Some Notes on Stochastic Growth Models (Notes for a Chapter of the Handbook of Economic Growth)

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1 Fluctuations and Growth

1.1 Introduction

In this section we describe the existing results on the effects of 'volatility,' both in technologies and policies, on the long-run growth rate. We start with a brief summary of the empirical research in this area, and we then describe some simple theoretical models that are useful in understanding the empirical results. We end with the description of some recent work based on the theoretical models but aimed at evaluating their ability to *quantitatively* match the growth observations. Things to do:

1. Cite De Hek and **** (2001) as a "general" description of conditions under which one gets growth in the stochastic case.

2. Be clear about what is excluded. So far, we have ignored all models with a Romer-type of externality, models with Dixit-Stiglitz preferences and monopolistic competition, and models with externalities. In addition, we have also included models with public goods in the production function with congestion effects.

1.2 Empirical Evidence

A relatively small (but growing) empirical literature has tried to shed light on the relationship between 'instability' and growth. This literature has concentrated on estimating reduced form models that try to capture, with varying degrees of sophistication, how 'volatility' (defined in a variety of different ways) affects long-run growth.

Kormendi and Meguire (1985) is probably the earliest study in this literature. They consider a sample of 47 countries with data covering the 1950-1977 period. Their methodology is to run a cross-country growth regression with the average (over the sample period) growth rate as the dependent variable, and a number of control variables, including the standard deviation of the growth rate (one measure of instability), as well as the standard deviations of policy variables such as the inflation rate and the money supply. Kormendi and Meguire (1985) find that the coefficient of the volatility measure (the standard deviation of the growth rate) is *positive* and significant. Thus, a simple interpretation of their results is that more volatile countries —as measured by the standard deviation of their growth rates— grow at a higher rate.

Grier and Tullock (1989) use panel data techniques on a sample of 113 countries covering a period from 1951 to 1980. Their findings on the effect of volatility on growth are in line with those of Kormendi and Meguire (1985). They find that the standard deviation of the growth rate is *positively,* and significantly, associated with mean growth rates. [Note: Say more about the sample and the technique.]

Ramey and Ramey (1995) first report the results of regressing mean growth on its standard deviation on a sample of 92 countries as well as a subsample of 25 OECD, covering (approximately) the 1950-1985 period. They find that for the full sample the estimated effect of volatility is negative and significant, while for the OECD subsample the point estimate is positive, but insignificant. In order to account that the variance of the innovations to the growth rate has to be jointly estimated with the effects of volatility, Ramey and Ramey (1995) posit the following statistical model

$$\gamma_{it} = \beta X_{it} + \lambda \sigma_i + u_{it} \tag{1}$$

where X_{it} is a vector of variables that affect the growth rate and

$$u_{it} = \sigma_i \epsilon_{it}, \qquad \epsilon_{it} \sim N(0,1). \tag{2}$$

The model is estimated using maximum likelihood. The control variables used were the (average) investment share of GDP (Average I/Y), average population growth rate (Average γ_{Pop}), initial human capital (measured as secondary enrollment rate, H_0), and the initial level of per capita GDP (Y_0). They study separately the full sample (consisting of 92 countries) as well as a subsample of 25 OECD (more developed) economies. Their results are reproduced in columns (1) and (3) of Table XXX.

Table XXX: Growth and Volatility				
Variables	(1)	(2)	(3)	(4)
	$(92\text{–}Country)$	$(92\text{–}Country)$	$(OECD)$	$(OECD)$
	N = 2,184	N = 2,184	N = 888	N = 888
Constant	0.07	0.08	0.16	0.16
	(3.72)	(3.73)	(5.73)	(4.48)
σ_i	−0.21	−0.109	−0.39	−0.401
	(−2.61)	(−1.22)	(−1.92)	(−1.93)
Average I/Y	0.13	0.12	0.07	0.071
	(7.63)	(6.99)	(2.76)	(2.67)
Average γ_{Pop}	−0.06	−0.115	0.21	0.230
	(−0.38)	(−0.755)	(0.70)	(0.748)
H_0	0.0008	0.0007	0.0001	0.0001
	(1.18)	(1.03)	(2.00)	(1.954)
Y_0	−0.009	−0.009	−0.017	−0.017
	(−3.61)	−3.53)	(−5.70)	(−4.7445)
$\sigma_{\ln(I/Y)}$	-	−0.023		0.007
		(1.81)		(0.22)

Note: t-statistics in parentheses
Source: Columns (1) and (3) Ramey and Ramey (1995)
Columns (2) and (4), Barlevy (2002)

For both sets of countries, Ramey and Ramey (1995) find that the standard deviation of the growth rate is *negatively* related to the average growth rate. However, for the OECD subsample, the coefficient is less precisely estimated (even though the point estimate is larger in absolute value). Ramey and Ramey (1995) also consider more 'flexible' specifications that try to capture differences across countries in the appropriate forecasting equations. Considering the most parsimonious version of their model, the estimated effect of volatility on growth is still positive. However, the strength of the estimated relationship is reversed: for the OECD subsample the point estimate is four times the size of the estimate for the full sample and highly significant.

In more recent work, Barlevy (2002) reestimates the Ramey and Ramey (1995) model with one change: he adds the standard deviation of the logarithm of the investment-output ratio ($\sigma_{\ln(I/Y)}$) as one of the explanatory variables. Barlevy (2002) hypothesizes that this variable can capture non-linearities in the investment function. His results, using the same basic data as Ramey and Ramey (1995) are in columns (2) and (4) of Table XXX.[1] For the full 92-country sample, the introduction of this measure of investment volatility halves the size of the coefficient of σ_i, and it is no longer significant at conventional levels. The coefficient on $\sigma_{\ln(I/Y)}$ is *negative* and significant (at 5%). For the OECD sample, the addition of $\sigma_{\ln(I/Y)}$ does not affect much the estimate of the effect of σ_i on growth. However, Barlevy (2002) points out that this is finding is not robust, since eliminating two outliers, Greece and Japan where high volatility of the investment share seems to be due to transitional dynamics,

[1] We thank Gadi Barlevy for providing us the estimated coefficient for the control variables.

implies that neither the volatility of the growth rate nor $\sigma_{\ln(I/Y)}$ are significant.[2]

One possible explanation for the differences in the estimates of the effects of volatility on growth found in Kormendi and Meguire (1985) and Grier and Tullock (1989) and Ramey and Ramey (1995), is —as pointed out by Ramey and Ramey (1985) and Barlevi (2002)— that Kormendi and Meguire (1985) and Grier and Tullock (1989) include among their explanatory variables the standard deviations of policy variables that could be proxying for $\sigma_{\ln(I/Y)}$.

Kroft and Lloyd-Ellis (2002) also start from the basic statistical model of Ramey and Ramey (1995) but offer a different way of decomposing volatility. They hypothesize that uncertainty can be split into two orthogonal components: uncertainty about changes in regime (e.g. expansion-contraction) and fluctuations within a given regime. To this end, they generalize the empirical specification of the Ramey and Ramey(1995) statistical model to account for this. They assume that

$$\gamma_{ist} = \beta X_{it} + \lambda_w \sigma_{iw} + \lambda_b \sigma_{ib} + \upsilon_{ist}, \tag{3a}$$

$$\upsilon_{ist} = \sigma_{iw}\epsilon_{it} + \mu_{is}, \qquad \epsilon_{it} \sim N(0,1), \tag{3b}$$

$$\mu_{is} = \begin{cases} \mu_{ie} & \text{with probability } p_i = \frac{T_{ie}}{T} \\ \mu_{ir} & \text{with probability } 1 - p_i \end{cases} \tag{3c}$$

Kroft and Lloyd-Ellis (2002) interpret the standard deviation of the random variable μ_{is}, σ_{ib} —which they assumed observed by the economic agents but unobserved by the econometrician— as a measure of variability *between* regimes, while σ_{iw} is viewed as the *within-regime* variability. Kroft and Lloyd-Ellis estimate their model by maximum likelihood using the same sample as Ramey and Ramey (1995). The results are in Table YYY

Table YYY: Growth and Volatility (Kroft and Lloyd-Ellis, 2002)

Independent Variable	92-Country Sample (2,208 observations)	OECD Sample (888 observations)
Constant	0.00132 (0.022)	0.095 (1.89)
Within-phase volatility (σ_{iw})	2.63 (4.69)	0.90 (1.44)
Between-phase volatility (σ_{ib})	−2.65 (−6.35)	−1.11 (−2.33)
Average investment share of GDP	−0.01 (−0.26)	−0.004 (−0.073)
Average population growth rate	0.58 (1.24)	0.28 (0.62)
Initial human capital	0.001 (0.66)	−0.00001 (−0.096)
Initial per capita GDP	0.002 (0.25)	−0.0008 (−1.30)

Note: t-statistics in parentheses.
Source: Kroft and Lloyd-Ellis (2002).

[2] The point estimates are negative but insignificant.

The major finding is that the 'source' of volatility matters: increases in σ_{iw} — the within phase standard deviation— have a positive impact on growth for the full sample. For the OECD, the coefficient estimate is still positive but about one third of the size. The effect of the between-phase volatility, σ_{ib}, is negative in both cases. However, the effects are stronger for the full sample. It is not easy to interpret the phases identified by Kroft and Lloyd-Ellis (2002) in terms of a switching model because their estimation procedure assumes that the econometrician can identify whether a particular period corresponds to either a recession or an expansion.[3]. Kroft and Lloyd-Ellis (2002) also use the same controls as Ramey and Ramey. However, they find that, when the two variances are allowed to differ, none of the control variables is significant. It is not clear why this is the case. One possibility is that the 'phases' that they identify are correlated with the control variables (this seems like a likely situation in the case of investment). Another possibility is that the control variables, in the Ramey and Ramey (1995) formulation, capture the non-linearity associated with the regime shift and that, once the shifts are taken into account, the control variables have no explanatory power. In any case, this illustrates a point that we will come back to: the fragility of the "growth" regressions suggest that better theoretical models are necessary to more provide restrictions that will allow to identify the parameters of interest.

