Stochastic Differential Equations and the Feynman-Kac Theorem

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Acknowledgement 1 *These notes contain some mathematical results taken from several different references (missing)*

1 Existence of Solutions to Stochastic Differential Equations

In this section we sufficient conditions for the existence of 'reasonable' solutions to stochastic differential equations following Karatzas and Shreve (Brownian Motion and Stochastic Calculus, Chapter 5).

1.1 Strong Solutions

Consider the following SDE

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t \tag{1}$$

where W_t is a standard Brownian Motion

Definition 2 (Strong Solution) *Given a probability space* $(\Omega, \mathcal{F}, P)$ *and given a (fixed) standard Brownian motion* $W = \{W_t, \{\mathcal{F}_t\}; 0 \leq t < \infty\}$ *and an initial condition* ξ*, a* ***strong solution*** *to (1) is a process* $X = \{X_t, \{\mathcal{F}_t\}; 0 \leq t < \infty\}$

1. *With continuous sample paths,*
2. *adapted to* $\{\mathcal{F}_t\}$*,*
3. $P[X_0 = \xi] = 1$*,*
4. $P[\int_0^t \{\mid \mu_i(s, X_s) \mid + \sigma_{ij}^2(s, X_s)\} ds < \infty] = 1$*, for all* $1 \leq i \leq d,\ 1 \leq j \leq r,\ 0 \leq t < \infty$*,*
5. *the integral version of (1)*
$$X_t = X_0 + \int_0^t \mu(s, X_s) ds + \int_0^t \sigma(s, X_s) dW_s, \qquad 0 \leq t < \infty \tag{2}$$
or
$$X_t^i = X_0^i + \int_0^t \mu_i(s, X_s) ds + \sum_{j=1}^r \int_0^t \sigma_{ij}(s, X_s) dW_s^j, \qquad 0 \leq t < \infty \tag{3}$$
hold almost surely.

There are two issues that are important to determine. First, if a solution exists, is it unique? Second, does a solution exist? Let's consider uniqueness first.

Definition 3 (Strong Uniqueness) *Let* $\mu(t, x)$ *and* $\sigma(t, x)$ *be given. Let* $(\Omega, \mathcal{F}, P)$ *and a (fixed) standard Brownian motion* $W = \{W_t, \{\mathcal{F}_t\}; 0 \leq t < \infty\}$ *be given as well. Let* ξ *be an independent d-dimensional vector. If* X *and* $\tilde{X}$ *are two strong solutions of (2) relative to* W *and given* ξ*, then if* $P[X_t = \tilde{X}_t,\ 0 \leq t < \infty] = 1$*, then the pair displays the* ***strong uniqueness*** *property.*

Theorem 4 (Karatzas and Shreve p 287)) *Suppose that* $\mu(t, x)$ *and* $\sigma(t, x)$ *satisfy the following locally Lipschitz condition. For every integer* $n \geq 1$*, there exists a constant* K_n *such that for all* $t \geq 0$*,* $\| x \| \leq n$*, and* $\| y \| \leq n$*,*

$$\| \mu(t, x) - \mu(t, y) \| + \| \sigma(t, x) - \sigma(t, y) \| \leq K_n \| x - y \| .$$

Then strong uniqueness holds for (2) [or (3)]

Since $\sigma(t,x)$ is an $d \times r$ matrix, we need to define the norm. In this case, we use the usual norm

$$\| \sigma(t,x) \|^2 = \sum_{i=1}^{d} \sum_{j=1}^{r} \sigma_{ij}^2(t,x).$$

Here is the basic theorem

Theorem 5 (Existence of Strong Solutions) *Assume that $\mu(t,x)$ and $\sigma(t,x)$ satisfy the following global Lipschitz and growth conditions*

$$\| \mu(t,x) - \mu(t,y) \| + \| \sigma(t,x) - \sigma(t,y) \| \leq K \| x - y \|, \tag{4}$$

