Topics in Macroeconomics: An Introduction to Dynamic Economics
Problem Set II

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Problem 1 (Exchange Rate Regimes) *Consider a model in which the exchange rate satisfies the following condition*

$$X_t = K_t + \alpha \frac{E[dX_t \mid \mathcal{F}_t]}{dt},$$

where X_t denotes the exchange rate at time t, K_t is an index of the fundamentals in this economy, and α is a coefficient that measures the responsiveness of the current exchange rate to expected changes in future exchange rates.

It can be shown that, under rational expectations, the 'solution' to this equation is given by the forward operator, i.e.

$$X_t = \alpha^{-1} \int_t^\infty e^{(t-s)/\alpha} E[K_s \mid \mathcal{F}_t] ds.$$

In order to 'solve' for the stochastic process followed by the exchange rate, it is necessary to assume a given process for the index of fundamentals. In this question you will explore the consequences of different regimes.

1. *(Floating) Assume that under a free float, fundamentals satisfy*

$$dK_t = \mu dt + \sigma dW_t,$$

where W_t is a standard Brownian Motion. Assume that $K_t = K$. Go as far as you can deriving an expression for X_t as a function of K_t.

2. *(Bounds on Fundamentals). In this section we assume that the government promises to 'freeze' fundamentals if they ever get outside a given band. More precisely, assume that there exist values $K_L < K_H$ such that if K_t ever gets to those values it will remain at them for the foreseeable future. Thus, both bounds are absorbing barriers for the process. Assume $K_L < K < K_H$. Thus, the*

stochastic process for fundamentals is a (μ, σ) Brownian motion in the continuation region, and it is constant outside that region. Go as far as you can to solve for the exchange rate process as a function of the fundamentals.

3. *(Costly Adjustment) Consider the following policy announcement: "When the exchange rate reaches $\bar{X}$, the government will peg it by an intervention that instantaneously changes the fundamental K_t by the required amount. From then on, the government will do whatever is necessary to maintain the exchange rate at $\bar{X}$."[1]. Go as far as you can describing the stochastic process followed by the exchange rate (assume that you start at a point in which the exchange rate is free to float, i.e. $X_0 < \bar{X}$).*

   Note: There is no lower bound. Thus, this regime is, in some sense, similar to the previous one when $K_L = -\infty$.

4. *(Comparing Exchange Rate Regimes). In this section you are asked to compare versions of the exchange rate regimes that you studied in 1-3. However, to simplify the analysis, consider the special case of the "Bounds on Fundamentals" regime when $K_L = -\infty$. Go as far as you can to compare both the levels of the exchange rate, and the 'slope' of the exchange rate as a function of K across regimes. Put it differently, try to graph the functions of K that give the exchange rate in all three cases. Go as far as you can describing the differences.*

Problem 2 (Entrepreneurial Ability and Project Quality)) *An entrepreneur's innate ability x_t evolves according to*

$$dx_t = \mu_x x_t dt + \sigma_x x_t dZ_{xt},$$

where Z_{xt} is a standard Brownian motion. If an entrepreneur is 'vacant' (i.e. he is not currently running a firm), he faces a potential project of quality y_t, where quality evolves according to

$$dy_t = \mu_y y_t dt + \sigma_y y_t dZ_{yt}.$$

It is assumed that Z_{yt} is a standard Brownian motion, and that $E[Z_{xt}Z_{yt}] = \eta t$. If the manager decides to implement project y when his 'ability' is x, net profits evolve according to

$$\pi_t = A y_t^{\alpha} x_t^{1-\alpha}.$$

Moreover, implementing a project entails an investment of $K \geq 0$ units of capital. Note that it is assumed that implementing a project does not 'fix' neither the manager's entrepreneurial ability or the 'quality' of the project. Assume that while the manager is vacant he has to pay q units of consumption for the right to see y_t. Once the manager has chosen a project, it has to operate forever. The payoff of the project

[1] Yes, I know, it sounds a little vague, but this is how policy announcements always sound. You get to give this statement some content (if you think that is needed).

is the expected presenbt discounted value of profits. Assume that the interest rate is fixed and equal to r.

