CHAPTER 5: USEFUL FORMULAS FOR BROWNIAN MOTIONS

In situations where action involves a fixed cost optimal policies have the property that control is exercised only occasionally. Specifically, optimal policies involve taking action when a state variable reaches an appropriately chosen threshold value. In this chapter methods are developed for analyzing models of this type.

To fix ideas, consider the following example. Suppose the profit flow $g(X)$ of a firm depends on its relative price $X = p - \overline{p}$, where p is the firm's own price and $\overline{p}$ is an aggregate price index, both in log form. Assume that $\overline{p}$ evolves as a Brownian motion. Then in the absence of action by the firm X also evolves as a Brownian motion, with a drift of opposite sign. But at any time the firm also has the option of changing its nominal price p, altering X by a discrete amount. Suppose that the firm adopts a policy of doing so when the relative price reaches either of two critical thresholds, a lower value b or an upper value B. Assume the initial condition $X(0) = x$ lies between the two thresholds, and let $v(x)$ denote the expected discounted return from following the stated policy, discounted at a constant rate r, conditional on the initial state.

Define the stopping time $T = T(b) \wedge T(B)$ as the first time the stochastic process X reaches b or B. The firm's policy involves doing nothing before T, and at T taking an action that may depend on whether b or B has been reached. Hence $v(x)$ can be written as the sum of three terms,

$$\begin{aligned} v(x) = &\text{ expected returns over } [0, T) \\ &+ \text{expected returns over } [T, +\infty) \text{ if } b \text{ is reached before } B \\ &+ \text{expected returns over } [T, +\infty) \text{ if } B \text{ is reached before } b. \end{aligned}$$

Recall from Theorem 3.7 that for a function $f(\cdot)$ that has a Brownian motion X as its argument, the integral of $e^{-rt}f(X)$ along any sample path up to a stopping time T can be written as an integral over states, where the latter integral uses the discounted local time $\hat{\ell}$ as a weighting function. For the first term in $v(x)$ above the expectation of $\hat{\ell}$ is needed. For the discussion here it is useful to indicate the initial condition for the process X. Let $\hat{\ell}(\xi, x; T, r)$ denote the discounted local time at level ξ, given the initial state x. In addition the stopping times of interest here always take the form $T = T(b) \wedge T(B)$, so it is convenient to write $\mathrm{E}[\hat{\ell}]$ in terms of b and B rather than T. For any $b < B$ let

$$\hat{L}(\xi; x, b, B; r) \equiv \mathrm{E}\left[\hat{\ell}(\xi, x, T(b) \wedge T(B); r)\right], \qquad \xi, x \in (b, B), \tag{1}$$

denote the expected local time at ξ before either threshold b or B is reached, given the initial state x.

Let $\mathrm{E}_x[\cdot]$ and $\Pr_x[\cdot]$ denote expectations and probabilities conditional on the initial state x, and let $w(x, b, B)$ denote the first term in $v(x)$, the expected returns before T. Then w can be written in terms of $\hat{L}$,

$$\begin{aligned} w(x, b, B) &\equiv \mathrm{E}_x\left[\int_0^T e^{-rt} g(X(t))dt\right] \\ &= \mathrm{E}\left[\int_{\mathbf{R}} \hat{\ell}(\xi, x, T; r)\, g(\xi)\, d\xi\right] \\ &= \int_b^B \hat{L}(\xi; x, b, B; r)\, g(\xi)\, d\xi. \end{aligned} \tag{2}$$

For the second and third terms define

$$\begin{aligned} \psi(x, b, B) &\equiv \mathrm{E}_x\left[e^{-rT} \mid X(T) = b\right] \Pr_x[X(T) = b], \\ \Psi(x, b, B) &\equiv \mathrm{E}_x\left[e^{-rT} \mid X(T) = B\right] \Pr_x[X(T) = B]. \end{aligned} \tag{3}$$

Thus, $\psi(x, b, B)$ is the expected discounted value of an indicator function for the event of reaching b before B is reached, given the initial state x. The value $\Psi(x, b, B)$ has a similar interpretation, with the roles of b and B reversed.

The functions $\hat{L}, \psi$ and Ψ in (2) and (3) can be used to describe the expected discounted profits of the firm for arbitrary thresholds b and B. Consequently they can also be used to characterize the optimal policy—the thresholds and the value to which the relative price is adjusted—and the associated value function $v(x)$.

Probabilities and long run averages can be characterized as well, by using an interest rate of zero in the expressions above. For example, setting $r = 0$ in (3) gives

$$\begin{aligned} \theta\left(x, b, B\right) &\equiv \Pr_x\left[X(T) = b\right], \\ \Theta\left(x, b, B\right) &\equiv \Pr_x\left[X(T) = B\right]. \end{aligned} \tag{4}$$

For an interpretation, consider a firm that operates over a long period with a fixed price adjustment policy. That is, for some S, b, B, with $b < S < B$, the firm always sets its relative price at $x = S$ when it makes an adjustment and always adjusts when the relative price reaches b or B. Then $\theta(S, b, B)$ is the fraction of adjustments from b in the long run and $\Theta(S, b, B)$ is the fraction from B.

Other features of the long run can also be described. For example, to calculate the average length of time between adjustments consider the function

$$\tau\left(x, b, B\right) \equiv \mathrm{E}_x\left[T(b) \wedge T(B)\right], \tag{5}$$

the expected time until the next adjustment conditional on the current state x. For a firm using the adjustment policy described above $\tau(S, b, B)$ is the average length of time between adjustments. Then note that setting $r = 0$ in (1) gives

$$L(\xi; x, b, B) \equiv \mathrm{E}\left[\ell(\xi, x, T(b) \wedge T(B)\right], \qquad \xi, x \in (b, B),$$

the (undiscounted) expected local time at level ξ. Normalizing this function by the expected time between adjustments gives $L(\cdot; S, b, B)/\tau(S, b, B)$, a density function for the time the firm's price is each level $\xi \in (b, B)$ in the long run.

In settings with a large number of such agents and idiosyncratic shocks the undiscounted functions also describe cross sectional averages. In particular, suppose that there is a continuum of agents with total mass one and that the shocks are i.i.d. across agents. (This assumption makes no sense for the shock in the menu cost model, but it is reasonable in other settings.) A setting of this type has a stationary cross-sectional distribution for the state variable, and the system converges to that distribution in the long run. Individual agents experience fluctuations as their own state rises and falls, but aggregates—the cross-sectional distribution of the state and the arrival rates at the two thresholds—are constant. That is, $\theta(S, b, B)$ and $\Theta(S, b, B)$ describe the fractions of adjustments at each threshold, $L(\cdot; S, b, B)/\tau(S, b, B)$ is the cross-sectional density for price, $1/\tau(S, b, B)$ is aggregate adjustment rate, and so on.

In this chapter closed form solutions are derived for the functions $\hat{L}, \psi, \Psi$, etc. for Brownian motions and geometric Brownian motions. For more general diffusions sharp characterizations of these functions are obtained. In section 1 conditions are provided under which the stopping time T is finite with probability one, and in section 2 this result is used to apply the Optional Stopping Theorem to Wald martingales associated with X. In section 3 the resulting relationship is used to obtain solutions for ψ, Ψ, θ and Θ, and properties of those functions are developed.

In section 4 a different approach is developed for Brownian motions, one that involves analyzing ODEs of a certain form. In section 5 the ODE is solved for $r = 0$ to obtain L, θ, Θ, and τ, and in section 6 it is solved for $r > 0$ to obtain ψ, Ψ, and L. In sections 7 - 9 the argument is extended to cover general diffusions, including geometric Brownian motions and Ornstein-Uhlenbeck processes. For the former closed form solutions are obtained.

1. Stopping times defined by thresholds

Let X be a Brownian motion with parameters (μ, σ^2) and (finite) initial value $X(0) = x$, and let b, B be threshold values satisfying $-\infty \leq b \leq x \leq B \leq +\infty$. As above, let E_x and $\Pr_x$ denote expectations and probabilities conditional on the initial value x. In this section conditions are described under which the stopping time $T = T(b) \wedge T(B)$ is finite with probability one.

The following assumption, which puts restrictions on b, B, and σ^2 for any μ is needed for the result.

ASSUMPTION 5.1: a. If $\mu > 0$, then $B < \infty$;

b. if $\mu < 0$, then $b > -\infty$;

c. if $\mu = 0$, then $\sigma^2 > 0$ and either $B < \infty$ or $b > -\infty$ or both.