$$\| \mu(t,x) \|^2 + \| \sigma(t,x) \|^2 \leq K^2(1 + \| x \|^2) \tag{5}$$

for all $0 \leq t < \infty$, $x \in \mathbb{R}^d$, $y \in \mathbb{R}^d$ and $0 < K < \infty$. Then, on a given probability space $(\Omega, \mathcal{F}, P)$, let $W = \{W_t, \{\mathcal{F}_t\}; 0 \leq t < \infty\}$ be a Brownian motion and let ξ be an independent d-dimensional vector such that $E[\| \xi \|^2] < \infty$. Then, there exists a continuous adapted process $X = \{X_t, \{\mathcal{F}_t\}; 0 \leq t < \infty\}$ which is a strong solution to

$$X_t = X_0 + \int_0^t \mu(s, X_s) ds + \int_0^t \sigma(s, X_s) dW_s, \qquad 0 \leq t < \infty.$$

Moreover, X is square integrable, and for a given (K,T) pair there exists a C such that

$$E[\| X_t \|^2] \leq C(1 + E[\| \xi \|^2]) e^{Ct}, \qquad 0 \leq t \leq T$$

1.1.1 One Dimensional Case

In the one dimensional case, it is possible to weaken the conditions. There is a number of alternative conditions that will work. One version of (4) is

$$\begin{aligned} &\| \quad \mu(t,x) - \mu(t,y) \| + \leq K \| x - y \|, \\ &\| \quad \sigma(t,x) - \sigma(t,y) \| \leq K h(\| x - y \|) \end{aligned}$$

where $h(0) = 0$ and

$$\int_{(0,\varepsilon)} \frac{1}{h^2(u)} du = \infty.$$

1.2 Weak Solutions

An alternative notion of a solution to a stochastic differential equation is that of a weak solution. Basically, the idea is that one gets to pick both the probability space as well as the Brownian motion (and the associated filtration) that determine the solution. This is formalized in the following definition

Definition 6 (Weak Solution) *A weak solution to equation (2) is a triple* $[(X,W),(\Omega,\mathcal{F},P),\{\mathcal{F}_t\}]$ *where*

1. $(\Omega,\mathcal{F},P)$ *is a probability space and* $\{\mathcal{F}_t\}$ *is a filtration satisfying the usual conditions.*

2. $W=\{W_t,\{\mathcal{F}_t\};0\le t<\infty\}$ *is an* r-dim*ensional Brownian motion and* $X=\{X_t,\{\mathcal{F}_t\};0\le t<\infty\}$ *is a continuous adapted* $\mathbb{R}^d$ *valued process.*

3. $P[\int_0^t\{\mid\mu_i(s,X_s)\mid+\sigma_{ij}^2(s,X_s)\}ds<\infty]=1,$ *for all* $1\le i\le d,\ 1\le j\le r,\ 0\le t<\infty,$

4. $$X_t=X_0+\int_0^t\mu(s,X_s)ds+\int_0^t\sigma(s,X_s)dW_s,\qquad 0\le t<\infty$$

   or

   $$X_t^i=X_0^i+\int_0^t\mu_i(s,X_s)ds+\sum_{j=1}^r\int_0^t\sigma_{ij}(s,X_s)dW_s^j,\qquad 0\le t<\infty$$

   hold almost surely.

Associated with the notion of weak solutions are two notions of uniqueness. The first parallels the definition that we used in the case of strong solution, while the second is, basically, uniqueness in distribution, which is a very "weak" notion. The formal definitions are:

Definition 7 (Pathwise Uniqueness) *Suppose that whenever* $[(X, W), (\Omega, \mathcal{F}, P), \{\mathcal{F}_t\}]$ *and* $[(\tilde{X}, W), (\Omega, \mathcal{F}, P), \{\tilde{\mathcal{F}}_t\}]$ *are weak solutions to (1) with common Brownian motion* W *(relative to possibly different filtrations) on a common probability space* $(\Omega, \mathcal{F}, P)$ *with common initial value, i.e.* $P[X_0 = \tilde{X}_0] = 1$*, the two processes* X *and* $\tilde{X}$ *are* indistinguishable: $P[X_t = \tilde{X}_t; 0 \leq t < \infty] = 1$. *Then we say that pathwise uniqueness holds for equation (1).*

