is given by a (μ,σ) Brownian Motion we have that

$$X_t = X_0 + \underbrace{\mu t}_{B_t} + \underbrace{\sigma W_t}_{M_t},$$

and Ito's formula is

$$df(t,X_t) = [\frac{\partial}{\partial t}f(t,X_t) + \frac{\partial}{\partial x}f(t,X_t)\mu + \frac{1}{2}\frac{\partial^2}{\partial x^2}f(t,X_t)\sigma^2]dt + \frac{\partial}{\partial x}f(t,X_t)\sigma dW_t,$$

where we have used the fact that if $M_t = \sigma W_t$, then $\langle M\rangle_t = \sigma^2\langle W\rangle_t = \sigma^2 t$.

4. Consider the *Geometric Brownian Motion* that satisfies the following differential equation

$$dX_t = \mu X_t dt + \sigma X_t dW_t.$$

Since this is just notation for the following stochastic integral

$$X_t = X_0 + \underbrace{\int_0^t \mu X_u du}_{B_t} + \underbrace{\int_0^t \sigma X_u dW_u}_{M_t},$$

it follows that $B_t \equiv \int_0^t \mu X_u du$ is a standard Lebesgue integral. Thus B_t is of bounded variation and, as such can be written as $B_t = A_t^+ - A_t^-$ (this is a property of functions that have bounded total variation) and is continuous. Since $M_t \equiv \int_0^t \sigma X_u dW_u$ is a stochastic integral, it is a martingale. In order to use Ito's formula we need to determine what is dB_t. Note that this is a regular Lebesgue integral and hence it is given by $dB_t = \mu X_t dt$. Thus, the differential version of Ito's lemma results in

$$df(t,X_t) = [\frac{\partial}{\partial t}f(t,X_t) + \frac{\partial}{\partial x}f(t,X_t)\mu X_t + \frac{1}{2}\frac{\partial^2}{\partial x^2}f(t,X_t)\sigma^2 X_t^2]dt + \frac{\partial}{\partial x}f(t,X_t)\sigma dW_t$$

5. The multi-dimensional case for diffusion processes. Let X_t satisfy

$$dX_t = \mu(t,\omega)dt + \sigma(t,\omega)dW_t$$

where

$$\begin{aligned}\mu &: [0,T]\times\Omega \to \Re^d, \\ \sigma &: [0,T]\times\Omega \to \Re^d \times \Re^m,\end{aligned}$$

where $\sigma(t,\omega)$ is a $d \times m$ matrix that is measurable with respect to $\{\mathcal{F}_t\}$ (non-anticipating), and W_t is an m-dimensional Wiener process. Let, $f(t,X)$ be as before. Then,

$$\begin{aligned} df(t,X_t) &= \frac{\partial}{\partial t}f(t,X_t) + \sum_{i=1}^{d}\frac{\partial}{\partial x_i}f(t,X_t)\mu_i(t,\omega) + \\ &\frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d}\frac{\partial^2}{\partial x_i \partial x_j}f(t,X_t)[\sigma(t)\sigma'(t)]_{ij} + \sum_{i=1}^{d}\frac{\partial}{\partial x_i}f(t,X_t)\sigma_i(t,\omega)dW_t. \end{aligned}$$

Note that the double summation can be written as

$$\sum_{i=1}^{d}\sum_{j=1}^{d}\frac{\partial^2}{\partial x_i \partial x_j}f(t,X_t)[\sigma(t)\sigma'(t)]_{ij} = tr\,(f_{xx}\sigma\sigma') = tr\,(\sigma\sigma' f_{xx})$$

3.2 A Digression: More General Processes

In this section, we state versions of Ito's Lemma that apply to processes that do not have continuous sample paths.

