

CHAPTER 6: EXERCISING AN OPTION

In the presence of a fixed cost of adjustment, optimally exercising control has two aspects: deciding at what point(s) action should be taken and choosing what the action(s) should be. In this chapter the simplest example of this sort is studied, the problem of exercising an infinitely lived, one-time option. The option problem is simple because action is taken only once and the action itself is fixed, so the only issue is timing—deciding when, if ever, to exercise the option. Its simplicity makes this problem a useful introduction to methods that are applicable more broadly.

Two approaches are studied. The first uses the functions \hat{L} and ψ defined in Chapter 5. Recall that $\hat{L}(z; x, b, B)$ is the expected discounted local time of a stochastic process at level z before the stopping time $T = T(b) \wedge T(B)$, given the initial state $x \in [b, B]$, and $\psi(x, b, B)$ is the expected discounted value of an indicator function for the event $T = T(b)$. The option problem involves only a lower threshold, so $B = +\infty$ and stops at an upper threshold are not involved. The expected discounted return from an arbitrary threshold policy can be written in terms \hat{L} and ψ , and finding the optimal policy then involves a straightforward maximization over the threshold value b .

The second approach uses the Hamilton-Jacobi-Bellman (HJB) equation for the firm's problem. Recall from Chapter 3 that the HJB equation is the stochastic analog to a continuous-time Bellman equation. The difference is that the variance term makes the HJB equation a second-order ordinary differential equation (ODE), while the Bellman equation is first order. The second approach involves determining ap-

appropriate boundary conditions for this ODE and solving it.

While mechanically quite different from each other, the two approaches are closely related. The connections between them are discussed after both have been studied in detail.

The rest of the chapter is organized as follows. In section 1 a deterministic version of the problem is studied using several approaches: as a control problem formulated in terms of the date when action should be taken, as a control problem formulated in terms of a threshold value for the state where action should be taken, and in terms of a Bellman equation. The second and third are analogs of the approaches to the stochastic problem studied next. In section 2 the stochastic problem is analyzed using the functions \hat{L} and ψ , and in section 3 it is studied using the HJB equation. The connections between the two are then discussed, and section 4 concludes with an example.

1. The deterministic problem

Consider a plant that generates a profit flow as long as it is operated. Profits depend on a state variable $X(t)$ that can reflect demand or plant capacity or both. The state has a stationary law of motion, which for now is assumed to be deterministic and linear,

$$X(t) = x_0 + \mu t, \quad \text{all } t \geq 0, \quad (1)$$

where the initial state $X(0) = x_0$ is given. $X(t)$ could be interpreted as the log of the capital stock, with depreciation at the constant rate $\delta = -\mu > 0$, in a setting with time-invariant demand. Or $X(t)$ could be interpreted as a measure of demand, which falls at the rate $|\mu|$.

Let $\pi(x)$ be the net profit flow from operating the plant if the current state is x . The function π is time invariant, and it is defined net of operating costs for labor,

materials, and other inputs. Let $r > 0$ be the (constant) interest rate.

The plant can be shut down at any time, and it cannot be reopened after it has been closed. When the plant is shut down it has a salvage value S , which might represent the scrap value of the capital stock. For now suppose that this value is independent of the state $X(t)$ at the time of the closing. The objective of the firm is to choose a shut-down policy that maximizes the discounted value of total returns, the profits from operating the plant plus the salvage value when it is shut down. Thus, the value from following an optimal policy, given the initial state x_0 , is

$$V(x_0) \equiv \sup_{T \geq 0} \left[\int_0^T e^{-rt} \pi(X(t)) dt + e^{-rT} S \right], \quad (2)$$

where $X(t)$ is as in (1), and $T = +\infty$ if the firm chooses to operate the plant forever.

The primitives for the problem are the constants r, μ , and S and the function π . The following assumption will be used throughout.

ASSUMPTION 6.1: (i) $r > 0$, $\mu < 0$, $S > 0$;

(ii) $\pi(\cdot)$ is bounded, continuous, and strictly increasing, with

$$\lim_{x \rightarrow -\infty} \pi(x) < rS < \lim_{x \rightarrow +\infty} \pi(x).$$

Part (ii) of the assumption insures that it is optimal to shut down immediately if the initial state is sufficiently low and to continue operating—at least for a while—if it is sufficiently high.

The restriction $\mu < 0$ is needed to avoid ‘corner’ solutions of the following sort. If $\mu > 0$ the incentive to continue operating grows over time, so it is never optimal to operate for a finite length of time and then to shut down. The following exercise shows that $T^* = 0$ or $+\infty$ if $\mu > 0$.