The logic behind these restrictions is as follows. Suppose the drift is positive, $\mu > 0$. As $b \to -\infty$ with B fixed, the probability that the lower threshold is reached first goes to zero, but the probability that the upper threshold is reached approaches unity. Hence T is finite with probability one. Moreover, the argument applies even if $\sigma^2 = 0$. However, if $B = +\infty$ with b finite, there are outcome paths where b is never reached, so $T = \infty$ with positive probability. An analogous argument applies for $\mu < 0$ with b and B reversed.

If $\mu = 0$, clearly the variance σ^2 must be strictly positive. Thus limits as $\sigma^2 \downarrow 0$ are well behaved if and only if $\mu \neq 0$. With $\mu = 0$ it suffices if one threshold is finite.

THEOREM 5.1: Let X be a (μ, σ^2) Brownian motion, let μ, σ^2, b, B satisfy Assumption 5.1, let the initial condition x satisfy $b \leq x \leq B$, and let T be the stopping time $T = T(b) \wedge T(B)$. Then $\Pr_x [T < \infty] = 1$.

PROOF: If $\mu > 0$ then $B < \infty$, so

$$\Pr_x [T > t] \leq \Pr_x [X(t) < B], \text{ all } t \geq 0.$$

If $\sigma = 0$, the right side is zero for all $t > (B - x) / \mu$. If $\sigma > 0$,

$$\Pr_x [X(t) < B] = F_N(B; x + \mu t, \sigma^2 t),$$

where $F_N(\cdot; m, s^2)$ denotes the c.d.f. for a normal distribution with parameters (m, s^2). Since $\mu > 0$, the term on the right goes to zero as $t \to \infty$.

If $\mu < 0$ then $b > -\infty$ and a symmetric argument applies.

If $\mu = 0$ then $\sigma^2 > 0$ and either $|b| < \infty$ or $|B| < \infty$ or both. Suppose both thresholds are finite. Then

$$\begin{aligned} \Pr_x[T > t] &\leq \Pr_x[b < X(t) < B] \\ &= F_N(B; x, \sigma^2 t) - F_N(b; x, \sigma^2 t). \end{aligned}$$

Since $\sigma^2 > 0$, the right side goes to zero as $t \to \infty$.

Suppose $|B| < \infty$ and $b = +\infty$. As will be shown below in section 3, for any finite b and $T = T(b) \wedge T(B)$

$$\Pr_x[X(T) = B] = \frac{x - b}{B - b}.$$

As $b \to -\infty$ this probability goes to one. Hence $\Pr_x[T < \infty] = 1$ for $b = -\infty$. A similar argument hold if $|b| < \infty$. ■

2. Expected values for Wald martingales

Recall from Chapter 4 that for any $\rho \neq 0$ and

$$q(\rho) \equiv \rho\mu + \frac{1}{2}(\rho\sigma)^2, \tag{6}$$

the stochastic process

$$M(t; \rho) \equiv \exp\{\rho X(t) - q(\rho)t\}, \quad \text{all } t, \tag{7}$$

is a **Wald martingale** with parameter ρ. For $\rho = 0$ let

$$\begin{aligned} M(t; 0) &\equiv \lim_{\rho \downarrow 0} \frac{1}{\rho} M(t; \rho) \\ &= X(t) - \mu t, \quad \text{all } t. \end{aligned} \tag{8}$$

In economic applications a discount rate $r \geq 0$ is typically given, and the issue is to find values ρ for which $q(\rho) = r$. From (6), these values are roots of the quadratic

$$\frac{1}{2}\sigma^2 R^2 + \mu R - r = 0. \tag{9}$$

If $\sigma^2 > 0$, these roots are

$$R_1 = \frac{-\mu - J}{\sigma^2} \leq 0, \quad \text{and} \quad R_2 = \frac{-\mu + J}{\sigma^2} \geq 0, \tag{10}$$

where

$$J \equiv \left(\mu^2 + 2r\sigma^2\right)^{1/2} \geq |\mu| \,. \tag{11}$$

There are then three cases:

(i) if $r > 0$, the roots are of opposite sign, $R_1 < 0 < R_2$;

(ii) if $r = 0$ and $\mu \neq 0$, the roots are $R_i = -2\mu/\sigma^2$ and $R_j = 0$, with the allocation depending on the sign of μ; and

(iii) if $r = 0$ and $\mu = 0$, then $R_1 = R_2 = 0$ is a repeated root.

If $\sigma^2 = 0$, then (9) has one root, $R = r/\mu$. Corresponding to this fact, one root in (10) converges to a finite limit as $\sigma \downarrow 0$, and the other diverges. Hence for $\sigma^2 = 0$,

$$\begin{aligned} R_1 &= \frac{r}{\mu} \leq 0, \text{ and } R_2 \text{ is undefined,} \quad \text{if } \mu < 0, \\ R_2 &= \frac{r}{\mu} \geq 0, \text{ and } R_1 \text{ is undefined,} \quad \text{if } \mu > 0. \end{aligned} \tag{12}$$

(If $\sigma^2 = \mu = 0$, a case excluded by Assumption 5.1, then (9) has no solution. Correspondingly, for $\mu = 0$ both of the roots in (10) diverge as $\sigma \downarrow 0$.)

Notice that the roots depend on r, μ and σ^2 only through their ratios. Since changing the time unit—for example, measuring time in months rather than years—requires proportionate changes in (μ, σ^2, r), such a change leaves the roots unaltered.

Exercise 5.1: Describe the effect on $|R_1|$, $|R_2|$, and $R_2 - R_1$ of changes in r, μ, and σ^2.

The next result shows that if both b and B are finite then the Optional Stopping Theorem applies to M. The assumption that both thresholds are finite is strong but it simplifies the proof. Cases where $b = -\infty$ or $B = +\infty$ will be treated in sections 4 - 7 with a different approach.

THEOREM 5.2: Let the hypotheses of Theorem 5.1 hold, and in addition assume $|b|, |B| < \infty$. Then

$$\mathrm{E}_x\left[M(T;R_i)\right] = M(0;R_i), \tag{13}$$

in the following cases:

a. if $\sigma^2 > 0,\ r > 0$, and $R_i,\ i = 1, 2$, are defined in (10);

b. if $\sigma^2 > 0,\ r = 0$, and $R_i = -2\mu/\sigma^2$;

c. if $\sigma^2 = 0,\ r \geq 0$, and $R_i = r/\mu$.

PROOF: It has been shown already that M is a martingale and T is a stopping time, so it suffices to show that the hypotheses of Theorem 4.4 hold, that

$$\begin{array}{ll} \text{(i)} & \Pr\{T < \infty\} = 1, \\ \text{(ii)} & \mathrm{E}_x\left[|M(T;R_i)|\right] < \infty, \\ \text{(iii)} & \lim_{t\to\infty} \mathrm{E}_x\left[|M(t;R_i)| \ \ I_{\{T>t\}}\right] = 0. \end{array}$$

Theorem 5.1 establishes (i). For the other conditions there are several cases.

If $\sigma^2 = 0$ then Assumption 5.1 requires $\mu \neq 0$. $X(t)$ is deterministic, $M(t;R_i) = x$ is constant, and (ii) - (iii) follow immediately.

If $\sigma^2 > 0$ and $R_i \neq 0$, let

$$A_i = \max\left\{e^{R_i b}, e^{R_i B}\right\}.$$

Then

$$|M(T;R_i)| \leq A_i e^{-rT} \leq A_i, \quad \text{all } \omega,$$

so (ii) holds. In addition $|M(t;R_i)| < A_i$, all $t < T$, so

$$\begin{aligned} \lim_{t\to\infty} \mathrm{E}_x\left[|M(t;R_i)| \ \ I_{\{T>t\}}\right] &\leq A_i \lim_{t\to\infty} \Pr{}_x\left[T > t\right] \\ &= 0, \end{aligned}$$

where the second line uses Theorem 5.1. Hence (iii) holds.

If $\sigma^2 > 0$ and $R_i = 0$, then $\mu = 0$. Hence $M(t;0) = X(t)$, and

$$|M(T;0)| = |X(T)| \leq A_i = \max\{|b|, |B|\}, \quad \text{all } \omega,$$

so (ii) holds. In addition $|M(t;0)| < A_i$, all $t < T$, and the argument for (iii) is as before. ■

3. The functions ψ and Ψ

Recall the functions $\psi(x, b, B; r)$ and $\Psi(x, b, B; r)$ defined in (3), the expected discounted values of indicator functions for reaching the lower threshold before the upper one (ψ) and the reverse (Ψ), and the functions $\theta(x, b, B)$ and $\Theta(x, b, B)$, the probabilities of these events. In this section explicit formulas are derived for these functions, and some of their properties are established. The proofs here use Theorem 5.2, so they require $|b|$,$|B| < \infty$. The results hold even without that assumption, however, as will be seen later.