Theorem 18 (Ito's Formula) *Let $f : \Re \to \Re$ be of class C^2 and let $X = \{X_t, \{\mathcal{F}_t\}; 0 \le t < \infty\}$ be semimartingale (not necessarily with continuous sample paths). Then , $P - as$*

$$\begin{aligned} f(X_t) &= f(X_0) + \int_0^t f'(X_s)dX_s + \frac{1}{2}\int_0^t f''(X_s)d\langle M\rangle_s + \\ &\sum_{s\le t}\{[f(X_s) - f(X_{s^-}) - f'(X_{s^-})]\Delta X_s - \frac{1}{2}f''(X_{s^-})(\Delta X_s)^2]\} \end{aligned}$$

or, using the standard decomposition of a semimartingale,

$$\begin{aligned} f(X_t) &= f(X_0) + \int_0^t f'(X_s)dM_s + \int_0^t f'(X_s)dB_s + \frac{1}{2}\int_0^t f''(X_s)d\langle M\rangle_s + \\ &\sum_{s\le t}\{[f(X_s) - f(X_{s^-}) - f'(X_{s^-})]\Delta X_s - \frac{1}{2}f''(X_{s^-})(\Delta X_s)^2]\} \end{aligned}$$

Remark 19 *Since the discontinuities of $\langle X\rangle_s = \langle M\rangle_s$ satisfy $\Delta\langle M\rangle_s = (\Delta X_s)^2$, then if the process has continuous quadratic variation (e.g. Brownian Motion or Poisson), then Ito's formula reduces to*

$$\begin{aligned} f(X_t) &= f(X_0) + \int_0^t f'(X_s)dM_s + \int_0^t f'(X_s)dB_s + \frac{1}{2}\int_0^t f''(X_s)d\langle M\rangle_s + \\ &\sum_{s\le t}[f(X_s) - f(X_{s^-}) - f'(X_{s^-})]\Delta X_s \end{aligned}$$

In this case, it is simple to provide the appropriate version of the integration by parts formula. To be precise we have

Theorem 20 (Integration by Parts I) *Let X and Y be semimartingales, then*

$$X_tY_t = X_0Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s + \langle X, Y\rangle_t$$

.Thus, we get the following integration by parts formula

$$\int_0^t X_s dY_s = X_tY_t - X_0Y_0 - \int_0^t Y_s dX_s - \langle X, Y\rangle_t.$$

Remark 21 *If one of the processes has continuous sample paths and the other is of finite variation, then $\langle X, Y\rangle_t = 0$. If the processes have no simultaneous jumps, it is also the case that $\langle X, Y\rangle_t = 0$.*

4 Girsanov's Theorem

- Consider a (μ, σ) Brownian Motion. In many applications, it turns out to be a lot easier to work with a Brownian Motion with no drift. There are two ways of doing this: change the stochastic process (i.e. work with $W_t - \mu t$) or change the probability measure so that the original process has no drift under the new probability measure. If the new probability measure is absolutely continuous with respect to the old, then almost sure statements are similar under either measure.

- Girsanov's theorem shows how to construct a new probability measure to modify the drift of a stochastic process without changing its instantaneous diffusion.

Theorem 22 (Lévy Characterization of Brownian Motion) *Let $X_t = (X_t^1, ..., X_t^n)$ be a continuous stochastic process on $(\Omega, \mathcal{F}, P)$. Then the following are equivalent:*

1. *X_t is Brownian Motion with respect to P.*

   (a) *X_t is a martingale with respect to P (and with respect to the filtration generated by X_t)*

   (b) *$X_t^i X_t^j - \delta_{ij} t$ is a martingale with respect to P.[Here $\delta_{ij} = 1$ if $i = j$, and 0 otherwise]*

Remark 23 *Condition 2 can be replaced by: The cross-variation process satisfies $\langle X^i, X^j \rangle_t = \delta_{ij} t$, where*

$$\langle X^i X^j \rangle_t = \frac{1}{2}\{\langle X^i + X^j, X^i + X^j \rangle_t - \langle X^i - X^j, X^i - X^j \rangle_t,$$

where, as before, $\langle X, X \rangle_t$ is the quadratic variation of the process, i.e.

$$\langle X, X \rangle_t = \lim_{\Delta t_k \to 0} \sum_{t_k \leq t} \mid X_{t_{k+1}} - X_{t_k} \mid^2 .$$

- To prove the result we first need to prove the appropriate version of Bayes' Theorem.