Exercise 6.1: Let r, π , and S satisfy Assumption 6.1, and consider the problem in (2). Show that if $\mu \geq 0$ there is a critical value x_c with the following property:

if $x < x_c$, the unique optimal policy is to shut down immediately ($T = 0$);
 if $x > x_c$, the unique optimal policy is to operate the plant forever ($T = +\infty$);
 if $x = x_c$, then both $T = 0$ and $T = \infty$ are optimal policies, and
 if $\mu > 0$ there are no others;
 if $\mu = 0$ any choice of T is optimal.

Under Assumption 6.1 the optimal policy is unique and it involves operating for a finite length of time. To see this, note that the first order condition for an optimum in (2) is

$$e^{-rT^*} [\pi(X(T^*)) - rS] \leq 0, \quad \text{with equality if } T^* > 0. \quad (3)$$

The expression in brackets in (3) represents the effects of operating a little longer: the additional profit flow less the foregone interest on the salvage value. At an interior optimum the two are equal. Under Assumption 6.1 there is a unique value b^* satisfying

$$\pi(b^*) - rS = 0, \quad (4)$$

and since π is increasing and $\mu < 0$ the problem is concave. Hence for any initial state x the unique optimal shut-down date is

$$T^* = \begin{cases} 0, & \text{if } x < b^*, \\ (b^* - x) / \mu, & \text{if } x \geq b^*, \end{cases}$$

and the value function V is

$$V(x) = \begin{cases} S, & \text{if } x < b^*, \\ \int_0^{(b^*-x)/\mu} e^{-rt} \pi(X(t)) dt + e^{-(b^*-x)r/\mu} S, & \text{if } x \geq b^*, \end{cases}$$

where for notational simplicity the subscript on the initial state x has been dropped.

The approach above involves using the shut-down date T as the control variable, but the form for the optimal policy suggests that it may be simpler to formulate the problem in terms of a critical value b for the state variable. To start, note that since π is increasing and $\mu < 0$, the optimal policy is a threshold policy. For the threshold

b , call the interval $(-\infty, b]$ the **action (or shut-down) region** and $(b, +\infty)$ the **inaction (or continuation) region**.

Fix the initial state x and suppose the firm uses a threshold policy with critical value $b \leq x$. Let $F(x, b)$ denote the returns from this policy, and use the change of variable $\xi = x + \mu t$ to write the problem in (2) as

$$\begin{aligned} V(x) &\equiv \max_{b \leq x} F(x, b) \\ &= \max_{b \leq x} \left[\frac{1}{-\mu} \int_b^x e^{(x-\xi)r/\mu} \pi(\xi) d\xi + e^{(x-b)r/\mu} S \right]. \end{aligned} \quad (5)$$

The follow result characterizes the solution.

PROPOSITION 6.1: Let r, μ, S , and π satisfy Assumption 6.1 and let X be as in (1). The solution to the problem in (5) is

$$b = \min \{x, b^*\},$$

where b^* is defined in (4). The associated value function is

$$V(x) = \begin{cases} S, & \text{if } x < b^*, \\ -\mu^{-1} \int_{b^*}^x e^{(x-\xi)r/\mu} \pi(\xi) d\xi + e^{(x-b^*)r/\mu} S, & \text{if } x \geq b^*. \end{cases} \quad (6)$$

PROOF: The first order condition for an optimum in (5) is

$$\frac{\partial F(x, b)}{\partial b} = \frac{1}{\mu} e^{(x-b)r/\mu} [\pi(b) - rS] \geq 0, \quad \text{with equality if } b < x. \quad (7)$$

If $x \geq b^*$, then $b = b^*$ and this condition holds with equality. If $x < b^*$, then $b = x$ and it holds with strict inequality. ■

Thus, the optimal policy is:

- (i) if $x \leq b^*$, shut down immediately; and
- (ii) if $x > b^*$, operate until the state reaches b^* and then shut down.

There is a third approach to this problem as well, which uses the Bellman equation. As before, begin by noting that the policy takes the form of a threshold. Let x be the initial state and fix a threshold $b \leq x$. If $b < x$ then the firm continues operating for a while, so for any sufficiently small time interval h the value from this policy, call it $F(x, b)$, satisfies

$$\begin{aligned} F(x, b) &= \int_0^h e^{-rt} \pi(X(t)) dt + e^{-rh} F(X(h), b) \\ &\approx \pi(x)h + \frac{1}{1+rh} [F(x, b) + \mu h F_x(x, b)], \end{aligned}$$

where $X(t)$ in the first line is as in (1), the second line uses a first-order Taylor series approximation to $F(X(h), b)$, and $F_x(x, b) = \partial F(x, b) / \partial x$. Rearrange terms, divide by h and let $h \rightarrow 0$ to get the Bellman equation

$$rF(x, b) = \pi(x) + \mu F_x(x, b). \quad (8)$$

Since b is fixed, (8) is a first-order ODE. In addition $F(\cdot, b)$ satisfies

$$\lim_{x \downarrow b} F(x, b) = S. \quad (9)$$

Integrate (8) and use the boundary condition in (9) to obtain the function $F(x, b)$ in (5).