1. *Assume $K > 0$. Go as far as you can describing the manager's optimal policy.*
2. *Go as far as you can characterizing the optimal strategy. (If necessary make additional assumptions about the parameters to get good answers. How 'creative' you are in specifying special cases will be taken into account to grade this section.)*
3. *How would your answer change if $K = 0$?*
4. *Consider an entrepreneur that is vacant. What is the expected time until he implement a project? How does this expected time vary with σ_x, σ_y and η? (As before, if necessary make special assumptions)*

Problem 3 (Default Risk) *Consider a firm that has K units of capital, and has issued M bonds, each promising to pay in perpetuity (or until the firm defaults) a coupon rate of b. Profits of the firm per unit of capital (while in operation) are given by*

$$d\pi_t = \mu\pi_t + \sigma\pi_t dz_t$$

where z_t is a standard Brownian motion. The cost of operating the firm are ωK. It follows that net profits are $(\pi_t - \omega)K - bM$. If the firm is liquidated, it sells its equipment at a price p. Thus, the liquidation value is pK. Assume that the instantaneous interest rate is constant and equal to $r > \mu$.

1. *Assume that $M = 0$. Let $V(\pi, K, p)$ be the value of the firm when current per unit profits are π. Go as far as you can describing the optimal liquidation policy and the value of the firm. Does your model imply that larger firms are more valuable per unit of K?*
2. *Assume now that $M > 0$. If the firm is declared bankrupt (a decision taken by the owners of the firm) shareholders receive $\max\{0, V(\pi, K, p) - bM/r\}$ and bondholders receive $\min\{X(\pi, K, p), bM/r\}$ where $X(\pi, K, p)$ is the value of a 100% equity financed firm run by the bondholders (instead of the original owners). The only difference between this value and the one computed in section 1 is that bondholders are less efficient than owners. Specifically, assume that if the firm is run by the bondholders, its profit flow is given by $(1 - \phi)(\pi_t - \omega)K$, $0 < \phi < 1$.*

   Assume that shareholders make the bankruptcy decision. Let $W(\pi, K, p, M)$ be the value of equity in a non-bankrupt firm that has issued M bonds, and let $B(\pi, K, p, M)$ be the total value *of the bonds issued by the firm. Go as far as you can describing the stochastic differential equations that B and W must satisfy.*

3. *In the model of section 2 argue that the optimal bankruptcy rule is to choose a value of π such that the firm liquidates itself when π drops below that level. Go as far as you can characterizing that threshold.*

4. *Let the market price of a bond issued by this firm be defined as $B(\pi, K, p, M)/M$. Let the market price of* riskless *debt be $1/r$. Go as far as you can describing how a "riskless" measure of the debt equity ratio given by bM/K affects the premium paid by this firm in the bond market (i.e. the difference between $1/r$ and $B(\pi, K, p, M)/M$). Go as far as you can describing the effect of σ on the same premium.*

5. *Consider the following claim: If the scrap value of the firm is sufficiently high (i.e. if p is sufficiently high), then the equity issued by this firm is riskless. In this case, the firm will be eventually liquidated but its bondholders will be paid the full value of their debt (i.e. bM/r).*

6. *Extra Credit: Suppose that the p is random (say it is a measure of the value of land and buildings owned by the firm). To be precise assume that p is a Poisson process with arrival rate λ. Moreover assume that p can take only two values $p_h > p_\ell$. (Thus, we are assuming that p is a two state process with switching probability equal to λdt in a small interval of time) Go as far as you can describing the impact of this type of variability in the price of bonds issued by this firm. More precisely, suppose that the price is initially high, and at some point it switches to low. Derive the impact of this change in the real estate market on the price of debt issued by this firm.*

Problem 4 (Pensions, Disability and Retirement) *Consider a worker who — while on the labor force— earns a wage w, where w_t follows a geometric Brownian motion given by*

$$dw_t = \mu w_t dt + \sigma w_t dz_t,$$

where z_t is a standard Brownian Motion. Assume that the individual maximizes the expected value of future discounted utility flows Let the discount rate be r. As usual, assume that $r > \mu$. The flow utility while employed is equal to the wage (linear preferences). Let q_t be a Poisson process with arrival rate λ. We interpret an arrival of this process as a shock to the individual's health. If there is an arrival when the current wage is w, the individual becomes disabled and his expected present discounted value of utility from then on is

$$\frac{bw}{r} + \frac{k}{r},$$

where k is the value of leisure for a disabled individual. The worker —if not disabled— can choose to retire. In this case, the expected present discounted value of utility from then on is

$$\frac{Bw}{r} + \frac{K}{r}.$$

Thus, we are assuming that, once retired, the worker does not return to work. Assume that $k < K$ and that the ratio k/K is a measure of the severity of the disability.