Lemma 24 (Bayes Rule) *Let μ and υ be two probability measures on a measurable space $(\Omega, \mathcal{F})$ such that*

$$\upsilon(d\omega) = f(\omega)\mu(d\omega)$$

for some nonzero $f \in L^1((\Omega, \mathcal{F}, \mu)$. *Let* X *be a random variable on* $(\Omega, \mathcal{F})$ *such that*

$$E_\upsilon[X] = \int_\Omega \mid X(\omega) \mid \upsilon(d\omega) = \int_\Omega \mid X(\omega) \mid f(\omega)\mu(d\omega) < \infty.$$

Let $\mathcal{H}$ *be a sigma algebra,* $\mathcal{H} \subset \mathcal{F}$. *Then*

$$E_\upsilon[X \mid \mathcal{H}] \times E_\mu[f \mid \mathcal{H}] = E_\mu[fX \mid \mathcal{H}],$$

or

$$E_\upsilon[X \mid \mathcal{H}] = \frac{E_\mu[fX \mid \mathcal{H}]}{E_\mu[f \mid \mathcal{H}]}$$

Proof. Recall the definition of conditional expectation. Given $\mathcal{H}$, the conditional expectation of X, $E_\upsilon[X \mid \mathcal{H}]$ is a random variable such that for all $H \in \mathcal{H}$,

$$\int_H E_\upsilon[X \mid \mathcal{H}](\omega)\upsilon(d\omega) = \int_H X(\omega)\upsilon(d\omega).$$

Let $H \in \mathcal{H}$,

$$\begin{aligned} \int_H E_\upsilon[X \quad | \quad \mathcal{H}](\omega)\upsilon(d\omega) = \int_H X(\omega)\upsilon(d\omega) = \\ \int_H X(\omega)f(\omega)\mu(d\omega) \quad &= \quad \int_H E_\mu[fX \mid \mathcal{H}](\omega)\mu(d\omega). \end{aligned}$$

Thus, for all $H \in \mathcal{H}$,

$$\int_H E_\upsilon[X \mid \mathcal{H}](\omega)\upsilon(d\omega) = \int_H E_\mu[fX \mid \mathcal{H}](\omega)\mu(d\omega).$$

On the other hand,

$$\begin{aligned} &\int_H E_\upsilon[X \quad | \quad \mathcal{H}](\omega)\upsilon(d\omega) = \int_H E_\upsilon[X \mid \mathcal{H}](\omega)f(\omega)\mu(d\omega) = \\ &E_\mu[E_\upsilon[X \quad | \quad \mathcal{H}]f\mathcal{X}_H] = E_\mu\{E_\mu[E_\upsilon[X \mid \mathcal{H}]f\mathcal{X}_H] \mid \mathcal{H}]\} \\ &\qquad = \quad E_\mu\{\mathcal{X}_H E_\upsilon[X \mid \mathcal{H}]E_\mu[f \mid \mathcal{H}]\} = \int_H E_\upsilon[X \mid \mathcal{H}]E_\mu[f \mid \mathcal{H}]\mu(d\omega). \end{aligned}$$

Taking both sets of equalities together, we have shown that for all $H \in \mathcal{H}$,

$$\int_H E_\mu[fX \mid \mathcal{H}](\omega)\mu(d\omega) = \int_H E_\upsilon[X \mid \mathcal{H}]E_\mu[f \mid \mathcal{H}]\mu(d\omega).$$

Since this holds for all $H \in \mathcal{H}$, it follows that

$$E_\mu[fX \mid \mathcal{H}] = E_\upsilon[X \mid \mathcal{H}]E_\mu[f \mid \mathcal{H}].$$