In the preceding argument b is fixed and arbitrary. An optimal policy involves choosing b to maximize $F(x, b)$, which as before leads to the first order condition in (7) involving $\partial F / \partial b$. But notice from (5) that

$$\begin{aligned} -F_b(x, b) &= -e^{(x-b)r/\mu} \frac{1}{\mu} [\pi(b) - rS] \\ &= -e^{(x-b)r/\mu} F_b(b, b) \\ &= +e^{(x-b)r/\mu} F_x(b, b), \quad \text{all } b \leq x, \text{ all } x. \end{aligned} \quad (10)$$

The interpretation is as follows. Suppose the firm uses the threshold policy b . For the first line, note that a small negative perturbation of the threshold, by $-\varepsilon < 0$, delays

the closing of the plant by $\varepsilon/(-\mu) > 0$ units of time. Over this interval the firm collects profits at the rate $\pi(b)$ but loses interest on the salvage value at the rate rS . If the initial state is $x > b$, the threshold b is reached at date $t = (x - b)/(-\mu) > 0$, and these changes in the firm's revenue must be appropriately discounted.

For the second line the interpretation is the same, except that the initial state is at the boundary, $x = b$. The discount factor must be appended to equate the second line with the first.

For the third line, suppose again that the initial state is $x = b$. A small *positive* perturbation of the state, by $\varepsilon > 0$, also delays the closing of the firm by $\varepsilon/(-\mu) > 0$ units of time, changing net revenue in the same way as a negative perturbation to the threshold.

Thus, conditional on $x = b$, positive perturbations of the state and negative perturbations of the threshold have identical effects. If $x > b$ a negative perturbation of the threshold also has the same effect, except that it is discounted.

It follows from (10) that the critical value b^* in (4) satisfies

$$\frac{\partial F(x, b^*)}{\partial x} \Big|_{x=b^*} = \frac{1}{\mu} [\pi(b^*) - rS] = 0. \quad (11)$$

In contrast to (7), which involves a derivative with respect to the threshold b holding the state x fixed, (11) involves a derivative with respect to the state x , evaluated at $x = b^*$, holding the threshold fixed. Hence, among all solutions $F(\cdot, b)$ to the ODE in (8) with the boundary condition in (9), the optimized value function $V = F(\cdot, b^*)$ is the one that in addition has the property in (11), $F_x(b^*, b^*) = V'(b^*) = 0$. The following result formalizes this conclusion.

PROPOSITION 6.2: Let r, μ, S and π satisfy Assumption 6.1 and let X be as in (1). The function V and threshold b^* defined in (6) and (7) have the following properties:

a. V satisfies

$$\begin{aligned} rV(x) &= \pi(x) + \mu V'(x), & x > b^*, \\ V(x) &= S, & x \leq b^*; \end{aligned} \tag{12}$$

b. V is continuous at b^* :

$$\lim_{x \downarrow b^*} V(x) = S; \tag{13}$$

c. V' is continuous at b^* :

$$\lim_{x \downarrow b^*} V'(x) = 0. \tag{14}$$

PROOF: There is a family of solutions to (12) and (13) indexed by b , the functions $F(\cdot, b)$ in (5). The unique member of this family that in addition satisfies (14) is the one with $b = b^*$, where b^* is defined by (4). ■

To summarize, there are two ways to arrive at the function $F(x, b)$ in (5), which describes the expected discounted value from using the arbitrary threshold policy b , given the initial state x . The first begins with the integral over time in (2) and converts it into the integral over states in (5). The second uses the fact that for any b , the function $F(\cdot, b)$ satisfies the Bellman equation in (8) on $(b, +\infty)$, with the boundary condition in (9).

There are also two ways to characterize the optimal threshold b^* . The first involves maximizing $F(x, b)$ with respect to b , as in (5), to get the first order condition in (7), a condition on $\partial F/\partial b$. The second involves using (10) to replace (7) with (11), a condition on the derivative $\partial F/\partial x$, evaluated at $x = b$.

Proposition 6.1 uses the first methods and Proposition 6.2 the second.

The following exercise extends the model to allow the salvage value $S(x)$ to depend on the state.

Exercise 6.2: Let r, μ and π satisfy Assumption 6.1, and assume in addition that

- (i) $S(x)$ is bounded and continuously differentiable;

(ii) the function

$$\phi(x) \equiv \pi(x) + \mu S'(x) - rS(x)$$

is strictly increasing in x , with

$$\lim_{x \rightarrow -\infty} \phi(x) < 0 < \lim_{x \rightarrow +\infty} \phi(x).$$

Use all three approaches to characterize the value function and optimal policy for the shut-down problem

$$V(x) \equiv \sup_{T \geq 0} \left[\int_0^T e^{-rt} \pi(X(t)) dt + e^{-rT} S(X(T)) \right],$$

where $X(t)$ is as in (1), with $X(0) = x$. What is the interpretation of the inequality restriction in (ii)?