The results of both Ramey and Ramey (1995) and Kroft and Lloyd-Ellis (2002) are consistent with the existence of nonlinearities in the relationship between measures of instability and growth. Fatás (2001) estimates a number of different specifications of the relationship between instability and growth. His approach is to run standard cross country regressions. His data set is taken from the most recent version of the Heston-Summers sample and includes 98 countries with information covering the period 1950-1998. His estimates (see Table ZZZ) support the view that the effect of volatility on growth is nonlinear. Using Fatás' (2001) basic estimate —shown in column (1) of Table ZZZ— the pure effect of volatility is *negative* with a coefficient of -2.772 indicating that a one standard deviation increase in volatility reduces the growth rate by over 2.5%. However, the interaction term, corresponding to the variable Volatility * GDP is positive and equal to 0.340. According to these estimates, the net effect of σ_i on γ_i for the *richest* countries in the data is *positive* and greater than 0.3. For the less developed countries the estimate of the effect of volatility is *negative*. Columns (2) and (3) use other measures of non-linearity (initial per capita GDP and M3/Y, a measure of financial development), with similar outcomes: In all cases there is a significant effect, and increases in volatility are less detrimental to growth —and could even have a positive effect— the more developed a country is according to the proxy variables.

[3] Kraft and Lloyd-Ellis estimate the probabilites p_i as the fraction of the time that an economy spends in the recession "phase," defined as periods of negative output growth. Thus, not only is the process assumed to be *i.i.d.* but the transition probabilites are not jointly estimated with the parameters.

Table ZZZ: Growth and Volatility (Fatás, 2001)			
Independent Variable	(1)	(2)	(3)
Volatility (σ_i)	−2.772 (0.282)	−1.700 (0.645)	−0.270 (0.091)
GDP per capita (1960)	−2.229 (0.235)	−1.856 (0.422)	−0.953 (0.220)
Human capital (1960)	0.037 (0.015)	0.040 (0.018)	0.026 (0.017)
Average investment share of GDP	0.083 (0.013)	0.143 (0.021)	0.120 (0.024)
Average population growth rate	−0.624 (0.153)	−0.562 (0.205)	−0.465 (0.465)
Volatility * GDP	0.340 (0.036)	−	-
Volatility * GDP (1960)	−	0.212 (0.082)	-
Volatility * M3/Y	-	-	0.004 (0.001)
R^2	0.77	0.58	0.57

Note: Sample 1950-1998. Robust standard errors in parentheses
Source: Fatás (2001

Martin and Rogers (2000) also study the relationship between the standard deviation of the growth rate and its mean, in a cross section of countries and regions. In both subsamples —European regions and industrialized countries— they find a *negative* relationship between σ_γ and γ. However, when they consider a sample of developing countries the point estimates are positive, but in general insignificant.

It is not easy to explain the differences between Ramey and Ramey (1995), Fatás (2001) and Martin and Rogers (2000). The period used to compute the growth rates (1962-1985 for Ramey and Ramey (1995), 1950-1998 for Fatás (2001) and 1960 to 1988 for Martin and Rogers (2000)), and the set of less developed countries included (68 in Ramey and Ramey's study, and 72 in Martin and Rogers') are fairly similar. The two studies differ on their definition of the growth rate (simple averages in the Ramey and Ramey (1995) and Fatás (2001) papers, and estimated exponential trend in Martin and Rogers (2000)), and in the variables that are used as controls. However, it is somewhat disturbing that what appear, in the absence of a theory, as ex-ante minor differences in definitions can result in substantial differences in the estimates.

Siegler (2001) studies the connection between volatility in inflation and growth rates and mean growth for the pre 1929 period. Specifically, he uses panel data methods for a sample of 12 (presently developed) countries over the 1870-1929 period. He finds that volatility and growth are negatively correlated, and this finding is robust to the inclusion of standard growth regression type of controls.

Dawson and Stephenson (1997) estimate a model similar to (1) and (2) applied to U.S. states. They use the average (over the 1970-1988 period) growth rate of gross state product per worker for U.S. states as their growth variable, and its standard deviation as a measure of volatility. In addition, they include in their cross-sectional regression the standard (in growth regressions) control variables (investment rate, initial level of gross state product per worker, labor force growth rate, and initial human capital). Dawson and Stephenson (1997) find that volatility has *no impact* on the growth rate, once the other effects are included. Unfortunately, they do not report the 'raw' correlation between mean growth and its standard deviation. Thus, it is not possible to determine if the lack of significance is due to the use of controls, or is a more robust feature of U.S. states growth performance.

Mendoza (1997) differs from the previous studies in terms of his definition of instability. Instead of the standard deviation of the growth rate, which, in general, is endogenous, he identifies instability with the standard deviation of a country's terms of trade. He estimates a linear model using a cross section of countries and finds a *negative* relationship between instability and growth. His sample is limited to only 40 developed and developing countries, and it only covers the period 1971-1991.

A fair summary of the existing results is that there is no sharp characterization of the relationship between fluctuations and growth. Variation across studies in samples or specifications yield fairly different results. Moreover, the findings do not seem robust to details of how the statistical model is specified.

Are the empirical findings of the channel through which uncertainty affects growth more robust? Unfortunately, the answer is negative. Ramey and Ramey (1995) find that volatility —measured as the standard deviation of the growth rate— does not affect the investment-output ratio. More recently, Aizenman and Marion (1999) find that volatility is negatively correlated with investment, when investment is disaggregated between public and private. Fatás (2001) estimates a non-linear model of the effect of volatility on investment. He finds that increases in volatility decrease investment in poor countries, but that the opposite is true in high income countries. Thus his findings are consistent with the view that changes in volatility affect mean growth rates through (at least partially) their impact upon investment decisions.

How should these empirical results be interpreted? Even though it is tempting to take one's preferred point estimate as a measure of the impact of fluctuations (or business cycles) on growth there are two problems with this approach. First, the empirical estimates are not robust to the choice of specification of the reduced form. Second, and more important in our view, is that from the point of view of policy design, the relevant measures of volatility is the —in general unobserved— volatility in policies and technologies. In most models, the growth rate (and its standard deviation) are endogenous variables and, as usual, the point estimate of one endogenous variable on another is at best difficult to interpret. One way of contributing to the interpretation of the empirical results is to study what simple theoretical models predict for the estimated relationships. In the next section we present a number of very simple models to illustrate the possible effects of volatility in fundamentals on mean growth. In the process, we find that it is very difficult to interpret the empirical findings. To put it simply, there are theoretical models that —depending on the sample— do not restrict the sign of the estimated coefficient of the standard deviation of the growth rate on its mean. Moreover, the sign and the magnitude of the coefficient is completely uninformative to determine the effect of volatility on growth.

1.3 A Simple Linear Endogenous Growth Model

We begin by presenting a stochastic analog of a standard Ak model with a 'twist.' Specifically, we consider the case in which there are multiple linear technologies, all producing the same good. In order to obtain closed-form results we specify that the utility of the representative household is given by

$$U = E\left[\int_0^\infty e^{-\rho t}\frac{c_t^{1-\theta}}{1-\theta}dt\,|F_0\right]. \tag{4}$$

We assume that each economy has two types of technologies to produce consumption (alternatively, the model can be interpreted as a two sector model with goods that are perfect substitutes). Output for each technology satisfies

$$dk_t = ((A-\delta_k)k_t - c_{1t})dt + \sigma_k k_t dW_t + \eta_k k_t dZ_t^k, \tag{5a}$$
$$db_t = ((r-\delta_b)k_t - c_{2t})dt + \sigma_b b_t dW_t + \eta_b b_t dZ_t^b, \tag{5b}$$

where (W_t, Z_t^k, Z_t^b) is a vector of three independent standard Brownian motion processes, k_t and b_t are two different stocks of capital. This specification assumes that each sector is subject to an aggregate shock, W_t, as well as sector (or technology) specific shocks, Z_t^j.

To simplify the algebra, we assume that capital can be costlessly reallocated across technologies, and we denote total capital by $x_t \equiv k_t + b_t$. Setting (without loss of generality) $k_t = \alpha_t x_t$ (and, consequently $b_t = (1-\alpha_t)x_t$) it follows that total capital evolves according to

$$\begin{aligned} dx_t &= [(\alpha_t(A-\delta_k) + (1-\alpha_t)(r-\delta_b))x_t - c_t]dt + \\ &\quad [(\alpha_t\sigma_k + (1-\alpha_t)\sigma_b)dW_t + \alpha_t\eta_k dZ_t^k + (1-\alpha_t)\eta_b dZ_t^b]x_t. \end{aligned} \tag{6}$$

Given the equivalence between equilibrium and optimal allocations in this convex economy, we study the solution to the problem faced by a planner who maximizes the utility of the representative agent subject to the feasibility constraint. Formally, the planner solves

$$\max E\left[\int_0^\infty e^{-\rho t}\frac{c_t^{1-\theta}}{1-\theta}dt\,|F_0\right],$$

subject to (6).

Let the value of this problem be $V(x)$. Then, it is standard to show that the solution to the planner's problem satisfies the Hamilton-Jacobi-Bellman equation

$$\rho V(x) = \max_{c,\alpha}\left[\frac{c^{1-\theta}}{1-\theta} + V'(x)(\mu(\alpha)x - c) + \frac{V''(x)x^2}{2}\sigma^2(\alpha)\right],$$

where

$$\begin{aligned} \mu(\alpha) &= r + \alpha(A - r) - (\alpha\delta_k + (1-\alpha)\delta_b), && (7a) \\ \sigma^2(\alpha) &= (\alpha\sigma_k + (1-\alpha)\sigma_b)^2 + \alpha^2\eta_k^2 + (1-\alpha)^2\eta_b^2. && (7b) \end{aligned}$$

It can be verified that the solution is given by $V(x) = v\frac{x^{1-\theta}}{1-\theta}$, where

$$v = \left[\frac{\rho - (1-\theta)[\mu(\alpha^*) - \delta(\alpha^*) - \theta\frac{\sigma^2(\alpha^*)}{2}]}{\theta} \right]^{-\theta} \tag{8}$$

and $\delta(\alpha) = \alpha\delta_k + (1-\alpha)\delta_b$.

