Some Results on Kolmogorov's Equations

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Acknowledgement 1 *These notes contain some mathematical results taken from several different references (missing).*

1 Introduction

In this note, I present some results on the properties of Kolmogorov's forward and backward equations, and their relation to the distribution of a stochastic process. The notes follow

2 Kolmogorov's Backward Equation

Let a function $u(t,x)$ be the solution to

$$0 = \frac{\partial u(t,x)}{\partial t} + \frac{\partial u(t,x)}{\partial x}\mu(x) + \frac{1}{2}\frac{\partial^2 u(t,x)}{\partial x^2}\sigma^2(x), \tag{1}$$

with boundary conditions

$$u(T,x) = \varphi(x). \tag{2}$$

Then, the Feynman-Kac theorem (for finite horizons) shows that $u(t,x)$ satisfies

$$u(t,x) = E[\varphi(X_T) \mid X_t = x],$$

where the stochastic process for $\{X_t\}$ satisfies

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \tag{3}$$

where W_t is a standard Brownian Motion or its integral version

$$X_t = X_0 + \int_0^t \mu(X_s)ds + \int_0^t \sigma(X_s)dW_s, \qquad 0 \leq t < \infty,$$

and given a probability space, $(\Omega, \mathcal{F}, P)$, and the filtration $\{\mathcal{F}_t\}$ (which we assume satisfies the usual conditions).

It is convenient to define the operator $\mathcal{L}$ by

$$\mathcal{L}u = \frac{\partial u(t,x)}{\partial x}\mu(x) + \frac{1}{2}\frac{\partial^2 u(t,x)}{\partial x^2}\sigma^2(x). \tag{4}$$

Using this notation the Backward Kolmogorov Equation (BKE) is

$$\frac{\partial u(t,x)}{\partial t} + \mathcal{L}u(t,x) = 0. \tag{5}$$

It follows that KBE is a special case of the Feynman-Kac theorem: It deals with the case in which the horizon is fixed (and finite), there is no running payoff and there is no discounting.

3 Kolmogorov's Forward Equation

In this sections we discuss the forward equation and show that its solution gives the probability density function of the process $\{X_t\}$. Let $p(z,s,x,t)$ be the probability density that, at time s, the process will take on the value z, given that at time t $(t \leq s)$ it has the value x.

We claim that p solves the following equation

$$\frac{\partial p(z,s,x,t)}{\partial s} = -\frac{\partial[p(z,s,x,t)\mu(x)]}{\partial z} + \frac{1}{2}\frac{\partial^2[p(z,s,x,t)\sigma^2(x)]}{\partial z^2} \tag{6}$$

with boundary conditions

$$\lim_{s\to t} p(z,s,x,t) = \delta_{\{x\}}(z)$$

which is just the measure that puts point mass on the point x. If the process has non-zero variance (and it is not absorbed) then

$$\lim_{z\to\infty} p(z,s,x,t) = 0.$$

It is convenient to define another operator to simplify the presentation. Let

$$\mathcal{L}^* p = -\frac{\partial[p(z,s,x,t)\mu(x)]}{\partial z} + \frac{1}{2}\frac{\partial^2[p(z,s,x,t)\sigma^2(x)]}{\partial z^2}. \tag{7}$$

For any probability density function $\rho(t,x)$ of the process at time t, it follows that the density at time s satisfies

$$\rho(s,z) = \int_{\mathbb{R}} p(z,s,x,t)\rho(t,x)dx. \tag{8}$$

Claim 2 *The density $\rho(s,z)$ satisfies the KFE.*

Remark 3 *Some intuition: Why should the KFE hold?*

By definition of the density, it must be the case that

$$\int_{\mathbb{R}} \rho(s,z)dz = 1.$$

Thus,

$$\frac{\partial}{\partial s}\int_{\mathbb{R}} \rho(s,z)dz = 0,$$

or, using Fubini,

$$\int_{\mathbb{R}} \frac{\partial \rho(s,z)}{\partial s}dz = 0.$$

If $\rho(s,z)$ satisfies the KFE, then

$$\int_{\mathbb{R}} \frac{\partial \rho(s,z)}{\partial s}dz = -\int_{\mathbb{R}} \frac{\partial[\rho(s,z)\mu(z)]}{\partial z}dz + \frac{1}{2}\int_{\mathbb{R}} \frac{\partial^2[\rho(s,z)\sigma^2(z)]}{\partial z^2}dz$$