■

Remark 25 *Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ be a filtered probability space. Fix $T > 0$ and let Q be another probability measure on $\mathcal{F}_T$ then Q is absolutely continuous with respect to P restricted to $\mathcal{F}_T$, denoted $Q << P \mid_{\mathcal{F}_T}$ if and only if $P(H) = 0 \rightarrow Q(H) = 0$, for all $H \in \mathcal{F}_T$. The Radon-Nikodym theorem (reference or state) shows that this holds if and only if there exists an $\mathcal{F}_T$-measurable function $Z_T(\omega)$, with $Z_T(\omega) \geq 0$ and such that$_t$dp*

$$Q(d\omega) = Z_T(\omega) P(d\omega).$$

Lemma 26 *Suppose that $Q << P \mid_{\mathcal{F}_T}$ with $dQ/dP = Z_T$. Then,*

$$Q \mid_{\mathcal{F}_t} << P \mid_{\mathcal{F}_t}, \ \forall t \in [0, T]$$

and if we define

$$Z_t = \frac{dQ \mid_{\mathcal{F}_t}}{dP \mid_{\mathcal{F}_t}}$$

then $\{Z_t\}$ is a martingale with respect to $\{\mathcal{F}_t\}$ and P.

Proof. Since $Q << P \mid_{\mathcal{F}_T}$ and $\mathcal{F}_t \subset \mathcal{F}_T$, it is obvious that $Q \mid_{\mathcal{F}_t} << P \mid_{\mathcal{F}_t}, \ \forall t \in [0, T]$. Choose $F \in \mathcal{F}_t$. Then

$$\begin{aligned} E_P[\mathcal{X}_F E_P[Z_T \quad | \quad \mathcal{F}_t]] &= E_P[E_P[\mathcal{X}_F Z_T \mid \mathcal{F}_t]] \\ &= E_P[\mathcal{X}_F Z_T] = E_Q[\mathcal{X}_F] = Q(F) = E_P[\mathcal{X}_F Z_t], \end{aligned}$$

since $Q \mid_{\mathcal{F}_t} << P \mid_{\mathcal{F}_t}$, $F \in \mathcal{F}_t$, and $dQ \mid_{\mathcal{F}_t} / dP \mid_{\mathcal{F}_t} = Z_t$. Thus, we have shown that

$$E_P[\mathcal{X}_F E_P[Z_T \mid \mathcal{F}_t]] = E_P[\mathcal{X}_F Z_t],$$

or that for all $F \in \mathcal{F}_t$

$$\int_F E_P[Z_T \mid \mathcal{F}_t](\omega) P(d\omega) = \int_F Z_t(\omega) P(d\omega),$$

which is the definition of conditional expectation. Thus

$$Z_t = E_P[Z_T \mid \mathcal{F}_t]$$

■

Theorem 27 (Girsanov I) *Let Y_t be an Ito process satisfying,*

$$dY_t = a(t,\omega)dt + dW_t, \qquad t \le T,\ Y_0 = 0,$$

where $T \le \infty$ is a given constant and W_t is a given n-dimensional Brownian Motion. Let $\{\mathcal{F}_t^{(n)}\}$ be the filtration generated by the Brownian Motion. Let

$$M_t = \exp\{-\int_0^t a(s,\omega)dW_s - \frac{1}{2}\int_0^t a^2(s,\omega)ds\}.$$

Assume that M_t is a martingale with respect to $\{\mathcal{F}_t^{(n)}\}$ and P. Define Q on $\mathcal{F}_T^{(n)}$ by

$$Q(d\omega) = M_T(\omega)P(d\omega).$$

Then

1. *Q is a probability measure on $\mathcal{F}_T^{(n)}$.*

2. *Y_t is a n-dimensional Brownian Motion with respect to Q, for $0 \le t \le T$.*

Remark 28

1. *How can one guarantee that M_t is a martingale? A sufficient condition is the **Novikov condition** which is*

$$E_P[\exp\{\frac{1}{2}\int_0^t a^2(s,\omega)ds\}] < \infty.$$

2. *What is $a^2(s,\omega)$ since $a(s,\omega) \in \mathbb{R}^n$? By convention this is just the Euclidean norm; i.e.*

$$a^2(s,\omega) = \sum_{i=1}^n a_i^2(s,\omega).$$

3. *Since M_t is a martingale then*

$$M_t = E_P[M_T \mid \mathcal{F}_t^{(n)}]$$

which iff

$$M_t dP = M_T dP \qquad \text{on } \mathcal{F}_t^{(n)}.$$

4. *Girsanov I says that for all Borel sets* $(F_1, ..., F_k)$ *and for all* $t_1, t_2, ..., t_k \leq T$, *then*

$$Q[Y_{t_1} \in F_1, ..., Y_{t_k} \in F_k] = P[W_{t_1} \in F_1, ..., W_{t_k} \in F_k],$$

since under the respective measures both processes are Brownian Motions. Moreover, absolute continuity implies that

$$P[Y_{t_1} \in F_1, ..., Y_{t_k} \in F_k] > 0 \rightarrow Q[Y_{t_1} \in F_1, ..., Y_{t_k} \in F_k] > 0.$$