The optimal decision rules are

$$\begin{aligned} \alpha^* &= \frac{\frac{A-\delta_k-(r-\delta_b)}{\theta} - \sigma_b(\sigma_k - \sigma_b) + \eta_b^2}{(\sigma_k - \sigma_b)^2 + \eta_b^2 + \eta_k^2}, && (9a) \\ c &= \frac{\rho - (1-\theta)[\mu(\alpha^*) - \delta(\alpha^*) - \theta\frac{\sigma^2(\alpha^*)}{2}]}{\theta} x. && (9b) \end{aligned}$$

It follows that for the solution to be well defined it is necessary that $\rho - (1 - \theta)[\mu(\alpha^*) - \delta(\alpha^*) - \theta\frac{\sigma^2(\alpha^*)}{2}] > 0$, which we assume. (In each case we make enough assumptions to guarantee that this holds.)[4]

It follows that the equilibrium stochastic differential equation satisfied by aggregate wealth is given by

$$\begin{aligned} dx_t &= [\frac{\mu(\alpha^*) - (\delta(\alpha^*) + \rho)}{\theta} - (1-\theta)\frac{\sigma^2(\alpha^*)}{2}]x_t dt + && (10) \\ &\quad [(\alpha^*(\sigma_k - \sigma_b) + \sigma_b)dW_t + \alpha_k^*\eta_k dZ_t^k + (1-\alpha^*)\eta_b dZ_t^b]x_t, \end{aligned}$$

and the instantaneous growth rate of the economy, γ, and its variance, σ_γ^2, satisfy

$$\begin{aligned} \gamma &= \frac{\mu(\alpha^*) - (\delta(\alpha^*) + \rho)}{\theta} - (1-\theta)\frac{\sigma^2(\alpha^*)}{2}, && (11a) \\ \sigma_\gamma^2 &= (\alpha^*(\sigma_k - \sigma_b) + \sigma_b)^2 + \alpha^{*2}\eta_k^2 + (1-\alpha^*)^2\eta_b^2. && (11b) \end{aligned}$$

One is tempted to interpret (10) as the theoretical analog of (1) by defining the stochastic growth rate as

$$\gamma_t = \frac{dx_t}{x_t}.$$

Given this definition, the discrete time —with period length equal to one— version of the stochastic process followed by the growth rate is

$$\begin{aligned} \gamma_t &= \frac{\mu(\alpha^*) - (\delta(\alpha^*) + \rho)}{\theta} - (1-\theta)\frac{\sigma_\gamma^2}{2} + \varepsilon_t, && (12a) \\ \varepsilon_t &= (\alpha^*(\sigma_k - \sigma_b) + \sigma_b)dW_t + \alpha^*\eta_k dZ_t^k + (1-\alpha^*)\eta_b dZ_t^b. && (12b) \end{aligned}$$

[4] Add a comment about boundedness. The issue is that the return function is unbounded above if $0 < \theta < 1$, and unbounded below if $\theta > 1$. Argue that $c > 0$ is equivalent to ensuring boundedness.

This simple model driven by i.i.d. shocks has a stark implication: the growth rate is i.i.d. and it is independent of other endogenous (or exogenous) variables, except through the joint dependence on the error term. Using panel data, it is relatively easy to reject this implication. This, however, is not an intrinsic weakness of this class of models. The theoretical setting *can* be generalized to include serially correlated shocks and a non-linear structure, which could account for "convergence" effects, and would provide a role for lagged dependent variables. However, generalizing the theoretical model comes at the cost of not being able to discuss the impact of different factors on the growth rate, except numerically. What are the theoretical models that we discuss useful for? We view the class of theoretical models that we study as more appropriate to discuss the implications of the theory for cross section regressions since, in this case, the constant $\frac{\mu(\alpha^*)-(\delta(\alpha^*)+\rho)}{\theta}-(1-\theta)\frac{\sigma_\gamma^2}{2}$ can be correlated with other variables like the investment-output ratio.

Even though there is a formal similarity between (12) and (1)-(2), the theoretical model suggests that the simple approach that ignores that the same factors that affect σ_γ, also influence the true value of β in (1) can result in incorrect inference. Alternatively, the "deep parameters" are not the means and the standard deviation of the growth rates. They are the means and standard deviations of the driving stochastic processes. In terms of those parameters, the "true" model is non-linear.

Whether the model in (12) implies a positive or negative relationship between fluctuations and growth depends on the sources of shocks. At this general level it is difficult to illustrate this point, but we will come back to it in the context of specific examples.

It is not obvious how to define the investment ratio in this model. The change in cumulative investment in k, X_k, is given by,

$$dX_{kt} = \delta_k k_t dt + dk_t,$$

while the change in total output can be defined as[5]

$$dY_t = \mu(\alpha^*)x_t dt + \sigma_\gamma dM_t,$$

where M_t is a standard Brownian motion defined so that

$$\sigma_\gamma dM_t = (\alpha^*(\sigma_k - \sigma_b) + \sigma_b)dW_t + \alpha^*\eta_k dZ_t^k + (1-\alpha^*)\eta_b dZ_t^b.$$

In order to avoid technical problems, we consider a discrete time approximation in which the capital stocks change only at the beginning of the period. The investment-output ratio (for physical capital) is given by

$$z_t = \frac{\gamma + \delta_k + \sigma_\gamma\varepsilon_t}{\mu(\alpha^*) + \sigma_\gamma\varepsilon_t},$$

[5]This is not the only possible way of defining output. It assumes that the economy two sectors (or technologies). However, another interpretation of this basic framework considers b_t as bonds, and k_t as the only real stock of capital. We will be precise about the notion of output in each application.

where ε_t is the same noise that appears in (12). Since the previous expression is non-linear, we approximate it by a second order Taylor expansion to obtain

$$z_t = \frac{\gamma + \delta_k}{\mu(\alpha^*)} + \frac{\sigma_\gamma[\mu(\alpha^*) - (\gamma + \delta_k)]}{\mu(\alpha^*)^2}\varepsilon_t - \frac{\sigma_\gamma^2[\mu(\alpha^*) - (\gamma + \delta)]}{\mu(\alpha^*)^3}\varepsilon_t^2. \tag{13}$$

The mean investment ratio, which we denote z, is given by

$$z = \frac{\gamma + \delta_k}{\mu(\alpha^*)}[1 + \frac{\sigma_\gamma^2}{\mu(\alpha^*)^2}] - \frac{\sigma_\gamma^2}{\mu(\alpha^*)^2}. \tag{14}$$

Given this approximation, the model implies that the covariance between the growth rate and the investment-output ratio is

$$cov(\gamma_t, z_t) = \frac{\sigma_\gamma^2[\mu(\alpha^*) - (\gamma + \delta_k)]}{\mu(\alpha^*)^2}, \tag{15}$$

while the standard deviation of z_t is

$$\sigma_z = \frac{\sigma_\gamma[\mu(\alpha^*) - (\gamma + \delta_k)]}{\mu(\alpha^*)^2}\frac{(1 + \mu(\alpha^*)^2)^{1/2}}{\mu(\alpha^*)}. \tag{16}$$

Simple algebra shows that, given that the existence condition (8) is satisfied, $cov(\gamma_t, z_t) > 0$. Thus, in a simple regression, the investment ratio has to appear to affect positively growth. At this general level it is more difficult to determine if high σ_z economies are also high γ economies. The problem is that there are a number of factors that jointly affect γ and σ_z. In order to be more precise, it is necessary to be specific about the sources of heterogeneity across countries. We will be able to discuss the sign of this relationship in specific contexts.

We now use this 'general' model to discuss —in a variety of special cases— the connection between the variability of the growth rate of output and its mean

1.3.1 Case 1: An Ak Model

Probably the simplest model to illustrate the role played by differences in the variability of the exogenous shocks across countries is the simple Ak model. Even though it is a special case of the model described in the previous section, it is useful to describe the technology in a slightly different way. Let the feasibility constraint for this economy be given by

$$\int_0^t \hat{A}k_s ds + \int_0^t \sigma_y \hat{A}k_s dW_s \geq \int_0^t (c_s ds + \int_0^t dX_{ks}.$$

The left hand side of this condition is the accumulated flow of output until time t, and the right hand side is the accumulated uses of output, consumption and investment. The law of motion of capital is

$$dk_t = -\delta_k k_t dt + dX_{kt},$$

where δ_k is the depreciation rate. Expressing the economy's feasibility constraint in flow form, and substituting in the law of motion for physical capital, the resource constraint satisfies

$$dk_t = [(\hat{A} - \delta_k)k_t - c_t]dt + \sigma_y \hat{A} k_t dW_t. \tag{17}$$

The planner's problem —which coincides with the competitive equilibrium in this economy— is to maximize (4) subject to (17). This problem resembles the more general model we introduced in the previous section if we set $\eta_b = \sigma_b = \eta_k = 0$, and

$$\begin{aligned} A &= \hat{A} - \delta_k, \\ \sigma_k &= \sigma_y \hat{A}. \end{aligned}$$

In addition, we need to make sure that the "b" technology is not used in equilibrium. A simple way of guaranteeing this is to view $r - \delta_b$ as endogenous, and to choose it so that, in equilibrium, $\alpha^* = 1$; that is, all of the investment is in physical capital. It is immediate to verify that this requires

$$r - \delta_b = \hat{A} - \delta_k - \theta \sigma_y^2 \hat{A}^2.$$

In this case is it follows that $x_t = k_t$ and the formulas in (11) imply that the mean growth rate and the variance of the growth rate satisfy

$$\begin{aligned} \gamma &= \frac{\hat{A} - (\rho + \delta_k)}{\theta} - (1 - \theta)\frac{\sigma_\gamma^2}{2}, \\ \sigma_\gamma^2 &= \sigma_y^2 \hat{A}^2. \end{aligned}$$

This result, first derived by Phelps (1962) and Levhari and Srinivasan (1969), shows that, in general, the sign of the relationship between the variance of the technology shocks, σ_y^2, and the growth rate is ambiguous:

- If preferences display less curvature than the logarithmic utility function, i.e. $0 < \theta < 1$, increases in σ_y are associated with decreases in the mean growth rate, γ.
- If $\theta > 1$, increases in σ_y are associated with increases in the mean growth rate, γ.
- In the case in which the utility function is the log (this corresponds to $\theta = 1$) there is no connection between fluctuations and growth.

The basic reason for the ambiguity of the theoretical result is that the total effect of a change in the variance of the exogenous shocks on the saving rate —and ultimately on the growth rate— can be decomposed in two effects that work in different directions:

- An increase in the variance of the technology makes acquiring future consumption less desirable, as the only way to purchase this good is to invest. Thus, an increase in variance of the technology shocks has a *substitution effect* that increases the demand for current (relative to future) consumption. This translates into a lower saving and growth rates.

- On the other hand, an increase in the variability of the exogenous shocks induces also an *income effect*. Intuitively, for concave utility functions, the fluctuations of the marginal utility decrease with the level of consumption. Thus, the (negative) effect of fluctuations is smaller when consumption is high. This income effect increases savings, as this is the only way to have a 'high' level of consumption (i.e. to spend more time on the relatively flat region of the marginal utility function).