Note that this definition is very similar to that of strong uniqueness. In fact, the only difference is that the notion of strong uniqueness is associated to the concept of a strong solution to (1) and, as part of the definition of strong solution, we take the filtration $\{\mathcal{F}_t\}$ as given (and, of course, unique). On the other hand, for a weak solution the filtrations need not coincide. However, for many results, the filtration is not used at all and, hence, the two notions should coincide.

An alternative uniqueness in distribution concept is the following:

Definition 8 (Uniqueness in Distribution) *We say that uniqueness in distribution holds for equation (1) if for any two weak solutions* $[(X, W), (\Omega, \mathcal{F}, P), \{\mathcal{F}_t\}]$ *and* $[(\tilde{X}, W), (\Omega, \mathcal{F}, P), \{\tilde{\mathcal{F}}_t\}]$ *with the same initial distribution, i.e.*

$$P[X_0 \in A] = P[\tilde{X}_0 \in A] = 1; \qquad A \in \mathcal{B}(\mathbb{R}^d),$$

the two processes have the same distribution.

Of course existence of a weak solution does not imply existence of a strong solution (since in the weak case we feel free to construct a Brownian motion so that the SDE holds)

There are a number of different theorems —associated with different conditions— that establish existence of weak solutions. Here is an incomplete list.

The first, assumes that the diffusion coefficient is just the identity matrix. Formally,

Proposition 9 *Consider the stochastic differential equation*

$$dX_t = \mu(t, X_t)dt + dW_t, \qquad 0 \leq t \leq T \tag{6}$$

where T is a fixed positive number, W is a d-dimensional Brownian motion, and $\mu(t,x)$ is a Borel measurable function which satisfies

$$\| \mu(t,x) \| \leq K(1 + \| x \|^2)$$

for some positive constant K. For any initial distribution λ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ equation (6) has a weak solution. Moreover, uniqueness in distribution holds for equation (6) if

$$P^{(i)}[\int_0^T \| \mu(t, X_t^{(i)}) \|^2 \, dt < \infty] = 1, \qquad i = 1, 2,$$

where the $P^{(i)}$ are the probability measures associated with each of the solutions $X^{(i)}$.

With some care (see Karatzas and Shreve (page 303) this result can be extended to all T.

In the time homogeneous case (however, nothing prevents one of the elements of the vector X_t to be just t! if some care is exercised verifying the properties of $\mu(x)$ and $\sigma(x)$) we have

Theorem 10 (Skorohod(1965), Stroock and Varadhan (1969)) *Consider the stochastic differential equation*

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t \tag{7}$$

where the coefficients μ_i and $\sigma_{ij} : \mathbb{R}^d \to \mathbb{R}$ are bounded and continuous functions. Then corresponding to every initial distribtuion λ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ such that

$$\int_{\mathbb{R}^d} \| x \|^{2m} \lambda(dx) < \infty, \; for \; some \; m > 1,$$

there exists a weak solution to (7)

1.2.1 One Dimensional Case

As in the case of strong solutions, the one dimensional case —which has quite a few applications in economics— requires weaker conditions. In this section we consider the one dimensional version of equation (7), namely

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t. \tag{8}$$

We can now give a definition of a weak solution restricted to I.