2. The stochastic problem: a direct approach

Next, suppose that r, μ, S and π satisfy Assumption 6.1, but that the state variable $X(t)$ is a Brownian motion with parameters μ and $\sigma^2 > 0$. The methods used in Propositions 6.1 and 6.2 to solve the deterministic problem can be applied in this case as well, requiring only modest modification to accommodate the fact that $X(t)$ is a stochastic process.

Begin by noticing that an optimal policy for the stochastic model has two basic properties. First, since the problem is stationary, it is clear that if an optimal policy requires (allows) shutting down in state b , it requires (allows) shutting down the *first* time the process reaches state b . Hence we can limit our attention to threshold policies. For any b , let $T(b)$ be the stopping time defined as the first time $X(t)$ takes the value b .

Second, since π is increasing and $\mu < 0$, an optimal policy cannot involve waiting for the state to increase and then shutting down. Thus, as in the deterministic case,

the optimal policy is characterized by a single threshold b , a lower bound on the region where the plant continues operating. As before this threshold divides the state space into an **action region** $(-\infty, b]$ and an **inaction region** $(b, +\infty)$. Hence the firm's problem can be written as

$$v(x) = \sup_{b \leq x} \mathbf{E}_x \left[\int_0^{T(b)} e^{-rt} \pi(X(t)) dt + e^{-rT(b)} S \right], \quad (15)$$

where $\mathbf{E}_x [\cdot]$ denotes an expectation conditional on the initial state.

First consider the expected return from an arbitrary policy. Let $f(x, b)$ be the total expected discounted return, given the initial state $X(0) = x$, from a policy of operating until the first time the state reaches $b \leq x$, and then shutting down. That is,

$$\begin{aligned} f(x, b) &\equiv \mathbf{E}_x \left[\int_0^{T(b)} e^{-rt} \pi(X(t)) dt + e^{-rT(b)} S \right] \\ &= \int_b^{\infty} \hat{L}(\xi; x, b, +\infty) \pi(\xi) d\xi + \psi(x, b, +\infty) S, \\ &= \frac{1}{J} \left[\int_b^x e^{R_1(x-\xi)} \pi(\xi) d\xi + \int_x^{+\infty} e^{R_2(x-\xi)} \pi(\xi) d\xi \right. \\ &\quad \left. - e^{R_1(x-b)} \frac{1}{J} \int_b^{+\infty} e^{R_2(b-\xi)} \pi(\xi) d\xi \right] + e^{R_1(x-b)} S \\ &\qquad\qquad\qquad \text{all } b \leq x, \end{aligned} \quad (16)$$

where the second line uses the expected discounted local time function \hat{L} to replace the expectation of the integral over time with an integral over states and uses ψ in place of $\mathbf{E}_x [e^{-rT(b)}]$; the third line uses the expressions for \hat{L} and ψ in sections 5.3 and 5.6; and the constants $J > 0$ and $R_1 < 0 < R_2$, which depend on the parameters r, μ and σ^2 , were defined in section 5.2. The first two terms in brackets in (16) represent the expected returns from operating the plant forever, neglecting profits that accrue when the state is below b , and the third subtracts the portion of those expected returns that accrues after b has been reached for the first time. The last is the salvage value.

The firm's problem is then

$$v(x) = \max_{b \leq x} f(x, b),$$

so the first order condition for a maximum at $b \leq x$ is

$$\begin{aligned} 0 &\leq \frac{\partial f(x, b)}{\partial b} \\ &= \int_b^\infty \hat{L}_b(\xi; x, b, +\infty) \pi(\xi) d\xi + \psi_b(x, b, +\infty) S, \\ &= \psi(x, b, +\infty) \left[\int_b^\infty \hat{L}_b(\xi; b, b, +\infty) \pi(\xi) d\xi + \psi_b(b, b, +\infty) S \right] \\ &= e^{R_1(x-b)} \left[(R_1 - R_2) \frac{1}{J} \int_b^{+\infty} e^{R_2(b-\xi)} \pi(\xi) d\xi - R_1 S \right], \end{aligned} \quad (17)$$

with equality if $b < x$,

where the second line uses the fact that $\hat{L}(b; x, b, +\infty) = 0$, and the third uses Propositions 5.5 and 5.6. Note that $\hat{L}_b < 0$ and $\psi_b > 0$, since raising the threshold decreases the expected discounted local time at every level and advances the hitting time for the threshold. Correspondingly, the first term in brackets in (17) is negative, representing the reduction in expected operating profits from increasing the shut-down threshold. These are profits that would have accrued along sample paths where the state fell almost to b , but not quite. Along these paths the state variable can rise to any level before falling again to b , so a small increase in the threshold strictly reduces the expected discounted local time at every level exceeding b . The second term, which is positive, represents the increase in the expected salvage revenue from shutting down earlier.