The formula we derived shows that the relative strength of the substitution and income effects depends on the degree of curvature of the utility function: if preferences have less curvature than the logarithmic function, the substitution effect dominates and increases in the variance of the exogenous shocks reduce growth. If the utility of the representative agent displays more curvature than the logarithmic function, the income effect dominates and the relationship between fluctuations and growth is positive.

In this simple economy, the variance of the technology shock, σ_y^2, and the variance of the growth rate of output, σ_γ^2, coincide up to scale factor $\hat{A}$.[6] If one views the differences across countries as due to differences in σ_y^2,[7] the theoretical model implies that the true regression equation is very similar to the one estimated in the empirical studies. The only difference is that the theory implies that it is σ_γ^2, and not σ_γ, that enters the right hand side of (1). If we use this model to interpret the results of Ramey and Ramey (1995), one must conclude that the negative relationship between mean growth and its standard deviation is evidence that preferences have less curvature than the logarithmic utility, i.e. $0 < \theta < 1$. On the other hand, the Kormendi and Meguire (1985) findings suggest that $\theta > 1$.

In this simple example, the mean investment ratio —the appropriate version of (14)— is

$$z = \frac{\gamma + \delta_k}{\hat{A}}[1 + \sigma_y^2] - \sigma_y^2$$

As was pointed out in the previous section, the covariance between the investment-ratio and the growth rate is positive. In this example, the appropriate version of (16) is

$$\sigma_z = \sigma_y \left(\frac{\rho - (1-\theta)(\hat{A} - \delta_k - \frac{\theta}{2}\sigma_y^2 \hat{A}^2}{\theta} \right) \left(1 + \hat{A}^2\right)^{1/2}.$$

[6] In general, this is not the case.

[7] This is not necessary. In addition to differences in preferences —which we will ignore in this chapter— countries can differ in terms of $(\hat{A}, \delta_k)$ as well.

In this case, the increases in σ_y are associated with increases (decreases) in σ_z if $\theta < (>)1$. Thus, if $\theta < 1$, the higher the (unobserved) variance of the technology shocks (σ_y^2), the higher the (measured) variances of both the growth rate, σ_γ^2, and the investment rate, σ_z^2, and the lower the mean growth rate. Moreover, in this stochastically singular setting the standard deviation of the growth rate and the investment rate are related (although not linearly). Thus, this simple model is consistent with the findings of Barlevy (2002) that the coefficient of σ_z is estimated to be negative, and that its introduction reduces the significance of σ_γ.

This simple model cannot explain the apparent non-linearity in the relationship between mean and standard deviation of the growth rate process which, according to Fatás (2001), is such that the effect of σ_γ on γ is less negative (and can be positive) for high income countries. In order to account for this fact it is necessary to increase the degree of heterogeneity, and to consider non-linear models.

Finally, the model can be reinterpreted as a multi-country model in which markets are incomplete and the distribution of the domestic shocks —the productivity shocks— is common across all countries.[8] More precisely, consider a market structure in which all countries can trade in a perfectly safe international bond market. In this case —which of course implies that mean growth rates are the same across countries— there is an equilibrium in which all countries choose to hold no international bonds, and the world interest rate is

$$r^* = \hat{A} - \delta_k - \theta\sigma_{y_i}^2\hat{A}^2.$$

If there is a common shock that decreases the variability of every country's technology shocks, this has a positive effect on the "world" interest rate, r^*, and an ambiguous impact on the world growth rate.

1.3.2 Case 2 : A Two Sector (Technology) Model

In the previous model, the variance of the growth rate is exogenous and equal to the variance of the technology shock. This is due, in part, to the assumption that the economy does not have another asset that can be used to diversify risk. In this section we present a very simple two-technology (or two sector) version of the model in which the variance of the growth rate is *endogenously* determined by the portfolio decisions of the representative agent. The main result is that, depending on the source of heterogeneity across countries, the relationship between σ_γ and γ need not be monotone. In particular, and depending on the source of heterogeneity across countries, the model is consistent with increases in σ_γ initially associated with increases in γ, and then, for large values of σ_γ, with decreases in the mean growth rate.

[8] It is possible to allow countries to share the same realization of the stochastic process. Even in this case, the demand for bonds is zero at the conjectured interest rate.

To keep the model simple, we assume that the second technology is not subject to shocks, and we ignore depreciation. Thus, formally, we assume that $\eta_b = \sigma_b = \eta_k = 0$,. However, unlike the previous case, the "safe" rate of return r satisfies

$$A - \theta\sigma_k^2 < r < A.$$

This restriction implies that $\alpha^* \ \varepsilon \ (0,1)$, and guarantees that both technologies will be used to produce consumption. Since this model is a special case of the results summarized in (11) (we set the depreciation rates equal to zero for simplicity), it follows that the equilibrium mean growth rate and its variance are given by

$$\gamma = \frac{r-\rho}{\theta} + \left(\frac{A-r}{\theta\sigma_k}\right)^2 \frac{1+\theta}{2}, \tag{18a}$$

$$\sigma_\gamma^2 = \left(\frac{A-r}{\theta\sigma_k}\right)^2. \tag{18b}$$

How can we use the model to interpret the cross country evidence on variability and growth? A necessary first step is to determine the variables that can potentially vary across countries. In the context of this example, a natural candidate is the vector (A, r, σ_k). Before we proceed, it is useful to describe the connection between γ and σ_γ implied by the model. The relationship is —taking a discrete time approximation—

$$\begin{aligned} \gamma_t &= \frac{r-\rho}{\theta} + \sigma_\gamma^2 \frac{1+\theta}{2} + \varepsilon_t, \\ \varepsilon_t &= \sigma_\gamma \omega_t, \qquad \omega_t \sim N(0,1). \end{aligned}$$

It follows that if the source of cross-country differences are differences in (A, σ_k) the model implies that —independently of the degree of curvature of preferences— the relationship between σ_γ^2 and γ is positive. To see why increases in σ_k result in such a positive association between the two endogenous variables σ_γ and γ, note that, as σ_k rises, the economy shifts more resources to the safe technology (α^* decreases) and this, in turn results in a decrease in the variance of the growth rate (which is a weighted average of the variances of the two technologies). Since the 'risky' technology has higher mean return than the 'safe' technology, the mean growth rate decreases. The reader can verify that changes in A have a similar effects.

If the source of cross-country heterogeneity is due to differences in r, the implications of the model are more complex. Consider the impact of a decrease in r. From (18b) it follows that σ_γ^2 increases and this tends to increase γ. However, as (18a) shows, this also decreases the growth rate, as it lowers the non-stochastic return. The total effect depends on the combined impact. A simple calculation shows that

$$\frac{\partial \gamma}{\partial r} \lesseqgtr 0 \qquad \Leftrightarrow \qquad r \lesseqgtr \hat{r},$$

where

$$\hat{r} = A - \frac{\theta\sigma_k^2}{1+\theta}.$$

To understand the implications of the model consider a "high" value of r; in particular, assume that $r > \hat{r}$. A decrease in r reduces σ_γ and, given that $r > \hat{r}$, it results in an increase in γ. Thus, for low σ_γ (high r) countries, the model implies a positive relationship between γ and σ_γ. If $r < \hat{r}$, decreases in the return to the safe technology still increase σ_γ, but, in this region, the growth rate decreases. Thus, in (σ_γ, γ) space the model implies that, due to variations in r, the relationship between σ_γ and γ has an inverted U-shape.

Can this model explain some of the non-linearities in the data? In the absence of further restrictions on the cross-sectional joint distribution of (A, r, σ_k) the model can accommodate arbitrary patterns of association between σ_γ and γ. If one restricts the source of variation to changes in the return r the model implies that, for high variance countries, variability and growth move in the same direction, while for low variance countries the converse is true. If one could associate low variance countries with relatively rich countries, the implications of the model would be consistent with the type of non-linearity identified by Fatás (2001).

1.3.3 Case 3: Aggregate vs Sectoral Shocks

The simple Ak model that we discussed in the previous section is driven by a single, aggregate, shock. In this section we consider a two sector (or two technology) economy to show that the degree of sectoral correlation of the exogenous shocks can affect the mean growth rate. To capture the ideas in as simple as possible a model, we specialize the specification in (5) by considering the case

$$\begin{aligned}
\sigma_k &= \sigma_b = \sigma > 0, \\
\eta_b &= 0, \ \eta_k = \eta, \\
\delta_k &= \delta_b = 0.
\end{aligned}$$

Note that, in this setting, there is an aggregate shock, W_t, which affects both sectors (technologies) while the A sector is also subject to a specific shock, Z_t^k. Using the formulas derived in (9) and (11) it follows that the relevant equilibrium quantities are

$$\begin{aligned}
\alpha^* &= \frac{A-r}{\theta\eta^2}, \\
\gamma &= \frac{r-\rho}{\theta} - (1-\theta)\frac{\sigma^2}{2} + \left(\frac{A-r}{\theta\eta}\right)^2 \frac{1+\theta}{2}, \\
\sigma_\gamma^2 &= \sigma^2 + \left(\frac{A-r}{\theta\eta}\right)^2.
\end{aligned}$$

As before, it is useful to think of countries as indexed by (A, r, σ, η). Since changes in each of these parameters has a different impact, we analyze them separately.

- `An increase in` σ. The increase in the standard deviation of the economy-wide shock affects both sectors equally, and it does not induce any 'portfolio' or sectoral reallocation of capital. The share of capital allocated to each sector (technology) is independent of σ. Since increases in σ increase σ_γ (in the absence of a portfolio reallocation, this is similar to the one sector case), the total effect of an increase in σ is to decrease the growth rate if $0 < \theta < 1$, and to increase it if $\theta > 1$.

- `A decrease in` r. The effect of a change in r parallel the discussion of the previous section. It is immediate to verify that a decrease in r results in an increase in σ_γ. However, the impact on γ is not monotonic. For high values of r, decreases in r are associated with increases in γ, while for low values the direction is reversed. Putting together these two pieces of information, it follows that the predicted relationship between σ_γ and γ is an inverted U-shape, with a unique value of σ_γ (a unique value of r) that maximizes the growth rate.

- `An increase in` η. This change increases the 'riskiness' of the A technology and results in a portfolio reallocation as the representative agent decreases the share of capital in the high return sector (technology). The change implies that σ_γ and γ decrease. Thus, differences in η induce a positive correlation between mean and standard deviation of the growth rate.

