Econ 879: Topics in Dynamic Economics

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Mathematical Preliminaries

- σ-algebra: Let Ω be a set. A class $\mathcal{F}$ of subsets of Ω is a σ-algebra if:
  - $A \in \mathcal{F} \rightarrow A^c \in \mathcal{F}$.
  - $A, B \in \mathcal{F} \rightarrow A \cup B \in \mathcal{F}$.
  - $\Omega \in \mathcal{F}$ ($\varnothing \in \mathcal{F}$)
  - if $A_i \in \mathcal{F}$, $i = 1, 2, \ldots$ then $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$.

- A set $A \in \mathcal{F}$ is called measurable.

- Example: Toss a coin twice. In this case

$$\Omega = \{(H,H),(H,T),(T,H),(T,T)\}$$

Let $\mathcal{F}_a$ be given by

$$\mathcal{F}_a = \{\Omega, \varnothing, \{(H,H),(H,T)\}, \{(T,H),(T,T)\}\}$$

Is it a σ-algebra?

- Example: Let $\mathcal{F}_b$ be given by

$$\mathcal{F}_b = \{\Omega, \varnothing, \{(H,H)\}, \{(H,T),(T,H),(T,T)\}\}$$

Is it a σ-algebra?

Theorem 1 (Monotone Convergence I) *Let $Y,\ X_1,\ X_2, ...$be random variables.*

1. *If $X_n \geq Y$, for all $n \geq 1$ and $E[Y] > -\infty$ and $X_n \nearrow X$, then $E[X_n] \nearrow E[X]$.*

2. *If $X_n \leq Y$, for all $n \geq 1$ and $E[Y] < \infty$ and $X_n \searrow X$, then $E[X_n] \searrow E[X]$.*

Theorem 2 (Monotone Convergence II) *Let $X_1,\ X_2, ...$be random variables. Then if $X_n \leq X_{n+1}$ then*

$$\int \lim_{n\to\infty} X_n d\mu = \lim_{n\to\infty} \int X_n d\mu.$$

Theorem 3 (Monotone Convergence III) *Let $X_1,\ X_2, \ldots$be random variables. Then if $X_n \leq X_{n+1}$ and if*

$$\int X_n d\lambda \leq B$$

for some B. Then there exists a random variable X such that $X_n \to X,\ \lambda{-}as$ and

$$\lim_{n\to\infty} \int X_n d\lambda = \int X d\lambda$$

Theorem 4 (Fatou's lemma) *Let $Y,\ X_1,\ X_2, \ldots$ be random variables.*

1. *If $X_n \geq Y$, for all $n \geq 1$ and $E[Y] > -\infty$, then $E[\underline{\lim} X_n] \leq \underline{\lim} E[X_n]$.*

2. *If $X_n \leq Y$, for all $n \geq 1$ and $E[Y] < \infty$, then $\overline{\lim} E[X_n] \leq E[\overline{\lim} X_n]$.*

3. *If $\mid X_n \mid \leq Y$, for all $n \geq 1$ and $E[Y] < \infty$, then $E[\underline{\lim} X_n] \leq \underline{\lim} E[X_n] \leq \overline{\lim} E[X_n] \leq E[\overline{\lim} X_n]$.*

Theorem 5 (Lebesgue Dominated Convergence) *Let* $Y,\ X_1,\ X_2,...$*be random variables such that* $\mid X_n \mid\leq Y$, $E[Y]<\infty$, *and* $X_n \overset{a.s.}{\to} X$, *then*

1. $E[\mid X \mid]<\infty$

2. $E[\mid X_n \mid]\to E[X]$

3. $E[\mid X_n - X \mid]\to 0.$

Corollary 6 *Let* $Y,\ X_1,\ X_2,...$*be random variables such that* $\mid X_n \mid\leq Y$, $E[Y^p]<\infty$, *for some* $p>0$, *and* $X_n \overset{a.s.}{\to} X$, *then* $E[\mid X \mid^p]<\infty$ *and* $E[\mid X_n - X \mid^p]\to 0.$

Theorem 7 (Fubini) *Let $X = X(\omega_1, \omega_2)$ be an $\mathcal{F}_1 \otimes \mathcal{F}_2$ measurable function, integrable with respect to the measure $\mu_1 \times \mu_2$. Then*

1. *The integrals*

$$\begin{array}{c} \int_{\Omega_1} X(\omega_1, \omega_2)\mu_1(d\omega_1) \\ \int_{\Omega_2} X(\omega_1, \omega_2)\mu_2(d\omega_2) \end{array}$$

are defined for all (ω_1, ω_2), and they are $\mathcal{F}_2$ and $\mathcal{F}_1$ measurable.