The following result characterizes ψ and Ψ for $r > 0$.

Proposition 5.3: Let X be a (μ, σ^2) Brownian motion with initial condition $x \in [b, B]$, let T be the stopping time $T = T(b) \wedge T(B)$, and let $r > 0$. If $\sigma^2 > 0$, then

$$\begin{aligned} \psi(x, b, B; r) &= \frac{e^{R_1 x}e^{R_2 B} - e^{R_2 x}e^{R_1 B}}{e^{R_1 b}e^{R_2 B} - e^{R_2 b}e^{R_1 B}}, \\ \Psi(x, b, B; r) &= \frac{e^{R_1 b}e^{R_2 x} - e^{R_2 b}e^{R_1 x}}{e^{R_1 b}e^{R_2 B} - e^{R_2 b}e^{R_1 B}}, \end{aligned} \tag{14}$$

where R_1 and R_2 are defined in (10).

If $\sigma^2 = 0$ and $\mu \neq 0$, then

$$\begin{aligned} &\psi(x, b, B; r) = 0, && \Psi(x, b, B; r) = e^{R_2(x-B)}, && \text{if } \mu > 0, \\ &\psi(x, b, B; r) = e^{R_1(x-b)}, && \Psi(x, b, B; r) = 0, && \text{if } \mu < 0, \end{aligned}$$

where R_1 and R_2 are defined in (12).

PROOF: Suppose $\sigma^2 > 0$. Since $r > 0$, the roots in (10) are $R_1 < 0 < R_2$. Let $M(t; R_i)$, $i = 1, 2$, be as in (7). Theorem 5.2 implies (13) holds for $i = 1, 2$. Break the expression on the left in (13) into two parts corresponding to stops at the lower and upper thresholds, and substitute from (7) to find that

$$\begin{aligned}
e^{R_i x} &= \mathrm{E}_x \left[M(T; R_i) \mid X(T) = b \right] \mathrm{Pr}_x \left[X(T) = b \right] \\
&\quad + \mathrm{E}_x \left[M(T; R_i) \mid X(T) = B \right] \mathrm{Pr}_x \left[X(T) = B \right] \\
&= \mathrm{E}_x \left[\exp\left\{ R_i X(T) - q(R_i) T \right\} \mid X(T) = b \right] \mathrm{Pr}_x \left[X(T) = b \right] \\
&\quad + \mathrm{E}_x \left[\exp\left\{ R_i X(T) - q(R_i) T \right\} \mid X(T) = B \right] \mathrm{Pr}_x \left[X(T) = B \right] \\
&= e^{R_i b} \mathrm{E}_x \left[e^{-rT} \mid X(T) = b \right] \mathrm{Pr}_x \left[X(T) = b \right] \\
&\quad + e^{R_i B} \mathrm{E}_x \left[e^{-rT} \mid X(T) = B \right] \mathrm{Pr}_x \left[X(T) = B \right] \\
&= e^{R_i b} \psi(x, b, B) + e^{R_i B} \Psi(x, b, B), \qquad i = 1, 2, \quad x \in [b, B]. \qquad (15)
\end{aligned}$$

This pair of linear equations in $\psi(x)$ and $\Psi(x)$ has the solution in (14).

If $\sigma^2 = 0$, let R_1 or R_2 be the one root in (12). Apply the argument above to the one (relevant) threshold to get the solution. ■

Figure 5.1 displays the function $\psi(x, b, B)$ and shows the effect of changes in the parameters r, μ, and σ. The function is in all cases decreasing, with $\psi(b) = 1$ and $\psi(B) = 0$. Notice that parameter changes alter $\psi(x)$ through three channels, by changing the probability that b is reached before B, by changing the time elapsed before b is reached, and by changing the discounting before b is reached. An increase in the interest rate r bends the curve downward, by increasing the discounting. An increase in μ reduces the probability that the lower threshold is reached first, and on paths where b still reached first it increases the time. A reduction in σ has similar effects.

For $r = 0$, the functions ψ and Ψ are simply the probabilities that the lower and upper thresholds are reached first, the functions $\theta(x, b, B)$ and $\Theta(x, b, B)$ defined in

(4). An analogous argument provides closed forms for these functions.

PROPOSITION 5.4: Let X be a (μ, σ^2) Brownian motion with initial condition $x \in [b, B]$, and let T be the stopping time $T = T(b) \wedge T(B)$. If $\sigma^2 > 0$, then

$$\begin{aligned} \theta(x, b, B) &= \frac{e^{-\delta B} - e^{-\delta x}}{e^{-\delta B} - e^{-\delta b}}, \\ \Theta(x, b, B) &= \frac{e^{-\delta x} - e^{-\delta b}}{e^{-\delta B} - e^{-\delta b}}, \quad \text{if } \mu \neq 0, \end{aligned} \tag{16}$$

where $\delta \equiv 2\mu/\sigma^2$, and

$$\begin{aligned} \theta(x, b, B) &= \frac{B - x}{B - b}, \\ \Theta(x, b, B) &= \frac{x - b}{B - b}, \quad \text{if } \mu = 0. \end{aligned}$$

If $\sigma^2 = 0$ and $\mu \neq 0$, then

$$\begin{aligned} &\theta(x, b, B) = 0, \quad \Theta(x, b, B) = 1, \quad \text{if } \mu > 0, \\ &\theta(x, b, B) = 1, \quad \Theta(x, b, B) = 0, \quad \text{if } \mu < 0. \end{aligned}$$

PROOF: In all cases Theorem 5.1 implies

$$1 = \theta(x, b, B) + \Theta(x, b, B). \tag{17}$$

If $\sigma^2 = 0$ the result is immediate. If $\sigma^2 > 0$, recall that for $r = 0$ the roots in (10) are $R_i = -\delta$ and $R_j = 0$. Theorem 5.2 applies to the root $R_i = -\delta$, so

$$\begin{aligned} e^{-\delta x} &= e^{-\delta b}\theta(x, b, B) + e^{-\delta B}\Theta(x, b, B), \qquad \text{if } \mu \neq 0, \\ x &= b\theta(x, b, B) + B\Theta(x, b, B), \qquad \text{if } \mu = 0. \end{aligned}$$

This equation and (17) are a linear pair with the solution in (16). ■

Figure 5.2 displays the function $\theta(x, b, B)$ and the effect of changes in δ. The function is decreasing, with $\theta(b) = 1$ and $\theta(B) = 0$. For $\delta = 0$ the function is linear. Positive values for δ bow the function ever more strongly downward, and negative values bow it upward. The function $\Theta(x,b,B) = 1-\theta(x,b,B)$ is the complement of the one displayed.

The functions ψ and Ψ have several important properties. All of these hold for $r=0$, so θ and Θ have these properties as well. First, notice that they can be written in terms of just the differences $(x-b)$ and $(x-B)$ (and their difference, $(B-b)$):

$$\begin{aligned}\psi(x,b,B) &= \frac{e^{R_1(x-B)}-e^{R_2(x-B)}}{e^{R_1(b-B)}-e^{R_2(b-B)}}, \\ \Psi(x,b,B) &= \frac{e^{R_1(x-b)}-e^{R_2(x-b)}}{e^{R_1(B-b)}-e^{R_2(B-b)}}, \quad x\in[b,B]\,.\end{aligned} \tag{18}$$

Since X is a Brownian motion, its increments do not depend on its current level, so translating $x,b,$ and B leaves ψ and Ψ unchanged. Thus, it is clear from (18) that

$$\psi_b+\psi_x+\psi_B=0, \qquad \text{and} \quad \Psi_b+\Psi_x+\Psi_B=0. \tag{19}$$

Next, note that $\psi_b(x)$ is the effect of an increase in the threshold b on the expected discounted value of an indicator function for reaching the threshold b before B is reached, given the initial state x. Proposition 5.5 shows that $\psi_b(x)$ is equal to the effect of the change conditional on arriving at b, $\psi_b(b)$, multiplied by the conditioning factor $\psi(x)$, which adjusts for the probability that this event occurs and the appropriate discounting. It also shows that conditional on $x=b$, increasing x and b together has no effect ψ. Similar conclusions hold for Ψ and at B.