- What is the impact of differences in the degree of correlation between sectoral shocks. Note that the correlation between the two sectoral shocks is

  $$\nu = \frac{\sigma}{(\sigma^2 + \eta^2)^{1/2}}.$$

  In order to isolate the impact of a change in correlation, let's consider changes in (σ, η) such that the variance of the growth rate is unchanged. Thus, we restrict (σ, η) to satisfy

  $$\sigma_\gamma^2 = \sigma^2 + \left(\frac{A-r}{\theta\eta}\right)^2,$$

  for a given (fixed) σ_γ. It follows that the correlation between the two shocks and the growth rate are

  $$\begin{aligned} \nu &= \left(1 + (\frac{A-r}{\theta})^2 \frac{1}{\sigma^2(\sigma_\gamma^2 - \sigma^2)}\right)^{-1}, \\ \gamma &= \frac{r-\rho}{\theta} - \sigma^2 + \frac{1+\theta}{2}\sigma_\gamma^2. \end{aligned}$$

Thus, lower correlation between sectors (in this case this corresponds to higher σ) unambiguously lower mean growth. If countries differ in this correlation then the implied relationship between σ_γ and γ need not be a function; it can be a correspondence. Put it differently, the model is consistent with different values of γ associated to the same σ_γ. [Note: Should we check if bringing in z is sufficient to 'control' for σ^2. The problem here is that it is not clear how to define z_t. Does it include investment in k and b?]

1.4 Physical and Human Capital

In this section we study models in which individuals invest in human and physical capital. The models vary in the details of the accumulation technology and the nature of the shocks. Each of them illustrates a specific dimension of the theoretical relationship between the standard deviation of the growth rate and mean growth.

1.4.1 Inelastic Labor Supply

We first consider a model in which the rate of utilization of human capital is constant. Even though the model is quite simple it is rich enough to be consistent with **any** estimated relationship between σ_γ and γ. This, of course, suggests that extreme caution should be used when interpreting the empirical results.

We assume that output can be used to produce consumption and investment, and that market goods are used to produce human capital. This is equivalent to assuming that the production function for human capital is identical to the production function of general output. The feasibility constraints are

$$\begin{aligned} dk_t &= ([F(k_t, h_t) - \delta_k k_t - x_t - c_t)\, dt + \sigma_y F(k_t, h_t) dW_t, \\ dh_t &= -\delta_h h_t + x_t dt + \sigma_h h_t dW_t + \eta h_t dZ_t, \end{aligned}$$

where (W_t, Z_t) is a vector of independent standard Brownian motion variables, and F is a homogeneous of degree one, concave, function. As in the previous sections, let $x_t = k_t + h_t$ denote total (human and non-human) wealth. With this notation, the two feasibility constraints collapse to

$$\begin{aligned} dx_t &= ([F(\alpha_t, 1-\alpha_t) - (\delta_k \alpha_t + \delta_h (1-\alpha_t))] x_t - c_t)\, dt + \sigma_y F(\alpha_t, 1-\alpha_t) x_t dW_t + \\ & \quad \sigma_h (1-\alpha_t) x_t dW_t + \eta (1-\alpha_t) x_t dZ_t. \end{aligned} \tag{19}$$

As in previous sections, the competitive equilibrium allocation coincides with the solution to the planner's problem. The planner maximizes (4) subject to (19). The Hamilton-Jacobi-Bellman equation corresponding to this problem is

$$\rho V(x) = \max_{c,\alpha} \left[\frac{c^{1-\theta}}{1-\theta} + V'(x)[(F(\alpha, 1-\alpha) - \delta(\alpha)) x_t - c_t] + \frac{V''(x) x^2}{2} \sigma^2(\alpha) \right],$$

where

$$\begin{aligned}\delta(\alpha) &= \delta_k\alpha + \delta_h(1-\alpha),\\ \sigma^2(\alpha) &= \sigma_y^2 F(\alpha,1-\alpha)^2 + \sigma_h^2(1-\alpha)^2 + \eta^2(1-\alpha)^2 + \sigma_y\sigma_h F(\alpha,1-\alpha)(1-\alpha).\end{aligned}$$

As before, a function of the form $V(x) = v\frac{x^{1-\theta}}{1-\theta}$ solves the Hamilton-Jacobi-Bellman equation. The solution also requires that

$$\begin{aligned}\rho &= \theta v^{-1/\theta} + (1-\theta)\{F(\alpha,1-\alpha) - \delta(\alpha) - \frac{\theta}{2}[\sigma_y^2 F(\alpha,1-\alpha)^2 + \sigma_h^2(1-\alpha)^2 + \\ &\quad \eta^2(1-\alpha)^2 + \sigma_y\sigma_h F(\alpha,1-\alpha)(1-\alpha)]\},\end{aligned}$$

where α is given by

$$\begin{aligned}\alpha &= \arg\max(1-\theta)\{F(\alpha,1-\alpha) - \delta(\alpha) - \frac{\theta}{2}[\sigma_y^2 F(\alpha,1-\alpha)^2 + \sigma_h^2(1-\alpha)^2 + \\ &\quad \eta^2(1-\alpha)^2 + \sigma_y\sigma_h F(\alpha,1-\alpha)(1-\alpha)]\}.\end{aligned}$$

It is clear that, for any homogeneous of degree one function F, the solution is a constant α. Moreover, α does not depend on v. Existence requires $v > 0$, and this has to be verified.

It follows that the growth rate and its variance are given by

$$\begin{aligned}\gamma &= F(\alpha,1-\alpha) - \delta(\alpha) - v^{-1/\theta},\\ \sigma_\gamma^2 &= \sigma_y^2 F(\alpha,1-\alpha)^2 + \sigma_h^2(1-\alpha)^2 + \eta^2(1-\alpha)^2 + \sigma_y\sigma_h F(\alpha,1-\alpha)(1-\alpha)\end{aligned}$$

It follows that, for the class of economies for which the planner problem has a solution (i.e. economies for which $\nu > 0$, and $\gamma > 0$), the conjectured form of $V(x)$ solves the HJB equation, for any homogeneous of degree one function F. However, in order to make some progress describing the implications of the theory, it will prove convenient to specialize the technology and assume that F is a Cobb-Douglas function given by

$$F(x,y) = Ax^\omega y^{1-\omega}, \qquad 0 < \omega < 1.$$

The next step is to characterize the optimal share of wealth invested in physical capital, α, and how changes in country-specific parameters affect the mean and standard deviation of the growth rate. It turns out that the qualitative nature of the solution depends on the details of the driving stochastic process. Thus, we separate our analysis in three different cases characterized by increased generality.

Case I: Deterministic Human Capital Technology Formally, this case corresponds to $\sigma_h = \eta = 0$. As indicated above, we assume that the production function is Cobb-Douglas. The first order condition for the optimal choice of α is simply

$$\phi(\alpha)\hat{F}(\alpha)[1 - \theta\sigma_y^2\hat{F}(\alpha)] = 0,$$

where

$$\hat{F}(\alpha) \equiv A\alpha^{\omega}(1-\alpha)^{1-\omega},$$
$$\phi(\alpha) = \frac{\omega}{\alpha} - \frac{1-\omega}{1-\alpha}.$$

The second order condition requires that

$$-\omega(1-\omega)[\alpha^{-2} + (1-\alpha)^{-2}]\hat{F}(\alpha)[1-\theta\sigma_y^2\hat{F}(\alpha)] - \theta\sigma_y^2\hat{F}(\alpha)^2\phi(\alpha) < 0.$$

Since $\hat{F}(\alpha) > 0$ in the relevant range, the solution is either $\phi(\alpha) = 0$, which corresponds to $\alpha^* = \omega$, or $\hat{F}(\alpha^*) = 1/\theta\sigma_y^2$. The latter, of course, does not result in a unique α^{*}[9].

The nature of the solution depends on the size of σ_y^2. There are two subcases characterized by

- *Case I.A*: $\sigma_y^2 \leq \frac{1}{\theta\hat{F}(\omega)}$.
- *Case I.B*: $\sigma_y^2 > \frac{1}{\theta\hat{F}(\omega)}$.

In *Case I.A*, the maximizer is given by $\alpha^* = \omega$, since $1 - \theta\sigma_y^2\hat{F}(\alpha) > 0$ for all feasible α. The second order condition is satisfied.

In *Case I.B*, there are two solutions to the first order condition. They correspond to the values of α, denoted α^- and α^+ that solve $\hat{F}(\alpha^*) = 1/\theta\sigma_y^2$. By convention, let's consider $\alpha^- < \omega < \alpha^+$. It can be verified that in both cases the second order condition is satisfied.[10] The implications of the model for the expected growth rate and its standard deviation in the two cases are

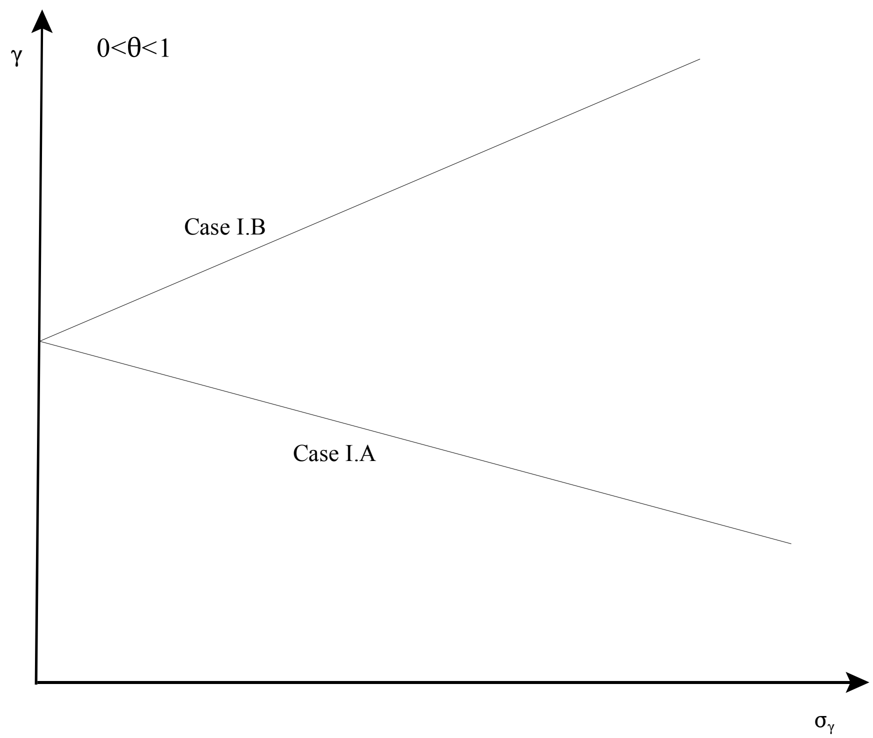
$$\gamma_A = \frac{\hat{F}(\omega) - (\rho+\delta)}{\theta} - \frac{1-\theta}{2}\sigma_y^2\hat{F}(\omega)^2,$$
$$\sigma_{\gamma_A} = \sigma_y\hat{F}(\omega),$$
$$\gamma_B = \frac{1}{\theta}[\frac{1+\theta}{2}\frac{1}{\theta\sigma_y^2} - (\rho+\delta)],$$
$$\sigma_{\gamma_B} = \frac{1}{\theta\sigma_y}.$$

It follows that for large σ_y^2, that is in *Case I.B*, the model predicts a positive relationship between mean growth and the standard deviation of the growth rate, while for small values of σ_y^2, *Case I.A*, the sign of the relationship depends on the magnitude of θ.