1. *Describe the determinants of the (voluntary) retirement decision. Make enough assumptions to guarantee that, at some wage, the individual chooses to retire.*
2. *Go as far as you can analyzing how economic instability (σ), the nature of the retirement regime (b, B), and the severity of the disability (k/K) affect the expected duration of the working life.*
3. *Does the theory have any predictions about the normal (not disability related) retirement age of workers as a function of the disability rate (λ)? If the answer is ambiguous, discuss under what conditions you would get a definitive answer.*

Problem 5 (Bonds and Default) *Let gross flow profits of a firm be given by $x_t - c$ where x_t satisfies*

$$dx_t = \mu x_t dt + \sigma x_t dz_t,$$

and z_t is a standard Brownian motion. This firm has issued one bond that pays (if not in default) $b > 0$ per unit of time. If the firm is liquidated when $x_t = \bar{x}$, the bondholders receive the net liquidation value, $\bar{x}(1 - \delta)/(r - \mu)$, where $0 < \delta < 1$ is a measure of liquidation costs. While the firm is in operation, shareholders receive dividends given by

$$\pi_t = x_t - c - b.$$

Once the firm is liquidated, $\pi_t = 0$, for all t. All agents (bondholders and shareholders) discount payments at the fixed rate $r > 0$. Assume that $r > \mu$ and that $x_0 > \bar{x}$.

1. *Let $L(x; b)$ be the market price of a bond. Go as far as you can characterizing this function. In particular, if the value of a riskless bond is b/r analyze the impact of economic instability (σ) and the growth rate of the industry (μ) on the market value of the bond.*
2. *Suppose that bondholders are given the option to liquidate the firm. Let $\bar{x}_b$ be the liquidation point. Go as far as you can analyzing the equilibrium value of $\bar{x}_b$. Discuss how economic factors affect the market value of a bond.*
3. *Suppose that shareholders are given the option to liquidate the firm. Let $\bar{x}_s$ be the liquidation point. Go as far as you can analyzing the equilibrium value of $\bar{x}_s$. Discuss how economic factors affect the market value of a bond.*
4. *Go as far as you can comparing $\bar{x}_b$ and $\bar{x}_s$.*

Problem 6 (Portfolio Choice and Insurance) *Consider an individual who has no labor income but who can invest in two assets: a safe bond and a risky asset. The price of the risky asset satisfies*

$$dS_t = \mu S_t dt + \sigma S_t dz_t,$$

where z_t is a standard Brownian Motion. The price of the bond satisfies

$$dB_t = rB_t dt.$$

Preferences given by

$$U = E\left\{\int_0^\infty e^{-\rho t}\frac{c_t^{1-\theta}}{1-\theta}dt\right\}.$$

The individual may become disabled. Disability is modeled as a Poisson process with arrival rate λ. After the individual has become disabled he invests 100% of his wealth in the safe asset. Thus, if his wealth at the time he becomes disabled is K, his continuation utility is

$$\frac{1}{\rho}\frac{(rK)^{1-\theta}}{1-\theta}.$$

The individual can buy disability insurance. If at instant t he pays a premium equal to x_t he receives (in case he becomes disabled at that moment) ηx_t. (This scheme —which simplifies the algebra— basically says that the amount that the insurance company pays is a function of your last premium). The consumer's budget constraint (before he becomes disabled) is

$$dW_t = (\alpha_t \mu W_t + (1-\alpha_t)rW_t - c_t - x_t)dt + \alpha_t \sigma W_t dz_t,$$

where W_t is wealth and α_t is the fraction invested in the risky asset. Note that if the individual becomes disabled at time t, his post-disability wealth is $K_t = W_t + \eta x_t$.