For the stochastic problem, b^* is the value for which the term in square brackets in (17) is equal to zero. To establish the existence of a solution, note that

$$\frac{R_1 - R_2}{R_1} \frac{1}{J} = \left(\frac{1}{R_2} - \frac{1}{R_1} \right) \frac{R_2}{J} = \frac{R_2}{r},$$

and define

$$\begin{aligned}\Pi(b) &\equiv R_2 \int_b^{+\infty} e^{R_2(b-\xi)} \pi(\xi) d\xi \\ &= R_2 \int_0^{+\infty} e^{-R_2\zeta} \pi(b+\zeta) d\zeta.\end{aligned}$$

Then (17) can be written as

$$-\Pi(b) + rS \geq 0, \quad \text{with equality if } b < x, \quad (18)$$

and the critical value b^* satisfies

$$\Pi(b^*) - rS = 0. \quad (19)$$

Under Assumption 6.1 Π is increasing, since π is, and in addition

$$\lim_{b \rightarrow -\infty} \Pi(b) < rS < \lim_{b \rightarrow +\infty} \Pi(b).$$

Hence (19) has a unique solution and f is concave in b at b^* . If the initial state x exceeds b^* the constraint $b \leq x$ is slack and $b = b^*$. Otherwise the constraint binds and $b = x$. The following proposition summarizes these arguments.

PROPOSITION 6.3: Let r, μ, S and π satisfy Assumption 6.1, and let X be a Brownian motion with parameters μ and $\sigma^2 > 0$. Then there exists a unique value b^* satisfying (19) and the optimal policy for the problem in (15) is

$$b = \min \{x, b^*\}.$$

The value from this policy is

$$\begin{aligned}v(x) &= S, & x \leq b^*, & \quad (20) \\ v(x) &= \frac{1}{J} \int_x^{+\infty} e^{R_2(x-\xi)} \pi(\xi) d\xi + \frac{1}{J} \int_{b^*}^x e^{R_1(x-\xi)} \pi(\xi) d\xi \\ &\quad - e^{R_1(x-b^*)} \frac{1}{J} \int_{b^*}^{+\infty} e^{R_2(b^*-\xi)} \pi(\xi) d\xi + e^{R_1(x-b^*)} S, & x > b^*.\end{aligned}$$

Thus, the optimal policy is:

- (i) if $x \leq b^*$, shut down immediately;
- (ii) if $x > b^*$, operate until the first time the state reaches b^* and then shut down.

The following exercise looks at the relationship between the stochastic and deterministic problem.

Exercise 6.3: Show that the function f in (16) and the first order condition (17) for the stochastic problem converge to their deterministic counterparts (5) and (7) as $\sigma^2 \rightarrow 0$.

3. Using the Hamilton-Jacobi-Bellman equation

The stochastic problem, like the deterministic one, can also be analyzed by solving an appropriate ODE. As before, begin by noting that the optimal shutdown region has the form $(-\infty, b]$. Fix any initial state x and any candidate threshold $b \leq x$, and let $f(x, b)$ be the return from the policy of operating until b is reached and then shutting down. If $x > b$ the policy requires continuing operation, at least for a while. In this case let h be a time interval that is short enough so that the probability of shutting down before h is negligible. Then

$$\begin{aligned} f(x, b) &\approx \mathbb{E}_x \left[\int_0^h e^{-rt} \pi(X(t)) dt + e^{-rh} f(X(h), b) \right] \\ &\approx \pi(x)h + \frac{1}{1+rh} \left[f(x, b) + f_x(x, b)\mu h + \frac{1}{2} f_{xx}(x, b)\sigma^2 h \right], \end{aligned}$$

where the second line uses a second-order Taylor series approximation to $f(X(h), b)$ and an application of Ito's lemma, as in section 5.4. The argument is like the one for the deterministic problem, except that a second-order approximation is needed to capture the variance term in ΔX . Rearrange terms, divide by h and let $h \rightarrow 0$ to get

$$rf(x, b) = \pi(x) + \mu f_x(x, b) + \frac{1}{2}\sigma^2 f_{xx}(x, b), \quad \text{all } x > b. \quad (21)$$

With b fixed this equation is a second-order linear ODE with constant coefficients, and it has exactly the same form as those in section 5.6.