[9] In the case of the Cobb-Douglas production function there are two values of α that satisfy $\hat{F}(\alpha^*) = 1/\theta\sigma_y^2$

[10] The reader can check that, in this case, the solution $\alpha^* = \omega$ does not satisfy the second order condition.

Much more interesting from a theoretical point of view is the fact that the model is consistent with two countries with different σ_y^2 to have *exactly the same* σ_γ. To see this, note that for any σ_γ in the range of feasible values —corresponding to the set $[0, \left(\frac{\hat{F}(\omega)}{\theta}\right)^{1/2}]$ in this example— there are two values of σ_y, one less than $\left(\frac{1}{\theta\hat{F}(\omega)}\right)^{1/2}$, and the other greater than this threshold that result in the same σ_γ. The relationship between σ_γ and γ is a correspondence. In Figure (**??**) we show the relationship between σ_γ and γ in the small risk aversion case, $0 < \theta < 1$.

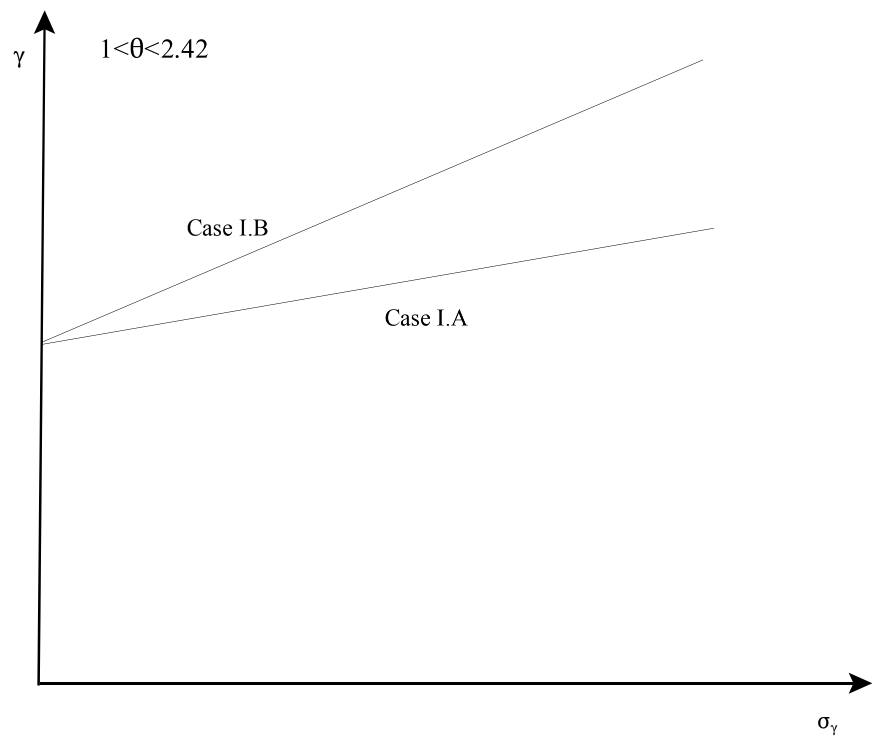


The mapping between σ_γ and γ. $[0 < \theta < 1]$

If the only source of cross-country heterogeneity are differences in the variability of the technology shocks, σ_y, the model implies that all data points should be in one of the two branches of the mapping depicted in Figure (**??**). By arbitrarily choosing the location of these points, the estimated relationship between σ_γ and γ can have any sign, and the estimated value says very little about the deep parameters of the model or, more importantly, about the effects of reducing the variability of shocks on the average growth rate (and welfare).

If we identify high growth countries (or countries with low variability of the exogenous shocks) with high relative income, this version of the model implies that, for low growth countries, the relationship between growth and its variability is negative, while for high growth countries is positive. This is consistent with the findings of Fatás (2001).

Does the nature of the result depend on the assumption $0 < \theta < 1$? It turns out that for almost all $\theta' s$ the relationship between σ_γ and γ is a correspondence, and hence that the model —in the absence of additional assumptions— does not pin down the sign of the correlation between σ_γ and γ.[11] In the case of $\theta > 1$, the size of θ matters only to determine which branch is steeper. If $\theta \; \epsilon \; [1, \; 2.41]$, the low σ_y^2-branch is steeper than the other one. Thus, the relevant Figure is



The mapping between σ_γ and γ. $[1 < \theta < 2.41]$

As in the previous case, if one views a sample as a set of points on these two branches, theory does not restrict the sign of the relationship between σ_γ and γ.

Case II: Shocks to the Human Capital Technology To be written

1.4.2 Elastic Labor Supply

So far the rate of utilization of human capital has been held constant; that is, the number of hours worked has been taken as fixed. In this section we generalize the model to allow for a variable number of hours. The main conceptual reason for

[11] At this point, we have not explored what are the consequences of adding the investment output ratio to the (theoretical) regression. However, to do this is a complete manner it seems necessary to model measurement errors, as the model is stochastically singular.

this generalization is that it opens up another channel through which volatility in fundamentals can affect growth: the choice between goods and leisure. So far, the only mechanism through which changes in uncertainty could affect mean return (and hence average growth) is reallocation of capital resources across technologies, with no change in the rate of utilization. By endogeneizing the labor supply decision we incorporate one additional margin that, we will show, is affected by changes in the variability of exogenous shocks. In terms of technology, we consider the symmetric case in which human and physical capital are produced using capital and (effective) labor in the same proportion as market goods.

To incorporate leisure we modify the utility functional (4) in a way that is consistent with the existence of a balanced growth path. We assume that preferences over consumption and leisure are given by

$$U = E\left[\int_0^\infty e^{-\rho t}\frac{[c_t m(1-n_t)]^{1-\theta}}{1-\theta}dt\,|F_0\right],$$

where m is an increasing function of $1-n$, which we interpret as leisure, and is such that the utility function is concave

The technology is

$$c_t dt + dX_{kt} + dX_{ht} = F(k_t, n_t h_t)dt + \sigma F(k_t, n_t h_t)dW_t,$$

where W_t is a standard Brownian Motion. It is assumed that the function F is homogeneous of degree one in its two arguments. The laws of motion for capital and human capital satisfy

$$\begin{aligned} dk_t &= -\delta_k k_t dt + dX_{kt}, \\ dh_t &= -\delta_h h_t dt + dX_{ht}. \end{aligned}$$

As in the previous case, let $x_t = k_t + h_t$. With this notation, the economy's feasibility constraint is given by,

$$dx_t = \left([F(\alpha_t, n_t(1-\alpha_t)) + \delta_k\alpha_t + \delta_h(1-\alpha_t)]x_t - c_t\right)dt + \sigma F(\alpha_t, n_t(1-\alpha_t))x_t dW_t,$$

where

$$\begin{aligned} \alpha_t x_t &= k_t, \\ (1-\alpha_t)x_t &= h_t. \end{aligned}$$

Thus, the planner's problem is

$$\max_{\{c_t, n_t, \alpha_t\}} E\left[\int_0^\infty e^{-\rho t}\frac{[c_t m(1-n_t)]^{1-\theta}}{1-\theta}dt\,|F_0\right],$$

subject to

$$dx_t = \left([F(\alpha_t, n_t(1-\alpha_t)) + \delta_k\alpha_t + \delta_h(1-\alpha_t)]x_t - c_t\right)dt + \sigma F(\alpha_t, n_t(1-\alpha_t))x_t dW_t.$$

Note that by choosing α as the control we are allowing (effectively) unbounded investment in the two stocks.[12] Let $V(x)$ be the value function of this problem, then the Hamilton-Jacobi-Bellman equation is

$$\rho V(x) = \max_{c,n,\alpha} \left[\frac{[cm(1-n)]^{1-\theta}}{1-\theta} + V'(x)(\mu(\alpha,n)x - c) + \frac{V''(x)x^2}{2}\sigma^2(\alpha,n) \right],$$

where

$$\begin{aligned} \mu(\alpha,n) &= F(\alpha, n(1-\alpha)) + \delta_k \alpha + \delta_h(1-\alpha), \\ \sigma^2(\alpha,n) &= \sigma^2[F(\alpha,n(1-\alpha))]^2. \end{aligned}$$

We consider the case in which the depreciation rate of the two stocks is the same, i.e. $\delta_k = \delta_h = \delta$, it follows that the first order conditions corresponding to the optimization problem are

$$\begin{aligned} \alpha &: \quad (F_k - nF_h)[V'(x)x + V''(x)x^2\sigma^2 F(\alpha, n(1-\alpha))] = 0 \\ c &: \quad c^{-\theta} m(1-n)^{1-\theta} = V'(x)x, \\ n &: \quad c^{1-\theta} m(1-n)^{-\theta} m'(1-n) = V'(x)xF_h(1-\alpha) + V''(x)x^2\sigma^2 FF_h(1-\alpha). \end{aligned}$$

Next, we conjecture that the value function is of the form $V(x) = v\frac{x^{1-\theta}}{1-\theta}$. Given this conjecture, an interior solution is characterized by the following three conditions

$$\begin{aligned} F_k &= nF_h, \\ m(1-n)^{\frac{1-\theta}{\theta}} v^{-1/\theta} &= \frac{\eta - (1-\theta)[\delta + F(1-\theta\frac{\sigma^2}{2}F)]}{\theta}, \\ m(1-n)^{\frac{1-\theta}{\theta}} v^{-1/\theta} \frac{m'(1-n)}{m(1-n)} &= F_h(1-\alpha)[1-\theta\sigma^2 F], \end{aligned}$$

where we have omitted the arguments of F when there is no risk of confusion. The previous system is a system of three equations and three unknowns which verifies the conjecture about the value function.