Proposition 5.5: For any $r\geq 0$, the functions ψ and Ψ satisfy

$$\begin{aligned}&\psi_b(x)=\psi(x)\psi_b(b)\,, && \Psi_b(x)=\psi(x)\Psi_b(b)\,,\\ &\psi_B(x)=\Psi(x)\psi_B(B), && \Psi_B(x)=\Psi(x)\Psi_B(B)\,,\\ &\psi_b(b)+\psi_x(b)=0, && \Psi_b(b)+\Psi_x(b)=0,\\ &\psi_B(B)+\psi_x(B)=0, && \Psi_B(B)+\Psi_x(B)=0, \qquad x\in(b,B)\,.\end{aligned}$$

Proof: Suppose $r>0$ and consider the claims for ψ, those in the first column. The first follows immediately from (14) or (18). For the second note that

$$\psi(x,b,B)=\frac{e^{R_1x}e^{(R_2-R_1)B}-e^{R_2x}}{e^{R_1b}e^{(R_2-R_1)B}-e^{R_2b}},$$

so

$$\psi_B(x) = \frac{(R_2 - R_1)\, e^{(R_2-R_1)B}}{e^{R_1 b} e^{(R_2-R_1)B} - e^{R_2 b}} \left[e^{R_1 x} - \psi(x) e^{R_1 b} \right].$$

Since $\psi(B) = 0$, it follows that

$$\psi_B(B) = \frac{(R_2 - R_1)\, e^{R_2 B}}{e^{R_1 b} e^{(R_2-R_1)B} - e^{R_2 b}}.$$

Using the latter expression and (15) in the expression for $\psi_B(x)$ establishes the second claim. Since $\psi(b) = 1$, it follows from the expression above that $\psi_B(b) = 0$. Hence the third claim follows from (19). A similar argument establishes the fourth. Analogous arguments establish the claims for Ψ and the claims when $r = 0$. ■

The arguments in this section apply when the underlying process is a Brownian motion, and similar arguments can be made for a geometric Brownian motion. But for more general diffusions a different approach is needed, and cases with only one threshold are also treated more easily by this second approach. This approach is developed for Brownian motions in sections 4 - 6, where it is used to derive the functions τ, L, and $\hat{L}$, and then extended to general diffusions in sections 7 - 9.

4. ODEs for Brownian motions

Let X be as above and consider the function $\psi(x)$. Fix an initial state in the interior of the interval of interest, $x \in (b, B)$, and consider an increment of time $h > 0$ sufficiently small so that the probability of reaching b or B is negligible. Then

$$\psi(x) \approx e^{-rh} \mathrm{E}_x \left[\psi(X(h)) \right], \quad x \in (b, B),$$

where $X(h)$ is the value of the state variable at date h. Use Ito's lemma and the approximation $e^{rh} \approx (1 + rh)$ to find that

$$\begin{aligned} (1+rh)\,\psi(x) &\approx \mathrm{E}_x \left[\psi(x) + \psi'(x)\Delta X + \frac{1}{2}\psi''(x) (\Delta X)^2 \right] \\ &\approx \psi(x) + \psi'(x)\mu h + \frac{1}{2}\psi''(x)\sigma^2 h, \end{aligned}$$

where $\Delta X = X(h) - x$ denotes the (random) increment to the state over the time increment h. The approximation is arbitrarily good as $h \to 0$, so

$$r\psi(x) = \mu\psi'(x) + \frac{1}{2}\sigma^2\psi''(x), \quad x \in (b, B).$$

The boundary conditions are obviously $\psi(b) = 1$ and $\psi(B) = 0$. Use this argument with $r = 0$ for $\theta(x)$, and use a similar construction with the boundary conditions reversed for $\Psi(x)$ and $\Theta(x)$.

For the function $w(x)$ defined in (2) an analogous argument establishes that

$$w(x) \approx g(x)h + \frac{1}{1+rh}\left[w(x) + w'(x)\mu h + \frac{1}{2}w''(x)\sigma^2 h\right],$$

so the relevant ODE is

$$rw(x) = g(x) + \mu w'(x) + \frac{1}{2}\sigma^2 w''(x), \quad x \in (b, B).$$

Clearly the boundary conditions are $w(b) = w(B) = 0$. For $r = 0$ and $g(x) = 1$, $w(x)$ is the expected value of the stopping time, the function

$$\tau(x, b, B) \equiv \mathrm{E}_x\left[T(b) \wedge T(B)\right]$$

defined in (5).

Thus, each example leads to an equation of the form

$$\frac{1}{2}\sigma^2 f''(x) + \mu f'(x) - rf(x) = -g(x), \quad x \in (b, B), \tag{20}$$

where the function $g(x)$ is given. This equation, which is a second order linear ODE with fixed coefficients, is called a Hamilton-Jacobi-Bellman equation. Equations of this type appear in many applications, with the solution depending on the function g and the boundary conditions.

If $\sigma^2 = 0$ and $\mu \neq 0$, this equation is first order, with solution

$$f(x)e^{-(r/\mu)x} = c_0 - \frac{1}{\mu}\int^x g(\xi)d\xi,$$

and if $\sigma^2 = \mu = 0$, the solution is $f(x) = g(x)/r$. The arguments below will assume $\sigma^2 > 0$. It will also be assumed throughout that g is piecewise continuous, so it can be integrated.

5. Solutions for Brownian motions when $r = 0$

If $r = 0$ (20) can be written as a first-order equation in the function $\phi \equiv f'$,

$$\phi'(x) + \delta\phi(x) = -\frac{2g(x)}{\sigma^2}, \quad x \in (b, B),$$

where as before $\delta \equiv 2\mu/\sigma^2$. Clearly solutions have the form

$$\phi(x)e^{\delta x} = c_1 - \frac{2}{\sigma^2}\int^x g(\xi)e^{\delta\xi}d\xi.$$

Integrate again to obtain f, and find that if $r = 0$ any solution to (20) has the form

$$f(x) = c_0 + \int^x \left[e^{-\delta z}c_1 - \frac{2}{\sigma^2}\int^z g(\xi)e^{\delta(\xi - z)}d\xi\right]dz, \quad x \in [b, B]. \tag{21}$$

As usual, the lower limits of integration in (21) can be chosen for convenience, and the constants c_0 and c_1 are determined by boundary conditions.

Using b for both limits, reversing the order of integration, and using the boundary condition at b to eliminate c_0 gives

$$\begin{aligned} f(x) &= f(b) - \frac{1}{\delta}\left(e^{-\delta x} - e^{-\delta b}\right)c_1 - \frac{1}{\mu}\int_b^x \left[1 - e^{\delta(\xi - x)}\right]g(\xi)d\xi, && \text{if } \mu \neq 0, \quad (22) \\ f(x) &= f(b) + (x - b)\,c_1 - \frac{2}{\sigma^2}\int_b^x (x - \xi)\,g(\xi)d\xi, && \text{if } \mu = 0, \end{aligned}$$

where c_1 is determined by the boundary condition at B. Reversing the roles of b and B produces similar expressions.

For the function θ use $g(\cdot) \equiv 0$ and the boundary conditions $\theta(b) = 1$ and $\theta(B) = 0$. For Θ reverse the boundary conditions.

Exercise 5.2: Verify that the solutions for θ and Θ obtained from (22) agree with those in Proposition 5.5.

Notice that the argument here requires $\sigma^2 > 0$ but it does not require Assumption 5.1. The results must be interpreted carefully if Assumption 5.1 fails, however. For example, suppose $\mu > 0$. Then $\delta > 0$, and Assumption 5.1 allows $b = -\infty$ but not $B = +\infty$. Consider both cases. Since $\lim_{b\to-\infty} e^{-\delta b} = +\infty$, (16) implies

$$\begin{aligned}\lim_{b\to-\infty} \theta(x,b,B) &= 0 \\ \lim_{b\to-\infty} \Theta(x,b,B) &= 1, \qquad \text{if } \mu > 0,\end{aligned}$$

in agreement with Theorem 5.1. On the other hand $\lim_{B\to+\infty} e^{-\delta B} = 0$, so (16) implies

$$\begin{aligned}\lim_{B\to+\infty} \theta(x,b,B) &= e^{-\delta(x-b)} > 0, \\ \lim_{B\to+\infty} \Theta(x,b,B) &= 1 - e^{-\delta(x-b)} > 0, \qquad \text{if } \mu > 0.\end{aligned}$$

Here $\Theta(x, b, +\infty) > 0$ is the probability that the lower threshold is never reached. The situation is symmetric for $\mu < 0$. If $\mu = 0$ the probabilities go to zero and one.