Example 29 *Let* Y_t *be a* $(\mu, 1)$ *Brownian Motion. Then,*

$$M_t = \exp\{-\mu W_t - \frac{1}{2}\mu^2 t\},$$

which satisfies the Novikov condition. Thus, under the measure Q *given by*

$$Q(d\omega) = \exp\{-\mu W_T(\omega) - \frac{1}{2}\mu^2 T\} P(d\omega),$$

Y_t *is a* $(0, 1)$ *Brownian Motion.*

Theorem 30 (Girsanov II) *Let* Y_t *be an Ito process satisfying,*

$$dY_t = \beta(t, \omega)dt + \theta(t, \omega)dW_t, \qquad t \leq T,$$

where $T \leq \infty$ *is a given constant and* W_t *is a given m-dimensional Brownian Motion. Let* $\{\mathcal{F}_t^{(m)}\}$ *be the filtration generated by the Brownian Motion. In this setting*

$$\begin{aligned} \beta(t, \omega) &\in \mathbb{R}^n, \\ \theta(t, \omega) &\in \mathbb{R}^{n \times m}. \end{aligned}$$

Suppose that there exists a process $u(t, \omega)$ *such that*

$$P[\int_0^T u^2(s, \omega)ds < \infty] = 1,$$

and a process $\alpha(t, \omega)$ *also satisfying*

$$P[\int_0^T \alpha^2(s, \omega)ds < \infty] = 1,$$

such that

$$\theta(t,\omega)u(t,\omega) = \beta(t,\omega) - \alpha(t,\omega).$$

Let

$$M_t = \exp\{-\int_0^t u(s,\omega)dW_s - \frac{1}{2}\int_0^t u^2(s,\omega)ds.$$

Assume that M_t is a martingale with respect to $\{\mathcal{F}_t^{(m)}\}$ and P. Define Q on $\mathcal{F}_T^{(m)}$ by

$$Q(d\omega) = M_T(\omega)P(d\omega).$$

Then

1. *Q is a probability measure on $\mathcal{F}_T^{(m)}$.*

2. *The process*

   $$\tilde{W}_t = \int_0^t u(s,\omega)ds + W_t, \qquad t \le T$$

   is a Brownian Motion with respect to Q. This condition is sometimes expressed as

   $$d\tilde{W}_t = u(t,\omega)dt + dW_t, \qquad t \le T.$$

3. *In terms of $\tilde{W}_t$ the process Y_t has the representation*

   $$dY_t = \alpha(t,\omega)dt + \theta(t,\omega)d\tilde{W}_t$$

Remark 31

1. *If $\alpha(t,\omega)$ is chosen to be zero, then*

   $$\theta(t,\omega)u(t,\omega) = \beta(t,\omega).$$

   If the $n \times m$ matrix has rank n, then

   $$u(t,\omega) = [\theta^T(t,\omega)\theta(t,\omega)]^{-1}\beta(t,\omega),$$

   which is the projection of $u(t,\omega)$ on the space generated by $\beta(t,\omega)$. Formally, it is the coefficient u of a regression

   $$\beta_i(t,\omega) = \theta_i(t,\omega)u(t,\omega) + noise.$$

2. *Since* $d\tilde{W}_t = u(t,\omega)dt + dW_t$ *then, given that*

$$dY_t = \beta(t,\omega)dt + \theta(t,\omega)dW_t, \qquad t \leq T,\ Y_0 = 0,$$

it follows that

$$\begin{aligned}
dY_t &= \beta(t,\omega)dt + \theta(t,\omega)[d\tilde{W}_t - u(t,\omega)dt], \\
dY_t &= \underbrace{[\beta(t,\omega) - \theta(t,\omega)u(t,\omega)]}_{\alpha(t,\omega)}dt + \theta(t,\omega)d\tilde{W}_t, \\
dY_t &= \alpha(t,\omega)dt + \theta(t,\omega)d\tilde{W}_t.
\end{aligned}$$