Definition 11 (Weak Solution in an Interval I) *A weak solution in an interval I of equation (8) is a triple $[(X,W),(\Omega,\mathcal{F},P),\{\mathcal{F}_t\}]$, where*

1. *$(\Omega,\mathcal{F},P)$ is a probability space, and $\{\mathcal{F}_t\}$ is a filtration of sub σ-fields of $\mathcal{F}$ satisfying the usual conditions,*

2. *$X = \{X_t,\{\mathcal{F}_t\}; 0 \leq t < \infty\}$ is a continuous, adapted, $[\ell,r]-$valued process, with $X_0 \in I$ almost surely, and $W = \{W_t,\{\mathcal{F}_t\}; 0 \leq t < \infty\}$ is an one-dimensional Brownian motion,*

3. *with $\{\ell_n\}$ and $\{r_n\}$ strictly monotone sequences satisfying $\ell < \ell_n < r_n < r$, and $lim_{n\to\infty}\ell_n = \ell$, and $lim_{n\to\infty}r_n = r$ and*

$$S_n \equiv \inf\{t \geq 0; X_t \notin (\ell_n, r_n)\},$$

we have

$$P[\int_0^{t\wedge S_n} \{\mid \mu(X_s) \mid + \sigma^2(X_s)\}ds < \infty] = 1,\ \ 0 \leq t < \infty,$$

and

$$\begin{aligned} P[X_{t\wedge S_n} &= X_0 + \int_0^t \mu(X_s)\mathcal{X}_{\{s\leq S_n\}}(s)ds \\ + \int_0^t \sigma(X_s)\mathcal{X}_{\{s\leq S_n\}}(s)dW_s;\ 0 &\leq t < \infty] = 1 \end{aligned}$$

hold.

4. *We refer to*

$$S = \lim_{n\to\infty} S_n = \inf\{t \geq 0; X_t \notin (\ell, r)\}$$

as the exit time *from I. The assumption $X_0 \in I$ guarantees that $P[S > 0] = 1$*

2 The Feynman-Kac Theorem I: Random Terminal Time

In this section we present a result that provides a probabilistic interpretation of the solution to some partial differential equations which arise in some economic problems. In general, the notes follow E. B. Dynkin, Markov Processes I and II (Springer Verlag, 1965).

Definition 12 (Canonical Diffusion Process) *A process X is a canonical diffusion if it satisfies*

1.
$$X_t(\omega) = X_0(\omega) + \int_0^t \mu(X_s(\omega))ds + \int_0^t \sigma(X_s(\omega))dW_s$$

   where X_t is a d-dimensional process, and W is a r-dimensional Brownian motion.

2. *Let the matrix*

$$a(x) \equiv \sigma(x)\sigma^T(x)$$

   and $\mu(x)$ satisfy the following Hölder condition

$$\| \ \mu(x) - \mu(y) \ \| \leq K \ \| \ x - y \ \|^{\lambda},$$
$$\| \ \sigma(x) - \sigma(y) \ \| \leq K \ \| \ x - y \ \|^{\lambda}$$

   where if $x \in \mathbb{R}^d$, $\| \ x \ \| \equiv (\sum_{i=1}^d x_i^2)^{1/2}$, while if a is a $d \times d$ matrix, $\| \ a \ \| \equiv (\sum_{i=1}^d \sum_{j=1}^d a_{ij}^2)^{1/2}$.

3. *Assume that* $\exists \gamma > 0$ *such that for all* $x \in E$ *(the feasible state space) and for a d-tuples* λ

$$\sum_{i=1}^{d} \sum_{j=1}^{d} a_{ij}(x) \lambda_i \lambda_j \geq \gamma \sum_{i=1}^{d} \lambda_i^2 = \gamma \parallel \lambda \parallel^2,$$

or

$$\hat{\lambda}^T a(x) \hat{\lambda} \geq \gamma; \qquad \parallel \hat{\lambda} \parallel = 1$$

Associated with a canonical diffusion there is an infinitesimal operator (generator) $\mathcal{A}$ defined by

$$\mathcal{A} f(x) \equiv \sum_{i=1}^{d} \mu_i(x) \frac{\partial f(x)}{\partial x_i} + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} a_{ij}(x) \frac{\partial^2 f(x)}{\partial x_i \partial x_j} \tag{9}$$