Simple algebra shows that the mean and the variance of the growth rates are given by,

$$\begin{aligned} \gamma &= \frac{F - \rho + (1-\theta)[\delta - \theta\frac{\sigma^2}{2}F^2]}{\theta}, \\ \sigma_\gamma^2 &= \sigma^2 F^2. \end{aligned}$$

[12] Formally, we are not ruling out the possibility of a negative growss investment in one stock to finance a postive groww investment in another. Imposing the obvious non-negativity constraint is laborious but does not change the basic approach. Moreover, our emphasis is on economies that are near the (stochastic) balanced growth, and subject to fluctuations that are small enough so that aggregate investment in the two types of capital is positive.

In the special case in which the technology is Cobb-Douglas, $F(x,y) = Ax^{\phi}y^{1-\phi}$, it follows that $\alpha = \phi$, and it is independent of the state x. In this case, the optimal level of n is the solution to,

$$\frac{m'(1-n)}{m(1-n)} = \frac{\theta F_h(1-\phi)[1-\theta\sigma^2 F]}{\eta - (1-\theta)[\delta + F(1-\theta\frac{\sigma^2}{2}F)]}.$$

Under standard assumptions (verify) the left side is an increasing function of n. Let $\bar{n}$ be the value of n such that $1 - \theta\sigma^2 F(\phi, \bar{n}(1-\phi)) = 0$. If $\bar{n} < 1$ then the right side of the expression is positive for $n \leq \bar{n}$. Given the Cobb-Douglas assumption, the right side goes to ∞ as $n \to 0$, and it goes to 0 as $n \to \bar{n}$. Thus, a solution exists if we assume that the left side goes to ∞ as $n \to 1$. [Need to check uniqueness; it is sort of a mess].

If the right side is decreasing (as it looks to be the case) then increases in σ^2 reduce the optimal level of n. What is the impact of changing σ^2 upon γ? A direct calculation shows that

$$\frac{\partial\gamma}{\partial n} > 0, \qquad \frac{\partial\gamma}{\partial\sigma^2} = -(1-\theta)\theta\frac{F^2}{2}.$$

Thus, if $0 < \theta < 1$ an increase in σ^2 decreases the mean growth rate. If $\theta > 1$ then the direct variance effect is positive, but we need to check the indirect effect.

Need to do:

- Establish that, at the fixed point, $\frac{\theta F_h(1-\phi)[1-\theta\sigma^2 F]}{\rho-(1-\theta)[\delta+F(1-\theta\frac{\sigma^2}{2}F)]}$ is decreasing.
- Argue how changes in σ move the function $\frac{\theta F_h(1-\phi)[1-\theta\sigma^2 F]}{\rho-(1-\theta)[\delta+F(1-\theta\frac{\sigma^2}{2}F)]}$
- Show that the impact of changes in σ depend on the value of θ.

1.5 The Opportunity Cost View

So far the models we discussed emphasize the idea that increases in the variability of the driving shocks can have positive or negative effects upon the growth rate depending on the relative importance of income and substitution effects. An alternative view is that recessions are "good times' to invest in human capital because labor —viewed as the single most important input in the production of human capital— has a low opportunity cost. In this section we present a model that captures these ideas. The model implies that the time allocated to the formation of human capital is independent of the cycle.[13] It also implies that shocks to the goods production technology

[13]The empirical relationship between investment in human capital and the cycle is mixed. Dellas and Sakellaris (1997) using CPS data for all individuals aged 18 to 22 find that college enrollment id procyclical. Christian (2002) also using the CPS but restricting the sample to 18-19 years olds

have no impact on growth, but that the variability of the shock process in the human capital technology decreases growth.

As before, we concentrate on a representative agent with preferences described by (4). The goods production technology is given by

$$c_t + x_t \leq z_t A k_t^{\alpha} (n_t h_t)^{1-\alpha},$$

where n_t is the fraction of the time allocated to goods production, k_t is the stock of physical capital, and h_t is the stock of human capital. The variable z_t denotes a stationary process. To simplify the theoretical presentation we assume that capital depreciates fully. Thus, goods consumption is limited by

$$c_t \leq z_t A k_t^{\alpha} (n_t h_t)^{1-\alpha} - k_t.$$ [14]

Human capital is produced using only labor in order to capture the idea that the opportunity cost of investing in human capital is market production. The technology is summarized by

$$dh_t = [1 - \delta + B(1 - n_t)] h_t dt + \sigma_h [1 - \delta + B(1 - n_t)] h_t dW_t,$$

where, as before, W_t is a standard Brownian motion.[15]

Given that the problem is convex[16] the competitive allocation solves the planner's problem. It is clear that, given $n_t h_t$, physical capital will be chosen to maximize net output. This implies that consumption is

$$c_t = A^* \hat{z}_t n_t h_t,$$

where $A^* = (A\alpha)^{1/(1-\alpha)}(\alpha^{-1} - 1)$ and $\hat{z}_t = z_t^{1/(1-\alpha)}$. We guess that the relevant state variable is the vector $(\hat{z}_t, h_t)$, and that the value function is of the form

$$V(\hat{z}_t, h_t) = v \frac{(\hat{z}_t h_t)^{1-\theta}}{1-\theta}.$$

(sa as to be able to control for family variables) finds no cyclical effects. Sakellaris and Spilimbergo (2000) study U.S. college enrollment of foreign nationals and conclude that, among those individuals coming from risch countries enrollment is countercyclical, while among students from less developed countries it is countercyclical. Moreover, college enrollment is only a partial measure of investment in human capital. Training (inside and outside business firms) is another (difficult to measure) component of increases in skill acquisition.

[14] This restriction makes it possible to derive the theoretical implications of the model in a simple setting. [Note: I suspect that if capital is allowed to depreciate the equilibrium n_t is no longer constant.]

[15] A special case of this model in which utility is assumed logarithmic, and the goods production function is not subject to shocks is analyzed in De Kek (1999).

[16] Even though our choice of notation somewhat obscures this, the convexity of the technology is apparent by defining $h_{mt} = n_t h_t$ and $h_{st} = (1 - n_t) h_t$, and adding the constraint $h_{mt} + h_{st} \leq h_t$.

Given this guess, the relevant Hamilton-Jacobi-Bellman equation is

$$\rho v \frac{(\hat{z}h)^{1-\theta}}{1-\theta} = \max_x \left\{ \frac{[\frac{A^*}{B}(\mu - x)\hat{z}h]^{1-\theta}}{1-\theta} + v(\hat{z}h)^{1-\theta}x - v(\hat{z}h)^{1-\theta}\theta\frac{\sigma_h^2}{2}x^2, \right\}$$

where $\mu \equiv 1-\delta+B$, and $x = 1-\delta+B(1-n)$. It follows that choosing x is equivalent to choosing n. The solution to the optimization problem is given by the solution to the following quadratic equation

$$x^2 = \frac{2(1+\mu\sigma_h^2)}{(1+\theta)\sigma_h^2}x + \frac{2(\rho-\mu)}{\theta(1+\theta)\sigma_h^2}.$$

In order to guarantee that utility remains bounded even in the case $\sigma_h = 0$ is is necessary to assume that $\rho - \mu > 0$. Simple algebra shows that the positive root of the previous equation is such that increases in σ_h decrease x. It follows that the stochastic process for h_t is given by

$$dh_t = xh_t dt + \sigma_h h_t dW_t$$

We now discuss the implications of the model for the growth rate of consumption (or output). Even though our results do not depend on the particular form of the z_t process, it is convenient to consider the case in which z_t is a geometric Brownian motion that is possibly correlated with the shock to the human capital. Specifically, we assume that

$$dz_t = z_t(\sigma_w dW_t + \sigma_m dM_t),$$

where M_t is a standard Brownian motion that is uncorrelated with W_t. Ito's lemma implies that

$$d\hat{z}_t = \frac{\alpha}{(1-\alpha)^2}\frac{\sigma_w^2+\sigma_m^2}{2}\hat{z}_t dt + \frac{\alpha}{(1-\alpha)}\hat{z}_t(\sigma_w dW_t + \sigma_m dM_t).$$

In equilibrium, consumption (and net output) is given by

$$c_t = \frac{A^*}{B}(\mu - x)\hat{z}_t h_t.$$

Applying Ito's lemma to this expression, we obtain that the growth rate of consumption

$$\frac{dc_t}{c_t} = \frac{dh_t}{h_t} + \frac{d\hat{z}_t}{\hat{z}_t} + \frac{\alpha x}{(1-\alpha)}\sigma_h\sigma_w dt,$$

or, taking a discrete time approximation,

$$\gamma_t = x(1+\frac{\alpha}{(1-\alpha)}\sigma_h\sigma_w) + \frac{\alpha}{(1-\alpha)^2}\frac{\sigma_w^2+\sigma_m^2}{2} + [(\frac{\alpha}{(1-\alpha)}\sigma_w + \sigma_h x)\tilde{W}_t + \frac{\alpha}{(1-\alpha)}\sigma_m\tilde{M}_t] \quad (20)$$

$$\gamma_t = \gamma + \varepsilon_t, \qquad \varepsilon_t \sim N(0,\sigma_\gamma^2),$$

$$\sigma_\gamma^2 = (\frac{\alpha}{(1-\alpha)}\sigma_w + \sigma_h x)^2 + (\frac{\alpha}{(1-\alpha)}\sigma_m)^2 \quad (21)$$

Equation (****) completely summarizes the implications of the model for the data. There are several interesting results. To simplify the notation, we will refer to W_t as the aggregate shocks and to M_t as the idiosyncratic component of the productivity shock in the goods sector.

- The share of the time allocated to human capital formation —the engine of growth in this economy— is independent of the variability of the technology shock in the goods sector, as measured by (σ_w, σ_m).
- High (σ_w, σ_m) economies are also high growth economies. Thus, if cross-country differences in σ_γ are mostly due to differences in (σ_w, σ_m), the model implies a positive correlation between the standard deviation of the growth rate and mean growth.
- It can be shown that increases in σ_h result in *decreases* in $\sigma_h x$. Thus, if countries differ in this dimension the model also implies a positive relationship between σ_γ and γ.
- In the model, investment in physical capital (as a fraction of output) is α, independently of the distribution of the shocks. Thus, there is no sense that a regression that shows that variability does not affect the rate of investment provides evidence against the role of shocks in development.
- This lack of (measured) effect on both physical and human capital investment should not be interpreted as evidence against the proposition that incentives for human or physical capital accumulation matter for growth. It is easy enough to include a tax/subsidy to the production of human capital —consider a policy that affects B— and it follows that this policy affects growth.
- De Hek's example uses logarithmic utility function and no productivity shocks. In this case the appropriate functions are [Note: this is for our consumption; it will not make it in the final version.]