or,

$$0 = \underbrace{-\lim_{x\to\infty} \rho(s,x)\mu(x)\mid_{-x}^{x}}_{=0} + \frac{1}{2}\frac{\partial[\rho(s,x)\sigma^2(x)]}{\partial x},$$

or

$$0 = \frac{1}{2}\lim_{x\to\infty} \rho(s,x)\sigma^2(x)\mid_{-x}^{x}$$

and these two should be 0!

Definition 4 *Let $v(x)$ and $w(x)$ two integrable functions, then by the linear operator $\langle v, w\rangle$ we mean*

$$\langle v, w\rangle = \int_{\mathbb{R}} v(x)w(x)dx.$$

Claim 5 *$\langle \mathcal{L}v, w\rangle = \langle v, \mathcal{L}^* w\rangle$, or that $\mathcal{L}^*$ is the dual of $\mathcal{L}$.*

Proof.

$$\begin{aligned}\langle \mathcal{L}v, w\rangle &= \int_{\mathbb{R}} (\mathcal{L}v)(x)w(x)dx \\ &= \int_{\mathbb{R}} [v'(x)\mu(x) + \frac{1}{2}v''(x)\sigma^2(x)]w(x)dx.\end{aligned}$$

Let

$$\begin{aligned}A &= \int_{\mathbb{R}} v'(x)\mu(x)w(x)dx, \\ B &= \int_{\mathbb{R}} \frac{1}{2}v''(x)\sigma^2(x)w(x)dx.\end{aligned}$$

Then, integration by parts, implies that

$$\begin{aligned}A &= \underbrace{\lim_{x\to\infty} v(x)\mu(x)w(x) \mid_{-x}^{x}}_{0} - \int_{\mathbb{R}} \frac{\partial[w(x)\mu(x)]}{\partial x} v(x)dx, \\ A &= -\int_{\mathbb{R}} \frac{\partial[w(x)\mu(x)]}{\partial x} v(x)dx\end{aligned}$$

and

$$\begin{aligned}B &= \underbrace{\frac{1}{2}\lim_{x\to\infty} v'(x)\sigma^2(x)w(x) \mid_{-x}^{x}}_{0} - \frac{1}{2}\int_{\mathbb{R}} \frac{\partial[w(x)\sigma^2(x)]}{\partial x} v'(x)dx \\ &= -\frac{1}{2}\int_{\mathbb{R}} \frac{\partial[w(x)\sigma^2(x)]}{\partial x} v'(x)dx \\ &= -\frac{1}{2}\{\underbrace{\lim_{x\to\infty} v(x)\frac{\partial[w(x)\sigma^2(x)]}{\partial x} \mid_{-x}^{x}}_{0} - \int_{\mathbb{R}} \frac{\partial^2[w(x)\sigma^2(x)]}{\partial x^2} v(x)dx\} \\ B &= \frac{1}{2}\int_{\mathbb{R}} \frac{\partial^2[w(x)\sigma^2(x)]}{\partial x^2} v(x)dx.\end{aligned}$$

It follows that

$$\begin{aligned}
\langle \mathcal{L}v, w\rangle &= -\int_{\mathbb{R}} \frac{\partial[w(x)\mu(x)]}{\partial x} v(x)dx + \frac{1}{2}\int_{\mathbb{R}} \frac{\partial^2[w(x)\sigma^2(x)]}{\partial x^2} v(x)dx \\
&= \int_{\mathbb{R}} [-\frac{\partial[w(x)\mu(x)]}{\partial x} + \frac{1}{2}\frac{\partial^2[w(x)\sigma^2(x)]}{\partial x^2}] v(x)dx \\
&= \int_{\mathbb{R}} (\mathcal{L}^* w)(x) v(x) dx.
\end{aligned}$$