Definition 13 (Regular Point) *Let G be a set and a a point on the boundary of G, ∂G. Let τ' be the first exist time from G, that is*

$$\tau' \equiv \inf\{t > 0 : X_t \in \partial G\}.$$

Then the point $a \in G$ is regular if

$$P[\tau' > 0 \mid X_0(\omega) = a] = 0$$

Definition 14 (Regular Boundary) *A set G with boundary ∂G has a* regular boundary *if $\forall a \in \partial G$, a is regular.*

Here is the main theorem

Theorem 15 (Dynkin (1965) Theorem 13.16) *Let X be a canonical diffusion process with differential generator $\mathcal{A}$. Let G be an open set with regular boundary ∂G. Let the stopping time T be the first exit time from G, that is,*

$$T \equiv \inf\{t : X_t \notin G\}.$$

Let φ be a continuous function on ∂G. Let g and k be functions on G that satisfy the following Hölder condition

$$\begin{aligned} \| \quad g(x) - g(y) \| &\leq K \| x - y \|^{\mu}, \\ \| \quad k(x) - k(y) \| &\leq K \| x - y \|^{\mu}, \\ k(x) &\geq 0. \end{aligned}$$

Then,

$$\begin{aligned} V(x) \equiv E\{\int_0^T e^{-\int_0^t k(X_s(\omega))ds} g(X_t(\omega))dt + \\ e^{-\int_0^T k(X_s(\omega))ds} \varphi(X_T(\omega)) \quad | \quad X_0(\omega) = x\} \end{aligned} \tag{10}$$

is

1. *A C^2 function on G,*
2. *and it is the unique solution to the partial differential equation*

$$k(x)v(x) = g(x) + \mathcal{A}v(x); \qquad x \in G, \tag{11a}$$

$$\lim_{x \to a} v(x) = \varphi(a); \qquad a \in \partial G \tag{11b}$$

where $\mathcal{A}v(x)$ is the differential generator of the process.

Proof. *: (See Dynkin)* ■

There are a number of related theorems that relax some conditions at the expense of others. One particularly useful extension —which will be discussed later— is the case in which T is fixed. In this case, it can be shown that the function V must depend on time. This case cannot be handled as a special case of the previous theorem by expanding the vector X since, if done in this way, the "strictly positive variance" condition is violated.

3 The Feynman-Kac Theorem II: Fixed Terminal Time

In this section we state a version of the Feynman-Kac theorem that fixes the time T

We consider the following stochastic integral equation

$$X_s = x + \int_t^s \mu(u, X_u)du + \int_t^s \sigma(u, X_u)dW_u \tag{12}$$

where $\mu_i(t, x)$ and $\sigma_{ij}(t, x)$ are continuous and satisfy the Lipschitz and growth conditions

$$\| \mu(t, x) - \mu(t, y) \| + \| \sigma(t, x) - \sigma(t, y) \| \leq K \| x - y \|, \tag{13}$$

$$\| \mu(t, x) \|^2 + \| \sigma(t, x) \|^2 \leq K^2(1 + \| x \|^2). \tag{14}$$

Assume that the equation (12) has a weak solution, and that this solution is unique. Associated with (12) is the infinitesimal generator

$$\mathcal{A}_t f(t, x) \equiv \sum_{i=1}^d \mu_i(t, x)\frac{\partial f(t, x)}{\partial x_i} + \frac{1}{2}\sum_{i=1}^d \sum_{j=1}^d a_{ij}(t, x)\frac{\partial^2 f(t, x)}{\partial x_i \partial x_j},$$

where, as before, $a(x) \equiv \sigma(x)\sigma^T(x)$ or

$$a_{ij}(t, x) = \sum_{k=1}^r \sigma_{ik}(t, x)\sigma_{kj}(t, x)$$

Let $T > 0$ be arbitrary but fixed, and let $L \geq 0$ and $\lambda \geq 1$ be appropriate constants. Let