For the function τ defined in (5) use $g(\cdot) \equiv 1$ and the boundary conditions $\tau(b) = \tau(B) = 0$ to obtain

$$\begin{aligned}\tau(x,b,B) &= \frac{1}{\mu}\frac{e^{-\delta x}(B-b) - e^{-\delta b}(B-x) - e^{-\delta B}(x-b)}{e^{-\delta B} - e^{-\delta b}}, \qquad \text{if } \mu \neq 0, \quad (23) \\ \tau(x,b,B) &= \frac{1}{\sigma^2}(B-x)(x-b), \qquad \text{if } \mu = 0.\end{aligned}$$

Exercise 5.3: Use (21) or (22) to obtain (23).

Again, this expression holds even if one of the thresholds diverges. For example, suppose the drift is positive. If the lower threshold diverges

$$\lim_{b\to-\infty} \tau(x) = \frac{1}{\mu}(B-x), \qquad \text{if } \mu > 0,$$

so the expected first hitting time for B is finite. If the upper threshold diverges

$$\lim_{B\to+\infty} \tau(x) = \lim_{B\to+\infty} \frac{1}{\mu}\left[1 - e^{-\delta(x-b)}\right] B = +\infty, \qquad \text{if } \mu > 0.$$

Since the probability that b is never reached is positive, the expected time is infinite. If $\mu = 0$ then $\tau(x) \to +\infty$ as either threshold diverges.

Figure 5.3a displays $\tau(x, b, B)$ and shows the effects of increasing the drift μ. In each case τ is hump shaped, and higher values for the drift flatten the hump. Figure 5.3b shows the effects of increasing the standard deviation σ, which also flattens the hump.

Finally, recall from section 2.5 the function

$$m(A, T, \omega) = \int_0^T 1_A (X(s, \omega)) \, ds, \quad \text{all } A \in \mathfrak{B}, \ t \geq 0, \ \omega \in \Omega,$$

the undiscounted occupancy measure for the set A up to the stopping time T. Here we are interested in the expected value of this occupancy measure for the sets of the form $[b, z]$, for the stopping time $T = T(b) \wedge T(B)$, conditional on the initial state $X(0) = x$. Call this function

$$\begin{aligned} M(z; x, b, B) &\equiv \mathrm{E}_x \left[\int_0^{T(b) \wedge T(B)} 1_{[b,z]}(X(t)) \, dt \right] \\ &= \mathrm{E}_x \left[m([b, z], T(b) \wedge T(B)) \right]. \end{aligned}$$

For fixed x, $M(\cdot\,; x, b, B)$ is like a c.d.f., with $M(B; x, b, B) = \tau(x, b, B)$.

To construct M, however, one must proceed the other way around, with a fixed function $g = 1_{[b,z]}$. That is, fix z, b, B and apply (22) with $g = 1_{[b,z]}$ and the boundary conditions $M(z; b) = M(z; B) = 0$. Doing so gives

$$\begin{aligned} M(z; x) &= \frac{1}{\mu} \Theta(x) \left[(z - b) - \frac{1}{\delta} e^{-\delta B} \left(e^{\delta z} - e^{\delta b} \right) \right] \\ &\quad - \frac{1}{\mu} \int_b^{\min\{x,z\}} \left[1 - e^{\delta(\xi - x)} \right] d\xi, \qquad \text{if } \mu \neq 0, \\ M(z; x) &= \frac{1}{\sigma^2} \Theta(x) (z - b) \left[(B - z) + (B - b) \right] \\ &\quad - \frac{2}{\sigma^2} \int_b^{\min\{x,z\}} (x - \xi) \, d\xi, \qquad \text{if } \mu = 0. \end{aligned}$$

Note that M is continuous and is differentiable except at $z = x$.

The expected local time function $L(\cdot; x, b, B)$ is found by differentiating M with respect to z. Hence

$$\begin{aligned} L(z; x, b, B) &= \frac{1}{\mu}\left\{\Theta(x)\left[1-e^{-\delta(B-z)}\right]-\left[1-e^{\delta\min\{(z-x),0\}}\right]\right\}, \qquad \mu \neq 0, \\ L(z; x, b, B) &= \frac{2}{\sigma^2}\left[\Theta(x)\left(B-z\right)+\min\left\{\left(z-x\right),0\right\}\right], \qquad \mu = 0. \end{aligned}$$

Note that L is continuous but has a kink at $z = x$. Figure 5.4a displays $L(\cdot; x_i, b, B)$, for $\sigma = 1$ and $\mu = 0.1$, for three initial conditions x_i. Figure 5.4b is a similar plot for a higher drift, $\mu = 0.3$.

6. Solutions for Brownian motions when $r > 0$

Similar constructions can be used when the discount rate is positive, when $r > 0$. The main difference is that the differential equation (20) is second order and requires a slightly different argument. Recall that all of the solutions of a second order ODE can be written as

$$f(x) = f_P(x) + a_1 h_1(x) + a_1 h_1(x),$$

where f_P is any particular solution, h_1, h_2 are homogeneous solutions, and a_1, a_2 are constant that are determined by the boundary conditions.

If $r > 0$, the homogeneous equation corresponding to (20) is

$$\frac{1}{2}\sigma^2 f''(x) + \mu f'(x) - rf(x) = 0.$$

Recall the constants $R_1 < 0 < R_2$ and $J > 0$ defined in (10) and (11). Clearly the functions

$$f_i(x) = e^{R_i x}, \quad i = 1, 2,$$

are solutions to the homogeneous equation. In addition, since

$$\frac{1}{2}\sigma^2\left(R_2 - R_1\right) = J,$$

it follows that

$$f_P(x) = \frac{1}{J}\left[\int^x e^{R_1(x-z)}g(z)dz + \int_x e^{R_2(x-z)}g(z)dz\right],$$

is a particular solution to (20). Consequently, if $\sigma^2 > 0$ and $r > 0$, any solution to (20) has the form

$$f(x) = \frac{1}{J}\left[\int^x e^{R_1(x-z)}g(z)dz + \int_x e^{R_2(x-z)}g(z)dz\right] + c_1 e^{R_1 x} + c_2 e^{R_2 x}. \qquad (24)$$

This equation can be solved—with different choices for g and the boundary conditions—to obtain ψ, Ψ and $\hat{L}$.

For the function ψ use $g(x) \equiv 0$ and the boundary conditions $\psi(b) = 1$ and $\psi(B) = 0$. For Ψ reverse the boundary conditions.

Exercise 5.4: a. Verify that the solutions obtained from (24) for ψ and Ψ agree with those in Proposition 5.3.

For the function $w(x, b, B)$ defined in (2), $g(\cdot)$ is arbitrary and the boundary conditions are $w(b) = w(B) = 0$. Use b and B for the limits of integration in (24), to find that

$$\begin{aligned} w(x) \quad &= \quad \frac{1}{J}\int_x^B e^{R_2(x-z)}g(z)dz + \frac{1}{J}\int_b^x e^{R_1(x-z)}g(z)dz \qquad (25) \\ &\quad -\Psi(x)\frac{1}{J}\int_b^B e^{R_1(B-z)}g(z)dz - \psi(x)\frac{1}{J}\int_b^B e^{R_2(b-z)}g(z)dz. \end{aligned}$$

Since (25) holds for any return function g, the expected discounted local time function $\hat{L}(z; x, b, B; r)$ is

$$\begin{aligned} \hat{L}(z; x, b, B; r) \quad &= \quad \frac{1}{J}\left[e^{R_i(x-z)} - \Psi(x)e^{R_1(B-z)} - \psi(x)e^{R_2(b-z)}\right], \\ \text{where} \quad i \quad &= \quad \begin{cases} 1, & \text{if } b \le z \le x, \\ 2, & \text{if } x \le z \le B. \end{cases} \end{aligned}$$

The next result shows that the functions $\hat{L}$ and w have the properties established for ψ and Ψ in Proposition 5.5. The interpretation here is the also the same. For example, the effect on w of a change in the threshold b, conditional on the current state x, is the product of $\psi(x)$ and the effect of the change conditional on the state b. That is $w_b(x) = \psi(x)w_b(b)$. In addition, conditional on $x = b$, increasing x and b together has no effect w. Similar conclusions hold for $\hat{L}\left(z;\cdot\right)$, for any z, and at B.