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Next we use this result to derive the KFE. Let ϕ be an arbitrary function (it has to be integrable!). Then, using Ito's Lemma we get that

$$\begin{aligned}
\phi(X_T, T) - \phi(x,t) &= \int_t^T [\frac{\partial \phi(s, X_s)}{\partial t} + \frac{\partial \phi(s, X_s)}{\partial x}\mu(X_s) + \frac{1}{2}\frac{\partial^2 \phi(t, X_s)}{\partial x^2}\sigma^2(X_s)]ds \\
&\quad + \int_t^T \frac{\partial \phi(s, X_s)}{\partial x}\sigma^2(X_s) dW_s.
\end{aligned}$$

It follows that

$$\begin{aligned}
E[\phi(X_T, T) \quad | \quad X_t = x] - \phi(x,t) &= \int_t^T [\frac{\partial \phi(s, X_s)}{\partial t} + \frac{\partial \phi(s, X_s)}{\partial x}\mu(X_s) + \\
&\quad \frac{1}{2}\frac{\partial^2 \phi(t, X_s)}{\partial x^2}\sigma^2(X_s)]ds \\
&= E[\int_t^T [\frac{\partial \phi(s, X_s)}{\partial t} + (\mathcal{L}\phi)(s, X_s)]ds \mid X_t = x]
\end{aligned}$$

or,

$$\begin{aligned}
\int_{\mathbb{R}} \phi(z,T)p(z,T,x,t)dz - \phi(x,t) &= \int_{\mathbb{R}}\int_t^T \frac{\partial \phi(s,z)}{\partial t} p(z,s,x,t)dsdz \\
&\quad + \int_{\mathbb{R}}\int_t^T (\mathcal{L}\phi)(s,z)p(z,s,x,t)dsdz.
\end{aligned}$$

Let the first term on the right side be

$$\begin{aligned}
\int_{\mathbb{R}}\int_t^T \frac{\partial \phi(s,z)}{\partial t} p(z,s,x,t)dsdz &= \int_{\mathbb{R}} \{\phi(s,z)p(z,s,x,t) \mid_t^T \\
&\quad - \int_t^T \phi(s,z)p_s(z,s,x,t)ds\}dz \\
&= -\int_{\mathbb{R}}\int_t^T \phi(s,z)p_s(z,s,x,t)dsdz.
\end{aligned}$$

Then, it follows that

$$\begin{aligned}\int_{\mathbb{R}} \phi(z,T)p(z,T,x,t)dz - \phi(x,t) &= \int_{\mathbb{R}} \phi(s,z)p(z,s,x,t) \mid_t^T dz \\ &- \int_{\mathbb{R}} \int_t^T \phi(s,z)p_s(z,s,x,t)dsdz \\ &+ \int_{\mathbb{R}} \int_t^T (\mathcal{L}^* p)(s,z,x,t)\phi(s,z)dsdz.\end{aligned}$$

Thus,

$$\begin{aligned}\int_{\mathbb{R}} \phi(z,T)p(z,T,x,t)dz - \phi(x,t) &= \int_t^T \int_{\mathbb{R}} [-p_s(z,s,x,t) + (\mathcal{L}^* p)(s,z,x,t)]\phi(s,z)dzds \\ + \int_{\mathbb{R}} \phi(s,z)p(z,s,x,t) &\mid {}_t^T dz\end{aligned}$$

Since this has to hold for any function, and the left hand side depends only on the value of the function at t and T,, it must be the case that

$$[-p_s(z,s,x,t) + (\mathcal{L}^* p)(s,z,x,t)] = 0.$$

Finally, taking limits as $T \to t$ the right hand side goes to 0 and, hence,

$$\lim_{T \to t} \int_{\mathbb{R}} \phi(z,T)p(z,T,x,t)dz - \phi(x,t) = 0$$

or,

$$\int_{\mathbb{R}} \phi(z,t)p(z,t,x,t)dz = \phi(x,t)$$

This argument shows that if $p(z,s,x,t)$ is a conditional probability density, then it must satisfy the KFE (and the boundary condition). Conversely, it can be shown that if a function satisfies the KFE (and the appropriate boundary conditions, including non-negativity and integration to one) then it is a conditional probability density.