Example 32 *Assume that the price of an asset evolves according to*

$$dS_t = \mu S_t dt + \sigma S_t dW_t.$$

Thus, with this notation:

$$\begin{aligned}
\beta(t,\omega) &= \mu S_t(\omega), \\
\theta(t,\omega) &= \sigma S_t(\omega).
\end{aligned}$$

Then u *and* α *satisfy*

$$\sigma S_t(\omega)u(t,\omega) = \mu S_t(\omega) - \alpha(t,\omega).$$

Let's choose $\alpha(t,\omega) = 0$. *This implies that*

$$u = \frac{\mu}{\sigma}.$$

Then, under Q

$$dS_t = \sigma S_t d\tilde{W}_t,$$

or, more precisely,

$$S_t = S_0 + \underbrace{\int_0^t \sigma S_s d\tilde{W}_s}_{martingale}.$$

Under Q *asset prices are martingales.*

Example 33 *Missing*

Theorem 34 (Girsanov III) *(Diffusion Version). Let X_t^x be a stochastic process that has initial value x. Assume that $X_t^x \in \mathbb{R}^n$ and $Y_t^x \in \mathbb{R}^n$ are an Ito diffusion and an Ito process respectively*

$$\begin{aligned} dX_t &= b(X_t)dt + \sigma(X_t)dW_t, \qquad X_0 = x, \\ dY_t &= [\gamma(t,\omega) + b(Y_t)]dt + \sigma(Y_t)dW_t, \qquad Y_0 = x, \end{aligned}$$

where

$$\begin{aligned} b &: \mathbb{R}^n \to \mathbb{R}^n \\ \sigma &: \mathbb{R}^n \to \mathbb{R}^{n\times m}, \\ \gamma &: \mathbb{R}^{n+1} \to \mathbb{R}^n, \end{aligned}$$

and such that

$$P[\int_0^T \gamma^2(s,\omega)ds < \infty] = 1.$$

Suppose that there exists a process $u(t,\omega)$ satisfying $P[\int_0^T u^2(s,\omega)ds < \infty] = 1$ such that

$$\sigma(Y_t)u(t,\omega) = \gamma(t,\omega).$$

Define

$$M_t = \exp\{-\int_0^t u(s,\omega)dW_s - \frac{1}{2}\int_0^t u^2(s,\omega)ds.$$

Assume that M_t is a martingale with respect to $\{\mathcal{F}_t^{(m)}\}$ and P. Define Q on $\mathcal{F}_T^{(m)}$ by

$$Q(d\omega) = M_T(\omega)P(d\omega),$$

and

$$d\tilde{W}_t = u(t,\omega)dt + dW_t, \qquad t \le T.$$

Then,

1. *Q is a probability measure on $\mathcal{F}_T^{(m)}$.*

2. The process

$$dY_t = b(Y_t)dt + \sigma(Y_t)d\tilde{W}_t, \qquad t \leq T$$

is such that the Q-law of Y_t is the same as the P-law of X_t.[Both processes have the same distribution.]

5 Martingale Representation Theorem

The definition of a stochastic integral shows that it is a martingale. Thus, every stochastic integral defines a martingale. The converse is also true.

Theorem 35 (Martingale Representation) *Let W_t be an n-dimensional Brownian Motion. Suppose that M_t is a square integrable martingale adapted to the filtration generated by W_t, $\{\mathcal{F}_t^{(n)}\}$. Then, there exists a unique stochastic process $g(t,\omega)$ such that $g(t,\omega)$ is $\{\mathcal{F}_t^{(n)}\}$ adapted and an element of $\mathcal{L}_T^*$, i.e. the class of processes for which*

$$E[\int_0^T g^2(t,\omega)dt] < \infty$$

such that

$$M_t(\omega) = E[M_0] + \int_0^t g(s,\omega)dW_s$$