$$\begin{aligned} f(x) &: \mathbb{R}^d \to \mathbb{R}, \\ g(t, x) &: [0, T] \times \mathbb{R}^d \to \mathbb{R}, \\ k(t, x) &: [0, T] \times \mathbb{R}^d \to \mathbb{R}_+, \end{aligned}$$

be continuous functions that satisfy,

$$| f(x) | \leq L(1 + \| x \|^2), \ or \ f(x) \geq 0 \tag{15a}$$

$$| g(t, x) | \leq L(1 + \| x \|^2), \ or \ g(t, x) \geq 0, \ t \in [0, T] \tag{15b}$$

Theorem 16 (Feynman-Kac) *Assume the conditions given in this section. Suppose that* $V(t,x) : [0,T] \times \mathbb{R}^d$ *is continuous and of class* $C^{1,2}([0,T] \times \mathbb{R}^d)$*, and it satisfies the Cauchy problem*

$$\begin{aligned} k(t,x) &= \frac{\partial V(t,x)}{\partial t} + g(t,x) + \mathcal{A}_t V(t,x), \qquad [0,T) \times \mathbb{R}^d && \text{(15pa)} \\ V(T,x) &= f(x) && \text{(15pb)} \end{aligned}$$

as well as the polynomial growth condition

$$\max_{0 \le t \le T} \mid V(t,x) \mid \le M(1+ \parallel x \parallel^{2\mu}), \qquad x \in \mathbb{R}^d \tag{15q}$$

for some $M > 0,\ \mu \ge 1$.

Then, $V(t,x)$ admits the stochastic representation

$$\begin{aligned} V(t,x) &= E\{\int_t^T e^{-\int_0^t k(u,Xu)du} g(s,X_s)ds && \text{(15r)} \\ +e^{-\int_0^T k(X_s(\omega))ds} f(X_T) &\quad \mid \quad X_t(\omega) = x\} \end{aligned}$$

Proof. (To be completed) ■

3.1 The Black-Scholes Formula: An Application of Feynman-Kac

Assume that the price of stocks satisfies

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

while the price of bonds is given by

$$dB_t = B_t r dt,$$

with $\mu > r$. It follows that (S_t, B_t) is a diffusion process. Assume that (a_t, b_t) is a pair of $\{\mathcal{F}_t\} - \textit{measurable}$ processes such that the stochastic integrals

$$\int_0^t a_u dS_u \ \textit{and} \ \int_0^t b_u dB_u \ \textit{are well defined.}$$

Recall that a sufficient condition for this is that the functions are continuous, $\{\mathcal{F}_t\} - measurable$ and such that

$$E[\int_0^t a_u^2 du] < \infty.$$

We now define a self-financing trading strategy. The basic idea is that a self-financing trading strategy is a portfolio that can be purchased with a given initial amount of wealth. Formally, a pair (a_t, b_t) is self-financing if

$$\underbrace{a_t S_t + b_t B_t}_{Value\ at\ t} = \underbrace{a_0 S_0 + b_0 B_0}_{Value\ at\ 0} + \underbrace{\int_0^t a_u dS_u}_{Gains\ from\ trading\ stock} + \underbrace{\int_0^t b_u dB_u}_{Gains\ from\ trading\ bonds}$$

Note that in discrete time this condition is simply

$$a_t S_t + b_t B_t = a_0 S_0 + b_0 B_0 + \sum_{j=0}^{t-1} a_j (S_{j+1} - S_j) + \sum_{j=0}^{t-1} b_j (B_{j+1} - B_j)$$

which clearly shows the sense in which the stochastic integrals correspond to the gains from trade. Consider a random variable X. If it is possible to replicate the payoff using stocks and bonds, then we say that X is a *redundant claim*. More specifically,

$$X = a_T S_T + b_T B_T.$$

In this case we define the *arbitrage value* of X, is the value of the portfolio that replicates the payoff. Thus, the arbitrage value of X is $a_0 S_0 + b_0 B_0$.