PROPOSITION 5.6: The functions $\hat{L}$ and w satisfy

$$\begin{aligned} &\hat{L}_b(z;x) = \psi(x)\hat{L}_b(z;b), \qquad \hat{L}_B(z;x) = \Psi(x)\hat{L}_B(z;B), \\ &\hat{L}_x(z;b) + \hat{L}_b(z;b) = 0, \qquad \hat{L}_x(z;B) + \hat{L}_B(z;B) = 0, \quad z,x \in [b,B]\,, \end{aligned}$$

and

$$\begin{aligned} &w_b(x) = \psi(x)w_b(b), \qquad w_B(x) = \Psi(x)w_B(B), \\ &w_x(b) + w_b(b) = 0, \qquad w_x(B) + w_B(B) = 0, \quad x \in [b,B]\,. \end{aligned}$$

PROOF: Since

$$\hat{L}_b(z;x) = -\frac{1}{J}\left[\Psi_b(x)e^{R_1(B-z)} + \psi_b(x)e^{R_2(b-z)} + R_2\psi(x)e^{R_2(b-z)}\right],$$

the first claim follows immediately from Proposition 5.5. And for $x = b < z$,

$$\hat{L}_x(z;b) = \frac{1}{J}\left[R_2e^{R_2(b-z)} - \Psi_x(b)e^{R_1(B-z)} - \psi_x(b)e^{R_2(b-z)}\right],$$

so the second claim is also immediate from Proposition 5.5. Similar arguments hold at B.

For w note that

$$\begin{aligned} w_b(x) \;=\; & \left[-e^{R_1(x-b)} + \Psi(x)e^{R_1(B-b)} + \psi(x)\right]\frac{1}{J}g(b) \\ & -\Psi_b(x)\frac{1}{J}\int_b^B e^{R_1(B-z)}g(z)dz - \psi_b(x)\frac{1}{J}\int_b^B e^{R_2(b-z)}g(z)dz \\ & -R_2\psi(x)\frac{1}{J}\int_b^B e^{R_2(b-z)}g(z)dz. \end{aligned}$$

Use (15) to find that the terms in the first line cancel. Then apply Proposition 5.5 to the remaining terms to establish the first claim. For the second note that

$$w_x(b) = \frac{1}{J}\int_b^B \left[R_2e^{R_2(b-z)} - \Psi_x(b)e^{R_1(B-z)} - \psi_x(b)e^{R_2(b-z)}\right]g(z)dz.$$

Apply Proposition 5.5 again to obtain the result. Similar arguments hold at B. ■

Next consider expected discounted returns over an infinite time horizon. If $r > 0$ and $|g|$ is bounded, then the integral in (2) is bounded as $T \to \infty$. Taking the limit in (25) as $b \to -\infty$ and $B \to \infty$ gives

$$\begin{aligned} V_P(x) &\equiv \mathrm{E}_x \left[\int_0^{\infty} e^{-rt} g\left(X(t)\right) dt \right] \\ &= \frac{1}{J} \int_x^{+\infty} e^{R_2(x-z)} g(z) dz + \frac{1}{J} \int_{-\infty}^{x} e^{R_1(x-z)} g(z) dz. \end{aligned} \tag{26}$$

Equation (26) provides an interpretation for the roots R_1 and R_2. Given the initial state x, states $z < x$ in (26) are weighted by $e^{R_1(x-z)}/J$ and states $z > x$ by $e^{R_2(x-z)}/J$. Both exponential terms have negative signs, so $|R_1|$ and $|R_2|$ measure how sharply more distant states are downweighted. Of course, the weights satisfy

$$\begin{aligned} \frac{1}{J} \left[\int_0^{\infty} e^{-R_2 \zeta} d\zeta + \int_{-\infty}^{0} e^{-R_1 \zeta} d\zeta \right] &= \frac{1}{J} \left(\frac{1}{R_2} - \frac{1}{R_1} \right) \\ &= \frac{1}{r}, \end{aligned}$$

reflecting the fact that for $g(x) \equiv 1$ (26) implies $V_P(x) = 1/r$.

Figure 5.5 displays $e^{-R_i z}/J$ and the effects of parameter changes. Note that, in accord with the results in Exercise 5.1,

—an increase in the interest rate r downweights all states, and the effect is greater for states farther from x;

—an increase in the variance σ^2 shifts weight away from states closer to x toward states that are farther away; and

—if $\mu > 0$, an increase in μ shifts weight away from states below x toward states above it.

The assumptions $r > 0$ and $|g| < M$ together insure that the integrals in (26) are finite, so V_P is well defined. With an infinite horizon $r = 0$ cannot be allowed, for obvious reasons, but the restriction on g can be relaxed. The next exercise shows that if $V_P(x)$ is finite for any x, then it is finite everywhere.

Exercise 5.5: Let $\sigma^2 > 0$ and $r > 0$. Show that if $|V_P(\hat{x})| < \infty$, for any $\hat{x}$, then $|V_P(x)| < \infty$, all x.

In addition, it is easy to show that if g is continuous, then V_P is twice continuously differentiable. Differentiate (26) to find that

$$\begin{aligned} V_P'(x) &= \frac{R_1}{J}\int_{-\infty}^{x} e^{R_1(x-u)} g(u)du + \frac{R_2}{J}\int_{x}^{+\infty} e^{R_2(x-u)} g(u)du \\ &+ \frac{1}{J}\left[\lim_{u\uparrow x} g(x) - \lim_{u\downarrow x} g(x)\right]. \end{aligned}$$

If g is continuous at x, then the last term is zero, and

$$V_P''(x) = \frac{R_1^2}{J}\int_{-\infty}^{x} e^{R_1(x-u)} g(u)du + \frac{R_2^2}{J}\int_{x}^{+\infty} e^{R_2(x-u)} g(u)du + \frac{R_1 - R_2}{J} g(x).$$

The next result is useful in applications, where it is convenient to know that V_P has a unique local maximum. It says that if g is single peaked and V_P is finite valued, then V_P is also single peaked.

Proposition 5.7: If g is continuous and single peaked and $|V_P(x)| < \infty$, then V_P is single peaked.

Proof: Note that V_P satisfies (20):

$$g(x) = rV_P(x) - \mu V_P'(x) - \frac{1}{2}\sigma^2 V_P''(x), \quad \text{all } x.$$

Suppose there exists values $x_1 < x_2 < x_3$ such that x_1 and x_3 are local maxima and x_2 is a local minimum. Then

$$V_P'(x_i) = 0, \quad i = 1, 2, 3,$$

and

$$V_P''(x_2) \geq 0, \quad \text{and} \quad V_P''(x_i) \leq 0, \quad i = 1, 3,$$

so

$$g(x_2) \leq rV_P(x_2) \leq rV_P(x_i) \leq g(x_i), \quad i = 1, 3,$$

contradicting the assumption that g is single peaked. ■

7. ODEs for diffusions

For more general diffusions the approach in sections 4 - 6 can be used to characterize the functions ψ, Ψ, w, etc. rather sharply, even if closed form solutions are not generally available. Geometric Brownian motions and Ornstein-Uhlenbeck processes provide useful examples. Not surprisingly, the former deliver closed form solutions that are closely related to those for Brownian motions.

Assume that $X(t)$ is a regular, stationary diffusion. That is,

a. its domain is an interval of the form (ℓ, r), $[\ell, r)$, $(\ell, r]$, or $[\ell, r]$, where we allow $\ell = -\infty$ and $r = +\infty$ if the endpoint is open;

b. its infinitesimal parameters $\mu(x)$ and $\sigma(x)$ are continuous functions, with $\sigma^2(x) > 0$, all x; and

c. for any points x, y in the interior of the state space,

$$\Pr_x\left[T(y) < \infty\right] > 0.$$

First note that the argument leading to the ODE in (20) still applies. The only change is that $\mu(x)$ and $\sigma^2(x)$ are functions of the state. Thus, each example leads to an ODE of the form

$$\frac{1}{2}\sigma^2(x)f''(x) + \mu(x)f'(x) - rf(x) = -g(x), \quad x \in (b, B)\,, \tag{27}$$

where property (c) for a diffusion implies $\sigma^2(x) > 0$, all x.