Recall from Chapter 5 the function

$$v_P(x) \equiv \mathbf{E}_x \left[\int_0^\infty e^{-rt} \pi(X(t)) dt \right], \quad \text{all } x, \quad (22)$$

the expected discounted value if the plant is operated forever. Under Assumption 6.1 π is bounded and $r > 0$, so v_P is bounded. As shown in Chapter 5, the function v_P is a particular solution of (21), and the functions $h_i(x) = e^{R_i x}$, $i = 1, 2$, are homogeneous solutions. Hence any solution has the form

$$f(x, b) = v_P(x) + c_1 e^{R_1 x} + c_2 e^{R_2 x},$$

where c_1, c_2 are constants that must be determined from boundary conditions. For the upper boundary, note that as $x \rightarrow \infty$ the stopping time $T(b)$ diverges to $+\infty$. Hence for any b

$$\lim_{x \rightarrow \infty} [f(x, b) - v_P(x)] = 0. \quad (23)$$

Since $R_2 > 0$ it follows that $c_2 = 0$. For the lower boundary note that for $x \leq b$ the policy involves stopping immediately, so

$$f(x, b) = S, \quad x \leq b. \quad (24)$$

Use this fact at $x = b$ to conclude that

$$c_1 = e^{-R_1 b} [S - v_P(b)].$$

Hence for any b ,

$$f(x, b) = \begin{cases} S, & x \leq b, \\ v_P(x) + e^{R_1(x-b)} [S - v_P(b)], & x > b. \end{cases} \quad (25)$$

When the firm shuts down it receives the salvage value S but loses the remaining profit stream $v_P(b)$. Hence for any fixed b the value of a firm that uses b for its

threshold can be written as the value of operating forever, the term $v_P(x)$, plus the expected discounted net gain from shutting down.

So far the threshold b has been arbitrary. Given x , the firm's decision problem is to choose the threshold b to maximize $f(x, b)$,

$$\begin{aligned} v(x) &= \max_{b \leq x} f(x, b) \\ &= v_P(x) + \max_{b \leq x} e^{R_1(x-b)} [S - v_P(b)]. \end{aligned} \quad (26)$$

Hence the condition for a maximum at b is

$$\begin{aligned} 0 &\leq \frac{\partial f(x, b)}{\partial b} \\ &= -e^{R_1(x-b)} \{v'_P(b) + R_1 [S - v_P(b)]\}, \quad \text{w/ eq. if } b < x. \end{aligned} \quad (27)$$

Substituting the expressions in section 5.6 for v_P and v'_P in (27) reproduces (17). As before b^* is the value for which (27) holds with equality.

Note that the maximized value of the second term in (26) is the *value of the option to shut down* in the present context or the *value of exercising control* in more general settings. Clearly for $x \geq b^*$,

$$p(x) = e^{R_1(x-b^*)} [S - v_P(b^*)]$$

is the price of the option in an appropriate market setting. Note that the term $[S - v_P(b^*)]$ is positive, reflecting the fact that the firm has the option to remain open. That is, a necessary condition for exercising control is that doing so raises the total return. The following exercise verifies that the approach in (26) leads to the solution obtained earlier.

- Exercise 6.5:** a. Verify that substituting for v_P and v'_P in (27) produces (17).
 b. Show that $[S - v_P(b^*)] > 0$.

Note that

$$e^{R_1(x-b)} = \psi(x, b, +\infty) = \mathbb{E}_x [e^{-rT(b)}],$$

so f can be written as

$$f(x, b) = v_P(x) + \psi(x, b, +\infty) [S - v_P(b)], \quad \text{all } b \leq x, \quad \text{all } x. \quad (28)$$

Recall from Proposition 5.5 that for any b, B ,

$$\begin{aligned} -\psi_x(b, b, B) &= \psi_b(b, b, B), \\ \psi_b(x, b, B) &= \psi(x, b, B)\psi_b(b, b, B), \quad x \in (b, B). \end{aligned}$$

Use (28) and the first fact about ψ to find that as in the deterministic model

$$-\frac{\partial f(x, b)}{\partial b}\Big|_{x=b} = +\frac{\partial f(x, b)}{\partial x}\Big|_{x=b}, \quad \text{all } b.$$

The interpretation of this fact is as before. Fix a threshold b , suppose that the state is at the threshold, $x = b$, and consider the effect of a small increase h in the state. The firm operates longer and receipt of the salvage value is delayed. Alternatively, reducing the threshold by h has exactly the same effect.

Then use (28) again with the second fact about ψ to find that

$$\begin{aligned} \frac{\partial f(x, b)}{\partial b} &= \psi_b(x, b, +\infty) [S - v_P(b)] - \psi(x, b, +\infty)v'_P(b) \\ &\quad \psi(x, b, +\infty) \{ \psi_b(b, b, B) [S - v_P(b)] - v'_P(b) \} \\ &= \psi(x, b, +\infty) \frac{\partial f(x, b)}{\partial b}\Big|_{x=b}, \quad \text{all } b \leq x, \quad \text{all } x. \end{aligned}$$

In the stochastic environment the function ψ plays the role of a discount factor: the effect of a change in the threshold b if the current state is x is the change it has when b is reached, adjusted for the expected discounted time until that event occurs.