Let $c_T = g(S_T)$, and assume that $\mid g(S_T) \mid$ is integrable. We now guess that

$$c_t = C(t, S_t), \qquad C \in C^{1,2}.$$

Given that the strategy (a_t, b_t) is self-financing it follows that

$$C(t, S_t) = C(0, S_0) + \int_0^t a_u dS_u + \int_0^t b_u dB_u,$$

or

$$C(t, S_t) = C(0, S_0) + \int_0^t a_u \mu S_u du + \int_0^t b_u r B_u du + \int_0^t a_u \sigma S_u dW_u \qquad (19)$$

On the other hand, Ito's lemma implies that

$$C(t,S_t) = C(0,S_0) + \int_0^t C_u du + \int_0^t C_S \mu S_u du + \frac{1}{2}\int_0^t C_{SS}\sigma^2 S_u^2 du + \int_0^t C_S \sigma S_u dW_u. \tag{20}$$

Since both (19) and (20) must hold for all t, it must be the case that

$$a_t \mu S_t + b_t r B_t = C_t + C_S \mu S_t + \frac{1}{2} C_{SS}\sigma^2 S_t^2, \tag{21a}$$

$$a_t \sigma S_t = C_S \sigma S_t. \tag{21b}$$

Then these conditions imply (i.e. solving the previous system for the optimal portfolio (a_t, b_t)

$$\begin{aligned} a_t^* &= C_S(t,S_t), \\ b_t^* &= \frac{C_t(t,S_t) + \frac{1}{2}C_{SS}(t,S_t)\sigma^2 S_t^2}{rB_t}. \end{aligned}$$

When this optimal portfolio is substituted into the condition $c_t = a_t S_t + b_t B_t = C(t,S_t)$ we obtain

$$C(t,S_t) = C_S(t,S_t)S_t + \frac{1}{r}[C_t(t,S_t) + \frac{1}{2}C_{SS}(t,S_t)\sigma^2 S_t^2],$$

or

$$rC(t,S_t) = C_t(t,S_t) + C_S(t,S_t) r S_t + \frac{1}{2}C_{SS}(t,S_t)\sigma^2 S_t^2 \tag{22}$$

Thus we have shown that $C(t,S_t)$ has to satisfy (22) and, in addition,

$$C(T,S) = g(S), \tag{23a}$$

$$C(t,0) = e^{-r(T-t)}g(0). \tag{23b}$$

In summary, the price of the security is a function $C(t,S_t)$ that satisfies (22) and (23). However, the Feynman-Kac Theorem shows that $C(t,S_t)$ corresponds to

$$C(t,S) = E\{e^{-r(T-t)}g(S_T) \mid S_t = S\}$$

if —and this is an interesting if— the process for S_t satisfies

$$dS_t = rS_t dt + \sigma S_t dW_t. \tag{24}$$

Thus, independently of the actual drift of the process for stock prices, the value of the security is similar to the value that it would have if the price of stocks grew at the safe rate of return.

There are many ways of finding a solution $C(t, S_t)$. A simple approach is to observe that given (24) it follows that

$$S_u = Se^{Z_u},$$

where

$$Z_u = (r - \frac{\sigma^2}{2})u + \sigma W_u$$

is a $(r - \frac{\sigma^2}{2}, \sigma)$ Brownian motion. Then,

$$C(t, S) = E\{e^{-r(T-t)} g(Se^{(r-\frac{\sigma^2}{2})(T-t)+\sigma(W_T - W_t)}) \mid S_t = S\}, \tag{25}$$

where $W_T - W_t$ is independent of $\mathcal{F}_t$ and has a normal distribution $N(0, T-t)$.