  1. x is the solution to
  $$\frac{\rho}{\mu - x} = 1 - \sigma_h^2 x$$
  which implies that $\partial x / \partial \sigma_h < 0$.
  2. A simple (but painful; check!) calculation shows that $\partial (x\sigma_h)/\partial \sigma_h < 0$.
  3. The growth rate process is the same as above.
  4. De Hek (and others) seem to claim that the negative relationship between σ_h and x implied by their models gives support to the findings of Ramey and Ramey. However, they are wrong. To see this, let's shut down other

forms of uncertainty, i.e. set $\sigma_w = \sigma_m = 0$, and let's consider the relationship between σ_γ and γ. In this case we have

$$\begin{aligned} \gamma &= x, \\ \sigma_\gamma &= \sigma_h x. \end{aligned}$$

The results we described show that $\partial\gamma/\partial\sigma_h < 0$, and $\partial\sigma_\gamma/\partial\sigma_h < 0$ and, hence, that the model implies a positive relationship between σ_γ and γ!

1.6 More on Government Spending, Taxation, and Growth

In this section we consider a simple Ak model in which a government uses distortionary taxes to finance an exogenously given stochastic process for government spending. Our analysis follows Eaton (1981).[17]

The representative household maximizes utility —given by (4)— by choosing consumption and saving in either capital or bonds. However, given that tax policy is exogenously fixed, it is not the case that the rate of return on bonds is risk free. On the contrary, since the government issues bonds to make up for any difference between revenue and spending it is necessary to let the return on bonds to be stochastic.

The representative household problem is

$$\max U = E\left[\int_0^\infty e^{-\rho t}\frac{c_t^{1-\theta}}{1-\theta}dt\,|F_0\right]. \tag{22}$$

subject to

$$\begin{aligned} dk_t &= (r_k k_t - c_{1t})dt + \sigma_k dW_t, && (23a) \\ db_t &= (r_b b_t - c_{2t})dt + \sigma_b dW_t, && (23b) \\ c_t &= c_{1t} + c_{2t}, && (23c) \end{aligned}$$

where k_t is interpreted as capital and b_t as bonds. As before, it is possible to simplify the analysis by using wealth as the state variable. Let $x_t \equiv k_t + b_t$. With this notation, the single budget constraint is given by,

$$dx_t = [(\alpha_t r_k + (1-\alpha_t)r_b)x_t - c_t]dt + (\alpha_t\sigma_k + (1-\alpha_t)\sigma_b)x_t dW_t.$$

Since this problem is a special case of the "general" two risky asset model, it follows that the optimal solution is characterized by

$$\begin{aligned} \alpha &= \frac{\frac{r_k - r_b}{\theta} - \sigma_b(\sigma_k - \sigma_b)}{(\sigma_k - \sigma_b)^2}, && (24a) \\ c_t &= \underbrace{\frac{\rho - (1-\theta)[\alpha r_k + (1-\alpha)r_b - \theta\frac{(\alpha\sigma_k + (1-\alpha)\sigma_b)^2}{2}]}{\theta}}_{c} x_t. && (24b) \end{aligned}$$

[17] For extensions of this model, see Turnovski (1995)

The set of feasible allocations is the set of stochastic process that satisfy

$$\begin{aligned} dk_t &= (Ak_t - c_t)dt + \sigma Ak_t dW_t - dG_t, \\ dG_t &= gAk_t dt + g'\sigma Ak_t dW_t. \end{aligned}$$

Thus, the government consumes a fraction g of the non-stochastic component of output, and a fraction g' of the stochastic component. Taxes are levied on the deterministic and stochastic components of output at (possibly) different rates. More precisely, the stochastic process for tax revenue satisfies

$$dT_t = \tau Ak_t dt + \tau' \sigma Ak_t dW_t.$$

It follows that the rate of return on capital satisfies

$$\begin{aligned} r_k &= (1-\tau)A, \\ \sigma_k &= (1-\tau')\sigma A. \end{aligned}$$

The government budget constraint requires that the excess of spending over tax revenue be financed through bond issues

$$B_t + dG_t - dT_t = p_t dB_t,$$

where $p_t B_t = b_t$ is the value of bonds issued. The stock of capital evolves according to

$$dk_t = ((1-g)A - \frac{c_t}{k_t})k_t dt + \sigma(1-g')Ak_t dW_t.$$

Note that

$$\frac{c_t}{k_t} = c\frac{x_t}{k_t} = c(1 + \frac{1-\alpha}{\alpha}) = \frac{c}{\alpha}.$$

Since, in equilibrium, it must be the case that, in all states of nature, the growth rate of private wealth and the growth rate of the capital stock are the same[18], it is necessary that

$$\begin{aligned} \alpha r_k + (1-\alpha_t)r_b - c &= (1-g)A - \frac{c}{\alpha}, \\ \alpha\sigma_k + (1-\alpha)\sigma_b &= \sigma(1-g')A. \end{aligned}$$

This is a system of two equations and two unknowns, r_b and σ_b, and its solution completely describes de equilibrium (along with the definition of α and c). In other to derive the equilibrium growth rate it is useful to define

$$\begin{aligned} \Delta_r &= r_k - r_b, \\ \Delta_\sigma &= \sigma_k - \sigma_b. \end{aligned}$$

[18] This, of course, depends on the fact that the solution to the individual agent problem is such that bonds and capital are held in fixed proportions.

Since (r_k, σ_k) are known, knowledge of $(\Delta_r, \Delta_\sigma)$ suffices to determine the equilibrium process for the growth rate. With this notation, it follows that

$$\alpha = \frac{\Delta_r}{\theta \Delta_\sigma} - \frac{\sigma_k}{\Delta_\sigma} + 1.$$

Imposing this in (***) we get that

$$\sigma_\gamma \equiv \sigma(1 - g')A = \frac{\Delta_r}{\theta \Delta_\sigma}.$$

The mean rate of return in the representative agent's portfolio is, in equilibrium,

$$\alpha r_k + (1 - \alpha) r_b = r_k + \theta(\sigma_\gamma^2 - \sigma_\gamma \sigma_k).$$

Given that the stochastic process for the growth rate of wealth is

$$\frac{dx_t}{x_t} = [\alpha r_k + (1 - \alpha) r_b - c]dt + [\alpha \sigma_k + (1 - \alpha)\sigma_b]dW_t,$$

it follows that, in equilibrium, the process for the growth rate is

$$\frac{dx_t}{x_t} = \left[\frac{\alpha r_k + (1 - \alpha) r_b - \rho}{\theta} - (1 - \theta)\frac{(\alpha \sigma_k + (1 - \alpha)\sigma_b)^2}{2}\right] dt + [\alpha \sigma_k + (1 - \alpha)\sigma_b]dW_t,$$

or,

$$\begin{aligned} \frac{dx_t}{x_t} &= \left[\frac{(1 - \tau)A - \rho}{\theta} + \theta \frac{\sigma_\gamma^2}{2} - \sigma_\gamma \sigma (1 - \tau')A\right] dt + \sigma_\gamma dW_t, \\ \sigma_\gamma &= \sigma(1 - g')A. \end{aligned}$$

as in the non-stochastic version of the model, increases in the mean tax rate, τ, unambiguously reduce growth. The effect of other parameters depends on the sources of cross country variation. If countries are identical in terms of the distribution of technology shocks, but differ only in terms of policy variables we obtain

- Holding other parameters constant, increases in τ', the tax rate at which the random component of the return is taxed, increases growth. The intuition for this result is simple: An increase in τ' reduces the private variance corresponding to investing in capital, and this induces private agents to allocate a larger share of their portfolio to capital which increases growth. [Note: check what this does to the saving ratio.]

- Holding other parameters constant, and increase in g' has ambiguous effects. The effect of a change in σ_γ on the mean growth rate is

$$\frac{\partial \gamma}{\partial \sigma_\gamma} = (1 + \theta)\sigma_\gamma - \sigma_k = \sigma A[\theta(1 - g') - (g' - \tau')].$$

Since the requirement that $\alpha \in (0,1)$ implies that $g' > \tau'$, the feasible set of g' is $(\tau', 1]$. It is easy to see that there exists one value of g', which we denote $\hat{g}'$, such that

$$\frac{\partial \gamma}{\partial \sigma_\gamma} \gtreqless 0 \qquad \Leftrightarrow \qquad g' \lesseqgtr \hat{g}'.$$

On the other hand it is the case that

$$\frac{\partial \sigma_\gamma}{\partial g'} < 0.$$

These two results combined imply that, for "small" values of g', increases in g' reduce σ_γ and increase γ, while for larger values the opposite is the case. Thus, if most of the differences across countries are due to differences in g' —which corresponds to the volatility of non-productive government spending— this simple model can explain the non-linearity in the relationship between σ_γ and γ found in the data: If relatively poor countries are small g' (high σ_γ) countries, the relationship between σ_γ and γ is negative. On the other hand, if rich countries are high g' countries (low σ_γ) then the model predicts a positive relationship between σ_γ and γ.

- The "true" model describing the relationship between the growth rate and its standard deviation is highly non-linear. It is given by,

$$\begin{aligned} \gamma_{it} &= \left[\frac{(1-\tau_i)A_i - \rho}{\theta} + \theta \frac{\sigma_{\gamma i}^2}{2} - \sigma_{\gamma i}\sigma_i(1-\tau_i')A_i \right] dt + \varepsilon_{it}, \\ \varepsilon_{it} &= \sigma_{\gamma i} dW_{it}, \qquad dW_{it} \sim N(0, dt). \end{aligned}$$

  Thus, it is not clear what the linear projection of $\frac{(1-\tau_i)A_i - \rho}{\theta} + \theta \frac{\sigma_{\gamma i}^2}{2} - \sigma_{\gamma i}\sigma_i(1 - \tau_i')A_i$ on variables that affect investment and $\sigma_{\gamma i}$ is. In particular, it seems to depend on the joint distribution of (τ', g'). [Note: Have to think some more about what the model says for the Ramey and Ramey regression.]

1.7 Missing

1. An incomplete markets model (role of market incompleteness in growth)
2. Quantitative papers. What do we include here?
   (a) Jones, Manuelli and Stacchetti (2003)
   (b) Jones, Manuelli and Siu and Stacchetti (2003)
   (c) Krebs (2002a), Krebs (2002b)
   (d) Barlevi (2002)
   (e) de Hek (1999)

(f) Collard (****)

(g) Fatas (****)

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