Combine the two results above to conclude that

$$f_b(x, b^*) = 0, \quad \text{all } x \geq b^* \quad \iff \quad f_x(b^*, b^*) = 0.$$

Hence the condition $f_x(b^*, b^*) = 0$ provides a convenient method for characterizing b^* . Specifically, among all the functions $f(x, b)$ satisfying (23) and (24), the optimized

value function $v = f(\cdot, b^*)$ is the one that has the additional property $f_x(b^*, b^*) = 0$. The following result summarizes these conclusions.

PROPOSITION 6.4: Let r, μ, S and π satisfy Assumption 6.1, and let X be a Brownian motion with parameters μ and $\sigma^2 > 0$. The threshold b^* and function v defined in (19) and (20) have the following properties:

a. v satisfies

$$\begin{aligned} rv(x) &= \pi(x) + \mu v'(x) + \frac{1}{2}\sigma^2 v''(x), & x > b^*, \\ v(x) &= S, & x \leq b^*; \end{aligned} \tag{29}$$

b. v has the limiting property

$$\lim_{x \rightarrow \infty} [v(x) - v_P(x)] = 0; \tag{30}$$

c. v is continuous at b^* :

$$\lim_{x \downarrow b^*} v(x) = S; \tag{31}$$

d. the derivative of v is continuous at b^* :

$$\lim_{x \downarrow b^*} v'(x) = 0. \tag{32}$$

PROOF: There is a family of solutions to (29)-(31), indexed by b , and they are as in (25). The only member of this family that in addition satisfies (32) is the one with $b = b^*$, where b^* is defined by (19). ■

Condition (31) is called the **value matching** condition and (32) is called the **smooth pasting** condition. Call (30) the **no bubble** condition. Thus, an alternative way to solve for the optimal value function v and threshold b^* is by using the HJB equation (29) together with the boundary conditions in (30) - (32).

For a heuristic argument that leads to the smooth pasting condition, suppose the threshold b has been chosen and the state is $x = b$, and consider the expected returns

from two strategies: (i) shutting down immediately, and (ii) continuing to operate for a short interval of time h and then deciding what to do. The return from the first strategy is simply $\Pi_1 = S$. The return from the latter can be calculated using the random walk approximation described in section 3.4. The payoff $\pi(b)$ is collected over the time interval h , and then a decision is made. The increment to the state over h is

$$\Delta X = \pm\sigma\sqrt{h},$$

and the probability of an upward jump is

$$p = \frac{1}{2} \left[1 + \frac{\mu\sqrt{h}}{\sigma} \right].$$

Assume that at the end of the time interval h , the firm keeps the plant open if X has increased and shuts it down if X has decreased. Then the expected return from the second strategy is approximately

$$\Pi_2 = \pi(b)h + (1 - rh) \left[pf(b + \sigma\sqrt{h}, b) + (1 - p)S \right].$$

Use a Taylor series expansion to evaluate f and find that

$$\begin{aligned} pf(b + \sigma\sqrt{h}, b) + (1 - p)S &\approx p \left[f(b, b) + f_x(b, b)\sigma\sqrt{h} \right] + (1 - p)S \\ &= S + pf_x(b, b)\sigma\sqrt{h}, \end{aligned}$$

where the second line uses the value matching condition $f(b, b) = S$. Hence the difference between the two payoffs is

$$\begin{aligned} \Pi_2 - \Pi_1 &= \pi(b)h + (1 - rh) \left[S + pf_x(b, b)\sigma\sqrt{h} \right] - S \\ &\approx [\pi(b) - rS]h + pf_x(b, b)\sigma\sqrt{h} \\ &\approx \frac{1}{2}f_x(b, b)\sigma\sqrt{h}, \end{aligned}$$

where the second line drops terms of order higher than h and the third drops those of order higher than \sqrt{h} . If the threshold b is optimal, the firm is indifferent between

the two strategies. This is true if and only if the threshold satisfies

$$f_x(b^*, b^*) = 0,$$

which is exactly the smooth pasting condition (32).

As in the deterministic model, there are two ways to characterize the function f . For the stochastic model the first uses the functions \hat{L} and ψ , while the second starts with the HJB equation.

There are also two methods for characterizing the optimal threshold b^* . The first involves a derivative of $f(x, b)$ with respect to b . The second uses Proposition 5.5 to replace $f_b(x, b^*)$ with the smooth pasting conditions $f_x(b^*, b^*) = 0$.

The next two exercises look at the effects of parameters changes in the shut-down problem and at a start-up problem that is the mirror image of the one studied here.