8. Solutions for diffusions when $r = 0$

As before, if $r = 0$ (27) becomes a first-order equation,

$$\phi'(x) + \delta(x)\phi(x) = -\hat{g}(x), \quad x \in (b, B)\,, \tag{28}$$

where $\phi = f'$, and

$$\delta(x) \equiv \frac{2\mu(x)}{\sigma^2(x)}, \quad \text{and} \quad \hat{g}(x) \equiv \frac{2g(x)}{\sigma^2(x)}.$$

For a Brownian motion $\delta(x)$ is constant and (28) has solutions of the form

$$\phi(x)e^{\delta x} = -\int^x \hat{g}(\xi)e^{\delta\xi}d\xi.$$

More generally,

$$s(x) = \exp\left\{\int^x \delta(\xi)d\xi\right\} \tag{29}$$

is an integrating factor for (28). That is, (28) can be written as

$$\begin{aligned}\frac{d}{dx}\left[\phi(x)s(x)\right] &= \left[\phi'(x) + \delta(x)\phi(x)\right]s(x) \\ &= -\hat{g}(x)s(x).\end{aligned}$$

Integrating and multiplying by $1/s(x)$ gives the particular solution

$$\phi_p(x) = -\int^x \hat{g}(\xi)\frac{s(\xi)}{s(x)}d\xi.$$

In addition, solutions to the homogeneous equation corresponding to (28) have the form

$$\phi_h(x) = \frac{c_1}{s(x)}.$$

Hence any solution to (28) can be written as

$$\phi(x) = \frac{c_1}{s(x)} - \int^x \hat{g}(\xi)\frac{s(\xi)}{s(x)}d\xi, \quad x \in (b, B).$$

Integrating again gives the function of interest. Thus, if $r = 0$, any solution to (27) can be written as

$$f(x) = c_0 + \int^x \left[\frac{c_1}{s(z)} - \int^z \hat{g}(\xi)\frac{s(\xi)}{s(z)}d\xi\right]dz, \quad x \in [b, B]. \tag{30}$$

Note that (30) is a generalization of (21). As before, the lower limits of integration can be chosen for convenience, and the constants c_0 and c_1 are determined by boundary conditions.

Using b for both limits and reversing the order of integration gives

$$\int_b^x \int_b^z \hat{g}(\xi)\frac{s(\xi)}{s(z)}d\xi\, dz = \int_b^x \hat{g}(\xi)s(\xi)\, H(\xi, x)d\xi,$$

where

$$H(x,y) \equiv \int_x^y s^{-1}(\xi)d\xi, \quad b \le x \le y \le B.$$

Thus, if $r = 0$, any solution to (27) can be written as

$$f(x) = f(b) + c_1 H(b,x) - \int_b^x g(\xi)\,\frac{2s(\xi)}{\sigma^2(\xi)}\,H(\xi,x)d\xi, \tag{31}$$

where g is returned to its original form and c_1 is determined by the boundary condition at B. Reversing the roles of b and B produces a similar expression.

It follows directly from (31) that the functions θ, Θ and τ are

$$\begin{aligned} \theta(x) &= \frac{H(x,B)}{H(b,B)}, \qquad \Theta(x) = \frac{H(b,x)}{H(b,B)} \\ \tau(x) &= \frac{H(b,x)}{H(b,B)}\int_b^B \frac{2s(\xi)}{\sigma^2(\xi)}\,H(\xi,B)d\xi - \int_b^x \frac{2s(\xi)}{\sigma^2(\xi)}\,H(\xi,x)d\xi. \end{aligned} \tag{32}$$

For the the undiscounted expected occupancy measure and local time, use the same procedure as in section 5.5. Let $M(z;x,b,B)$ denote the expected occupancy measure for the set $[b,z]$ up to the stopping time $T = T(b) \wedge T(B)$, conditional on the initial state $X(0) = x$. Recall that M can be found by fixing z, b, B, and using (31) with $g = 1_{[b,z]}$ and the boundary conditions $M(z,b) = M(z,B) = 0$. Hence

$$M(z;x) = \frac{H(b,x)}{H(b,B)}\int_b^z \frac{2s(\xi)}{\sigma^2(\xi)}\,H(\xi,B)d\xi - \int_b^{\min\{x,z\}} \frac{2s(\xi)}{\sigma^2(\xi)}\,H(\xi,x)d\xi.$$

The expected local time function is then the derivative of M with respect to z,

$$L(z;x,b,B) = \frac{2s(z)}{\sigma^2(z)}\left[\Theta(x)H(z,B) - H(z,x)1_{z<x}\right].$$

Notice that L is continuous at $z = x$, but it has a kink at that point.

Exercise 5.6: Verify that for a Brownian motion these expression agree with those in (16) and (23).

Geometric Brownian motion.—

If X is a geometric Brownian motion with parameters $\mu \neq 0$ and $\sigma^2 > 0$, then $\delta(x) = 2\mu/\sigma^2 x$, and

$$s(x) = x^{\omega+1}, \qquad H(x,y) = \frac{y^{-\omega} - x^{-\omega}}{-\omega},$$

where $\omega \equiv 2\mu/\sigma^2 - 1$. Hence if $\mu \neq 0$,

$$\begin{aligned}
\theta(x) &= \frac{B^{-\omega} - x^{-\omega}}{B^{-\omega} - b^{-\omega}}, \\
\tau(x) &= \frac{1}{\omega+1}\left[\Theta(x)\left(\ln B - \ln x\right) + \theta(x)\left(\ln b - \ln x\right)\right], \\
L(z; x, b, B) &= \frac{2z^{\omega}}{\omega\sigma^2}\frac{1}{z}\left[\Theta(x)\left(z^{-\omega} - B^{-\omega}\right) - \left(z^{-\omega} - x^{-\omega}\right)1_{z<x}\right] \\
&= \frac{1}{\mu - \sigma^2/2}\frac{1}{z}\left\{\Theta(x)\left[1 - \left(\frac{z}{B}\right)^{\omega}\right] - \left[1 - \left(\frac{z}{x}\right)^{\omega}\right]1_{z<x}\right\}.
\end{aligned}$$

If the drift is zero, then $\delta(x) = 0$, and

$$s(x) = x, \qquad H(x,y) = \ln y - \ln x.$$

Hence if $\mu = 0$,

$$\begin{aligned}
\theta(x) &= \frac{\ln B - \ln x}{\ln B - \ln b}, \\
\tau(x) &= \frac{1}{\sigma^2}\left(\ln B - \ln x\right)\left(\ln x - \ln b\right), \qquad \text{if } \mu = 0.
\end{aligned}$$

$$L(z; x, b, B) = \frac{2}{\sigma^2}\frac{1}{z}\left[\Theta\left(x\right)\left(\ln B - \ln z\right) - \min\left\{\left(\ln z - \ln x\right), 0\right\}\right].$$

Exercise 5.7: Let $Y = \ln X$, where X is as above. Show that the expressions in above agree with the corresponding values for the process Y.

Ornstein-Uhlenbeck processes.—

For an Ornstein-Uhlenbeck process $\mu(x) = -\alpha x$, where $\alpha > 0$, and $\sigma^2 > 0$ is constant. Hence $\delta(x) = -2\eta x$, where $\eta = \alpha/\sigma^2 > 0$. An integrating factor is

$$\begin{aligned}
s(x) &= \exp\left\{-\int^x 2\eta\xi d\xi\right\} \\
&= \exp\left\{-\eta x^2\right\},
\end{aligned}$$

so

$$H(x,y) = \int_x^y e^{\eta\xi^2} d\xi.$$

Hence

$$\theta(x) = \frac{\int_x^B e^{\eta\xi^2} d\xi}{\int_b^B e^{\eta\xi^2} d\xi}, \qquad \Theta(x) = \frac{\int_b^x e^{\eta\xi^2} d\xi}{\int_b^B e^{\eta\xi^2} d\xi}, \qquad \text{etc.}$$

9. Solutions for diffusions when $r > 0$

If $r > 0$, solutions to (27) can be characterized by using the 'variation of parameters' method. Suppose that two linearly independent solutions f_1, f_2 of the homogeneous equation

$$\frac{1}{2}\sigma^2(x) f''(x) + \mu(x) f'(x) - r f(x) = 0, \tag{33}$$

have already been obtained. The ease or difficulty of this step depends on the infinitesimal parameters $\mu(x)$ and $\sigma(x)$.