Consider, as an example, the standard call option. In this case,

$$g(S) = \max(S - K, 0).$$

In this case, let W^* satisfy

$$Se^{(r-\frac{\sigma^2}{2})(T-t)+\sigma W^*} = K,$$

or,

$$W^* = \frac{\ln(\frac{K}{S}) - (r - \frac{\sigma^2}{2})(T-t)}{\sigma}. \tag{26}$$

Using this last equation in (25) we get

$$e^{-r(T-t)}\{\underbrace{\int_{W^*}^{\infty} Se^{(r-\frac{\sigma^2}{2})(T-t)+\sigma x}\tilde{\Phi}(dx)}_{A} - \underbrace{K\int_{W^*}^{\infty} \tilde{\Phi}(dx)\}}_{B},$$

where $\tilde{\Phi}(x)$ is the cumulative distribution function of a $N(0, T-t)$; i.e.

$$\tilde{\Phi}(dx) = \frac{e^{-\frac{x^2}{2(T-t)}}}{\sqrt{2\pi(T-t)}}dx.$$

Note that

$$\int_{W^*}^{\infty} \tilde{\Phi}(dx) = P[Z \geq W^*] = 1 - \Phi(\frac{W^*}{T-t}) = \Phi(-\frac{W^*}{T-t}) \qquad (symmetry).$$

Thus,

$$B = K\Phi(-\frac{\ln(\frac{K}{S}) - (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}).$$

Next we work with A. The key term is

$$\int_{W^*}^{\infty} \frac{e^{\sigma x} e^{-\frac{x^2}{2(T-t)}}}{\sqrt{2\pi(T-t)}} dx \tag{27}$$

Let's find a term a such that

$$(\frac{x}{\sqrt{T-t}} - a)^2 = \frac{x^2}{T-t} - 2a\frac{x}{\sqrt{T-t}} + a^2.$$

If $a = \sigma\sqrt{T-t}$, this square is

$$(\frac{x}{\sqrt{T-t}} - \sigma\sqrt{T-t})^2 = \frac{x^2}{T-t} - 2\sigma x + \sigma^2(T-t).$$

It follows that $\frac{x^2}{T-t} - 2\sigma x = (\frac{x}{\sqrt{T-t}} - \sigma\sqrt{T-t})^2 - \sigma^2(T-t)$. Using this in (27) we obtain

$$\int_{W^*}^{\infty} \frac{e^{\sigma x} e^{-\frac{x^2}{2(T-t)}}}{\sqrt{2\pi(T-t)}} dx = e^{\sigma^2(T-t)/2} \int_{W^*}^{\infty} \frac{e^{-(\frac{x}{\sqrt{T-t}} - \sigma\sqrt{T-t})^2/2}}{\sqrt{2\pi(T-t)}} dx,$$

where $u = \frac{x}{\sqrt{T-t}} - \sigma\sqrt{T-t}$. Given $du = dx/\sqrt{T-t}$, and $u^* = \frac{W^*}{\sqrt{T-t}} - \sigma\sqrt{T-t}$, the previous integral is

$$\int_{W^*}^{\infty} \frac{e^{-(\frac{x}{\sqrt{T-t}} - \sigma\sqrt{T-t})^2/2}}{\sqrt{2\pi(T-t)}} dx = \int_{[\frac{W^*}{\sqrt{T-t}} - \sigma\sqrt{T-t}]}^{\infty} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} du.$$

Thus, A is

$$A = Se^{r(T-t)} \int_{[\frac{W^*}{\sqrt{T-t}} - \sigma\sqrt{T-t}]}^{\infty} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} du = Se^{r(T-t)}\Phi[\frac{\ln(\frac{S}{K}) + r(T-t)}{\sigma\sqrt{T-t}}]. \tag{28}$$

Thus, putting together both terms (A and B) we obtain,

$$C(t, S_t) = S\Phi[\frac{\ln(\frac{S}{K}) + r(T-t)}{\sigma\sqrt{T-t}}] - Ke^{-r(T-t)}\Phi(-\frac{\ln(\frac{K}{S}) - (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}) \tag{29}$$

which is the celebrated Black-Scholes formula.