Exercise 6.5: a. Describe the qualitative effects of small changes in the parameters μ , σ^2 , S and r on the optimal threshold b^* and the maximized value function v .

b. Describe the qualitative effect of each of these parameters on the expected length of time until the option is exercised, conditional on a fixed initial condition $x_0 > b$.

Exercise 6.6: Consider an unemployed worker who continuously receives wage offers. Suppose that the stochastic process $\{X(t)\}$ describes these offers. The worker can accept a job at any time. When he does accept an offer, his wage remains constant forever afterward. The worker receives an unemployment benefit s (a flow) as long as he remains unemployed. The worker lives forever and is interested in maximizing the expected discounted value of his lifetime income, discounted at the constant rate $r > 0$.

a. Describe his optimal strategy when $X(t)$ is a Brownian motion with parameters (μ, σ^2) . What restrictions (if any) are needed to make the problem well behaved?

b. Repeat (a) assuming that $X(t)$ is an Ornstein-Uhlenbeck process with parameters (α, σ^2) .

4. An example

Suppose the profit function is

$$\pi(x) = ae^{\eta x},$$

with $a > 0$ and $0 < \eta < R_2$. Then Assumption 6.1 holds for any salvage value $S > 0$.

In addition

$$\begin{aligned} v_P(x) &= \frac{a}{J} \left[\int_0^\infty e^{-R_2 z} e^{\eta(x+z)} dz + \int_{-\infty}^0 e^{-R_1 z} e^{\eta(x+z)} dz \right] \\ &= aCe^{\eta x}, \end{aligned}$$

where the restriction on η insures that

$$C \equiv \frac{1}{J} \left[\frac{1}{R_2 - \eta} - \frac{1}{R_1 - \eta} \right] > 0$$

is finite.

Hence for an arbitrary threshold b , the value function is

$$\begin{aligned} f(x, b) &= v_P(x) + e^{R_1(x-b)} [S - v_P(b)] \\ &= aCe^{\eta x} + e^{R_1(x-b)} [S - aCe^{\eta b}]. \end{aligned}$$

Figure 6.1 displays the function $f(x, b)$ for $x \in [31, 41]$ and $b \in [30, 40]$, for the parameter values

$$\begin{aligned} \mu &= -.001, & \sigma &= 10, & r &= .045, \\ a &= .03, & \eta &= .02, & S &= 4. \end{aligned}$$

For high values of x , $f(x, b)$ exceeds S and is strictly concave in b over the displayed range. For intermediate values of x it exceeds S , is strictly concave in b for $b < x$, and is constant at $f(x, b) = S$ for $b > x$. For low values of x and b , with $b < x$, the function lies below S , rising to S at $b = x$.

The optimal threshold b^* solves

$$0 = (R_1 - \eta) aC e^{\eta b^*} - R_1 S$$

so

$$\begin{aligned} b^* &= \frac{1}{\eta} \ln \left(\frac{S}{aC} \frac{R_1}{R_1 - \eta} \right) \\ &= \frac{1}{\eta} \ln \left(\frac{rS}{a} \frac{R_2 - \eta}{R_2} \right) = 34.69. \end{aligned}$$

The restriction on η insures that there is a solution, and the value function is

$$v(x) = f(x, b^*) = aC e^{\eta x} + e^{R_1(x-b^*)} [S - aC e^{\eta b^*}].$$

Figure 6.2 displays three cross-sections of f , for $b_1 < b^* < b_2$. Notice that $f(\cdot, b^*)$ lies above the other two curves and is smooth at $x = b^*$; that $f(\cdot, b_1)$ falls below S and has a kink at $x = b_1$; and that $f(\cdot, b_2)$ has a kink at $x = b_2$.

Figure 6.3 displays v_P, v and two additional functions. Note that v approaches v_P from above as x gets large. If the no bubble condition (30) is ignored, then for any threshold b the constants c_1 and c_2 in (25) can be chosen so that the value matching and smooth pasting conditions (31) and (32) hold. Figure 6.3 displays two such “pseudo value functions,” f_1 and f_2 , for $b_1 < b^* < b_2$. Both are similar to v except that neither approaches v_P as $x \rightarrow +\infty$.

5. Notes

The literature on option pricing in financial markets is vast. See Duffie (1988, 1996) and the references there for an introduction. Cox and Ross (1976) and Cox, Ross and Rubinstein (1979) are early and highly readable treatments based on discrete approximation to a Brownian motion.

Option theory has been employed in a wide variety of other areas as well. For example, see MacDonald and Siegel (1985, 1986) for applications to the timing of

entry and exit—investment and shutting down; see Weisbrod (1964), Arrow and Fisher (1974), and Henry (1974) for applications to environmental protection; and see Brennan and Schwartz (1985) for the role of option values in assessing natural resource projects.