If f_1 and f_2 are available, conjecture that (27) has a particular solution of the form

$$f_p(x) = \sum\nolimits_{i=1}^{2} \gamma_i(x) f_i(x),$$

where the γ_i's are functions that must be determined. The key step is an additional conjecture that γ_1 and γ_2 satisfy

$$\sum\nolimits_{i=1}^{2} \gamma_i'(x) f_i(x) = 0. \tag{34}$$

If this conjecture is correct, then

$$\begin{aligned} f_p'(x) &= \sum\nolimits_{i=1}^{2} \gamma_i(x) f_i'(x), \\ f_p''(x) &= \sum\nolimits_{i=1}^{2} \left[\gamma_i'(x) f_i'(x) + \gamma_i(x) f_i''(x)\right]. \end{aligned}$$

Substitute f_p and its derivatives into (27) and use the fact that $f_i,\ i = 1, 2$ satisfy (33). Most of the terms cancel, and what remains is

$$\frac{1}{2}\sigma^2(x) \sum\nolimits_{i=1}^{2} \gamma_i'(x) f_i'(x) = -g(x). \tag{35}$$

Hence the conjecture is correct if functions γ_1 and γ_2 satisfying (34) and (35) can be found. That is, γ_1' and γ_2' must satisfy

$$\begin{pmatrix} f_1(x) & f_2(x) \\ f_1'(x) & f_2'(x) \end{pmatrix} \begin{pmatrix} \gamma_1'(x) \\ \gamma_2'(x) \end{pmatrix} = \begin{pmatrix} 0 \\ -\hat{g}(x) \end{pmatrix}, \qquad x \in (b, B)\,,$$

where as before $\hat{g}(x) \equiv 2g(x)/\sigma^2(x)$. Hence

$$\gamma_1'(x) = \frac{f_2(x)\hat{g}(x)}{W(f_1, f_2)(x)}, \quad \gamma_2'(x) = -\frac{f_1(x)\hat{g}(x)}{W(f_1, f_2)(x)},$$

where

$$W(f_1, f_2)(x) \equiv f_1(x)f_2'(x) - f_1'(x)f_2(x), \qquad x \in (b, B)\,,$$

is the Wronskian. Integrate to get

$$\begin{aligned} \gamma_1(x, b) &= \int_b^x \frac{f_2(z)\hat{g}(z)}{W(f_1, f_2)(z)} dz, \\ \gamma_2(x, B) &= \int_x^B \frac{f_1(z)\hat{g}(z)}{W(f_1, f_2)(z)} dz, \end{aligned} \tag{36}$$

where the limits of integration have been chosen in a specific way. Sum the particular and homogeneous solutions to find that if f_1, f_2 satisfy (33), then any solution to (27) can be written as

$$f(x, b, B) = [\gamma_1(x, b) + c_1]\, f_1(x) + [\gamma_2(x, B) + c_2]\, f_2(x), \quad x \in (b, B)\,, \tag{37}$$

where γ_1 and γ_2 are defined in (36), and c_1, c_2 incorporate the constant terms from (36). Note that while f_1 and f_2 are functions of x only, γ_1 and γ_2 each depend in addition on one threshold.

For the function ψ use $\hat{g}(\cdot) \equiv 0$ and the boundary conditions $\psi(b) = 1$ and $\psi(B) = 0$, and for Ψ reverse the boundary conditions. In either case $\gamma_1 = \gamma_2 = 0$, so (37) implies

$$\begin{aligned} \psi(x, b, B) &= \frac{1}{D} \left[f_1(x)f_2(B) - f_1(B)f_2(x) \right], \\ \Psi(x, b, B) &= \frac{1}{D} \left[f_1(b)f_2(x) - f_1(x)f_2(b) \right], \end{aligned} \tag{38}$$

where

$$D \equiv f_1(b)f_2(B) - f_1(B)f_2(b).$$

For the function $w(x)$, $\hat{g}(\cdot)$ in (36) is arbitrary, and the boundary conditions are $w(b) = w(B) = 0$. Hence c_1 and c_2 satisfy

$$\begin{pmatrix} f_1(b) & f_2(b) \\ f_1(B) & f_2(B) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -\gamma_2(b,B)f_2(b) \\ -\gamma_1(B,b)f_1(B) \end{pmatrix}.$$

Solve for the constants and use the expressions above for ψ and Ψ to find that

$$\begin{aligned} w(x,b,B) &= \gamma_1(x,b)f_1(x) + \gamma_2(x,B)f_2(x) \\ &\quad -\Psi(x,b,B)\gamma_1(B,b)f_1(B) - \psi(x,b,B)\gamma_2(b,B)f_2(b). \end{aligned} \tag{39}$$

The next result shows that for any regular diffusion $\psi, \Psi, \hat{L}$ and w have the properties established in Propositions 5.5 and 5.6 for a Brownian motion. The interpretations are as before.

PROPOSITION 5.8: The functions $f = \psi, \Psi, \hat{L}\,(z;\cdot)\,, w$ satisfy

$$\begin{aligned} &f_b(x) = \psi(x)f_b\,(b)\,, && f_B(x) = \Psi(x)f_B(B), \quad x \in (b,B)\,, \\ &f_b(b) + f_x(b) = 0, && f_B(B) + f_x(B) = 0. \end{aligned}$$

PROOF: Use (36) and evaluate the derivatives in (38) and (39). ■

Exercise 5.8: Show that

$$f_i(x) = f_i(b)\psi(x,b,B) + f_i(B)\Psi(x,b,B), \quad i = 1,2.$$

The expected value over an infinite horizon can be found by taking limits in (39) as $b \to -\infty$ and $B \to +\infty$. Doing so, one find that

$$\begin{aligned} V_P(x) &\equiv \mathrm{E}_x\left[\int_0^\infty e^{-rt}g\left(X(t)\right)dt\right] \\ &= f_1(x)\int_{-\infty}^x \frac{f_2(z)\hat{g}(z)}{W(f_1,f_2)(z)}dz + f_2(x)\int_x^\infty \frac{f_1(z)\hat{g}(z)}{W(f_1,f_2)(z)}dz. \end{aligned}$$

Geometric Brownian motion.—

For a geometric Brownian motion with parameters μ and $\sigma^2 > 0$, the solutions to the homogenous equation (33) are

$$f_i(x) = x^{R_i}, \quad i = 1, 2,$$

where $R_1 < 0 < R_2$ are the roots of the quadratic

$$\frac{1}{2}\sigma^2 R^2 + \left(\mu - \frac{1}{2}\sigma^2\right) R - r = 0.$$

Hence

$$\begin{aligned} \psi(x) &= \frac{x^{R_1} B^{R_2} - B^{R_1} x^{R_2}}{b^{R_1} B^{R_2} - B^{R_1} b^{R_2}}, \\ \Psi(x) &= \frac{b^{R_1} x^{R_2} - x^{R_1} b^{R_2}}{b^{R_1} B^{R_2} - B^{R_1} b^{R_2}}. \end{aligned}$$

The Wronskian in this case is

$$W(f_1, f_2)(x) \equiv (R_2 - R_1)\, x^{R_1 + R_2 - 1},$$

and $\hat{g}(z) = 2g(z)/\sigma^2 z^2$, so

$$\begin{aligned} \gamma_1(x, b) &= \frac{1}{J} \int_b^x z^{-R_1 - 1} g(z) dz, \\ \gamma_2(x, B) &= \frac{1}{J} \int_x^B z^{-R_2 - 1} g(z) dz, \end{aligned}$$

where

$$\begin{aligned} \frac{1}{J} &= \frac{1}{R_2 - R_1} \frac{2}{\sigma^2}, \\ J &\equiv \pm\sqrt{\left(\mu - \sigma^2/2\right)^2 - 2\sigma^2 r}. \end{aligned}$$

Hence for a geometric Brownian motion

$$\begin{aligned} w(x) = \frac{1}{J} \Bigg[& \int_b^x \left(\frac{x}{z}\right)^{R_1} g(z) \frac{dz}{z} + \int_x^B \left(\frac{x}{z}\right)^{R_2} g(z) \frac{dz}{z} \\ & - \Psi(x) \int_b^B \left(\frac{B}{z}\right)^{R_1} g(z) \frac{dz}{z} - \psi(x) \int_b^B \left(\frac{b}{z}\right)^{R_2} g(z) \frac{dz}{z} \Bigg]. \end{aligned}$$

Exercise 5.8: Let $Y = \ln X$, where X is a geometric Brownian motion. Show that the expressions for ψ, Ψ and w above agree with those in sections 5.3 and 5.6. What is the relationship between the roots for the two processes?

Ornstein-Uhlenbeck processes.—

An Ornstein-Uhlenbeck process has drift $\mu(x) = -\alpha x$ and variance $\sigma^2 > 0$, so (33) takes the form

$$\frac{1}{2}\sigma^2 f''(x) - \alpha x f'(x) - r f(x) = 0. \tag{40}$$

Although this equation does not have a closed form solution, it is easy to verify that if h satisfies

$$\frac{1}{2}\sigma^2 h''(x) + \alpha x h'(x) + (\alpha - r)\, h(x) = 0,$$

then

$$f(x) = h(x)e^{\eta x^2}, \qquad \text{where } \eta = \alpha/\sigma^2,$$

is a solution to (40).

10. Notes

The arguments in section 1-3 follow Harrison (1985, sections 1.5 and 3.2), where the term Wald martingale is introduced, and the arguments in sections 4 - 9 follow Karlin and Taylor (1981, section 15.3). Borodin and Salminen (2002) is a useful compendium of a vast number of formulas related to Brownian motion.