Topics in Macroeconomics: An Introduction to Dynamic Economics
Problem Set III

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Problem 1 (Indexed Bonds) *Let preferences of a consumer be given by,*

$$\max E\left\{\int_0^\infty e^{-\rho t}\frac{c_t^{1-\theta}}{1-\theta}dt\right\}.$$

To maximize utility, the individual chooses a portfolio that can include three assets: a (risky) asset (equity) a nominal bond and an indexed bond. Assume that the price level satisfies

$$dP_t = \pi P_t dt + \sigma_\pi P_t dZ_{Pt},$$

the ***nominal*** *value of equity evolves according to*

$$dS_t = R_S S_t dt + \sigma_S S_t dZ_{St},$$

the real return on the indexed bond is given by r_I*,and the value of the nominal bond satisfies*

$$dB_t^N = R_N B_t^I dt.$$

Assume that the correlation between Z_{Pt} *and* Z_{St} *is* η*; that is,* $E\{dZ_{Pt}dZ_{St}\} = \eta dt$*. Let* $W_0 > 0$ *be the initial value of wealth in nominal terms. Assume that the consumer has no other form of income.*

1. *Formulate the optimal saving-consumption problem.*
2. *Go as far as you can analyzing the composition of the portfolio. In particular, make sure that you describe how uncertainty in inflation affects real returns and the shares of each asset in the optimal portfolio.*
3. *Assume now that the consumer has a fixed income* $y > 0$*. Go as far as you can analyzing this problem.*

4. *Go as far as you can analyzing the finite horizon case. In particular, does the model deliver the usual recommendation that as individuals get close to retirement (say when t is close to T —the end of the planning horizon) they should invest a larger fraction of their wealth in safe bonds?*

Problem 2 (Optimal Consumption of an Exhaustible Resource) *In the standard version of this problem (e.g. see Dasgupta and Heal) the planner's problem is (ignore the expected value operator for now)*

$$\max E\left\{\int_0^\infty e^{-\rho t}u(c_t)dt\right\}$$

subject to

$$dR_t = -c_t dt,$$

with $R_0 > 0$, given. Consider now the stochastic version in which there is uncertainty about the stock. Specifically, assume that

$$dR_t = -c_t dt + \sigma R_t dZ_t.$$

1. *Formulate the planner's problem and describe the solution for an arbitrary $u(c)$ function that satisfies all the "nice" properties (i.e. concavity, differentiability, and anything else you need). You may assume that the value function is differentiable.*

2. *Assume that*

$$u(c) = \frac{c^{1-\theta}}{1-\theta}.$$

Go as far as you can characterizing the solution to the planner's problem. In particular, describe the effect of uncertainty on the rate of extraction.

3. *Let R_L be an arbitrarily small level of the resource. Using the specification of utility form the previous paragraph compute the expected time until that level is reached for the second time.*

Problem 3 (Insurance and Utility) *Let $S(t)$ be the probability that an individual lives at least t years. Thus,*

$$S(t) = 1 - F(t) = \int_t^{\bar{T}} \lambda(s)ds,$$

where $\lambda(s)$ is the density associated with dying and time s, and $\bar{T}$ is maximum life span. Expected utility is given by

$$E\left\{\int_0^T e^{-\rho t}u(c_t)dt\right\},$$

where T is the random length of life. It follows that

$$E\left\{\int_0^T e^{-\rho t}u(c_t)dt\right\} = \int_0^{\bar{T}} \lambda(t)[\int_0^t e^{-\rho s}u(c_s)ds]dt.$$

Assume that each consumer is endowed with $A > 0$ assets at time 0. The rate of return is constant and equal to r.In addition, the consumer earns $y > 0$ units of consumption each period he is alive.

1. *Show that*

$$E\left\{\int_0^T e^{-\rho t}u(c_t)dt\right\} = \int_0^{\bar{T}} S(t)e^{-\rho t}u(c_t)dt.$$

2. *Assume that $\rho = r = 0$ and that there are perfect insurance (annuity) markets. Go as far as you can computing the optimal consumption rule. In particular, show how it depends on life expectancy..*

3. *Consider now the case in which an individual is told —at the time of birth— his life span; that is, immediately after birth, the consumer knows when he is going to die. Compute the optimal consumption in this case.*

4. *Now compare the ex-ante (i.e. before being born) utility of the consumers in sections 2 and 3. Can you describe what accounts for the difference?*