1. *Analyze the individual portfolio problem. Argue that, at the optimum, $\alpha_t = \alpha$.*

2. *Suppose that the insurance industry is competitive. Thus, the 'value' of selling an insurance contract (which yields x_t while the individual is not disabled and $-\eta x_t$ in the event that the agent becomes disabled) is zero. Go as far as you can characterizing η.*

3. *Let x_t/W_t be the share of wealth invested in disability insurance (you may assume it is positive). It is claimed that this share is increasing in σ. The argument is that in a more unstable economy individuals purchase more insurance. Discuss this claim.*

4. *Assume that $\eta = 0$ (i.e. no insurance). How does the optimal composition of the portfolio compare to the one you derived in 1?*

5. ***Extra Credit*** *(Hard): Assume that the individual (before he becomes disabled) also earns labor income. The stochastic process for labor income satisfies*

$$dy_t = \gamma y_t dt + \upsilon y_t dz_t.$$

When the individual becomes disabled $y_t = 0$ *from that time on. Go as far as you can analyzing the portfolio and insurance problems. Discuss how economic factors affect the demand for insurance.*

Problem 7 (Timing of Births) *Consider the problem faced by a woman who wants to have (at most) a single child. If she has a child, she decides whether to take time off work (maternity leave) and whether to return back to work. Thus, in this model the timing of first (and only) birth and the length of maternity leave are endogenous. Assume that a woman's wage is given by the following stochastic process:*

$$dw_t = \mu w_t dt + \sigma w_t dz_t,$$

where z_t *is a standard Brownian Motion. A woman is risk neutral and discounts flows of utility at the rate* $r > 0$ $(r - \mu > 0)$. *Her planning horizon is infinite. If she does not have a child, her utility is linear in income. If she decides to have a child, the additional utility she derives from the child depends on the "child regime" she is in. If she is not working (i.e. immediately after birth) the payoff of allocating 100% of the time to child rearing is b. Thus, the flow utility while out of the labor force is b. When the mother decides to go back to work, flow utility utility is given by* $\alpha w_t + mb$, *where* α *and* m *are between 0 and 1. One interpretation of* α *is that it is either a tax on mothers (discrimination) or part time employment. In this setting a higher value of* m *corresponds to better day care facilities.*

1. *Analyze the woman's decision to go back to work after having a child. Argue that she will reenter the job market when the wage is sufficiently high.*

2. *Go as far as you can characterizing the effect of changes in* α *and* σ *on the expected duration of maternity leave conditional on some (arbitrary) wage* w_t.

3. *Go as far as you can describing the decision when to have a child. What does the model say about the relationship between motherhood and wages? To be precise, consider a large number of identical women who draw independent copies of the wage process (i.e. each of them gets her own realization) but all of whom start at the same wage* w_0. *What does the model predict will be the sign of the correlation between wages and age at first birth?*

4. *Go as far as you can analyzing the effect of economic instability (i.e.* σ*), discrimination (*α*) and day care quality (*m*), on the age at first birth.*

5. ***Extra Credit***: *Let the value of utility of a woman who earns initial wage w, has child preferences given by b, and who has not had a child be $V(w,b)$. Let the cost of 'acquiring' an initial wage w, be given by $C(w)$, where $C(w)$ has the usual properties of cost functions (i.e. increasing and strictly convex; also differentiable). Interpret the initial wage as a measure of human capital. Thus, high wage individuals are high human capital persons. Let a woman choose her initial wage to solve*

$$\max_w V(w,b) - C(w).$$

Go as far as you can describing the relationship between preferences for children (b) and economic instability (σ) on the human capital acquisition decision.

Note: *Whenever you are computing an expectation, e.g. the expected duration of maternity leave, assume that the parameters are such that the expectation is finite.*

Problem 8 (Executing an Option) *Suppose that the period t value of a bottle of wine is x_t, where x_t satisfies*

$$dx_t = \mu_x x_t dt + \sigma_x x_t dZ_t^x,$$

where Z_t^x is a standard Brownian motion. The owner of the bottle of wine is interested in maximizing the present discounted value of a sale. The interest rate is stochastic. To be precise, the value at time zero of a unit of consumption at time t is p_t, where p_t is the solution to

$$dp_t = \mu_p p_t dt + \sigma_p p_t dZ_t^p,$$

where Z_t^p is a standard Brownian motion. Assume that $E[Z_t^x Z_t^p] = \eta t$. Thus, the shocks to the value of wine and the interest rate are correlated.

1. *Argue that the solution of the problem is to find a stopping time T to maximize*

$$E\{p_T x_T \mid p_0, x_0\}$$

2. *Provide conditions on the parameters to guarantee that the problem is well-defined.*

3. *Go as far as you can describing the effect of changes in σ_x, σ_p and η on the nature of the solution?*