2. $\mu_2\{\omega_2 : \int_{\Omega_1} \mid X(\omega_1, \omega_2) \mid \mu_1(d\omega_1) = \infty\} = 0$*, and* $\mu_1\{\omega_1 : \int_{\Omega_2} \mid X(\omega_1, \omega_2) \mid \mu_2(d\omega_2) = \infty\} = 0$*, and*

$$\begin{aligned} \int_{\Omega_1 \times \Omega_2} X(\omega_1, \omega_2)\mu_1 \times \mu_2(d\omega_1, d\omega_2) &= \int_{\Omega_1} [\int_{\Omega_2} X(\omega_1, \omega_2)\mu_2(d\omega_2)]\mu_1(d\omega_1) \\ &= \int_{\Omega_2} [\int_{\Omega_1} X(\omega_1, \omega_2)\mu_1(d\omega_1)]\mu_2(d\omega_2). \end{aligned}$$

Remark 8 *If* $\Omega_1 = [0, \infty)$, *and* $\mathcal{F}_1 = \mathcal{B}(\Re)$, $\mu_1 =$ *Lebesgue, and* $\mu_2, \Omega_2, \mathcal{F}_2$ *arbitrary it says that if*

$$E\left[\int_0^\infty \mid X(t,\omega) \mid dt\right] < \infty$$

then,

$$E\left[\int_0^\infty X(t,\omega) dt\right] = \int_0^\infty E[X(t,\omega)] dt$$

Theorem 9 (Tonelli) *If* $X(t,\omega) \geq 0$ *and*

$$\int_{[0,\infty)\times\Omega} X(t,\omega)\lambda \times \mu(dt, d\omega)$$

is defined (but it could be infinite), we still have

$$E\left[\int_0^\infty X(t,\omega)dt\right] = \int_0^\infty E[X(t,\omega)]dt$$

although it could be equal to infinity.

- Let Ω be a space of events, or sample space, and P a probability measure. Let $\mathcal{F}$ be σ-algebra of subsets. Let $S \subseteq \Re^d$ be the state space, and $\mathcal{S}$ the Borel σ-algebra. A stochastic process is a function

$$X : [0, \infty) \times \Omega \to S$$

such that, for each t, $X(t, \bullet)$ is $\mathcal{F}$-measurable. The mapping

$$X(\bullet, \omega) : [0, \infty) \to S$$

is a sample path.

- $X(t, \bullet)$ is $\mathcal{F}$-measurable means that $B \in \mathcal{S} \to \{\omega : X(t, \omega) \in B\} \in \mathcal{F}$.

Definition 10 (Filtration) *A filtration is a non-decreasing family of sub σ-fields of $\mathcal{F}$, $\mathcal{F}_s \subseteq \mathcal{F}_t$, for $0 \le s \le t < \infty$. Let $\mathcal{F}_\infty = \sigma(\cup_{t\ge 0}\mathcal{F}_t)$*

$\mathcal{F}_{t-} \equiv \sigma(\cup_{s<t}\mathcal{F}_s)$,

$\mathcal{F}_{t+} \equiv \cap_{\epsilon>0}\mathcal{F}_{t+\epsilon}$

A filtration is right continuous if $\mathcal{F}_t = \mathcal{F}_{t+}$.

Definition 11 (Adapted) *A process X is adapted to $\{\mathcal{F}_t\}$ if X_t is $\mathcal{F}_t$-measurable*

- Example: Coin toss. In this case

$$\Omega = \{(H, H), (H, T), (T, H), (T, T)\}$$

Let's consider the natural filtration, i.e. the filtration generated by each coin toss. Before we toss the coin, $\mathcal{F}_0$ is

$$\mathcal{F}_0 = \{\Omega, \varnothing, \}.$$

After the first toss (at time $t = 1$) the information available (i.e. whether the first toss is a head or a tail) is summarized by $\mathcal{F}_1$ where

$$\mathcal{F}_1 = \{\Omega, \varnothing, \{(H,H),(H,T)\}, \{(T,H),(T,T)\}\}.$$

After the second toss (i.e. in the second period) we have that

$$\begin{aligned}\mathcal{F}_2 \;=\; & \{\Omega, \varnothing, \{(H,H),(H,T)\}, \{(T,H),(T,T)\}, \\ & \{(H,H)\}, \{(H,T)\}, \{(T,H)\}, \{(T,T)\}, \\ & \text{plus other sets}\}\end{aligned}$$

- Let X be $\{\mathcal{F}_t\}$ adapted. Then, it must be the case that be given by

$$X_1(H,H) = X_1(H,T) = x_1^H,$$

and

$$X_1(T,H) = X_1(T,T) = x_1^T.$$

In this case

$$\begin{aligned} X_1^{-1}(x_1^H) &= \{(H,H),(H,T)\} \in \mathcal{F}_1, \\ X_1^{-1}(x_1^T) &= \{(T,H),(T,T)\} \in \mathcal{F}_1. \end{aligned}$$

- Let Y be such that

$$\begin{aligned} Y_1(H,H) &= 1, \\ Y_1(H,T) &= Y_1(T,H) = Y_1(T,T) = 0, \end{aligned}$$

and

$$Y_2(\omega) = Y_1(\omega).$$

Is Y $\{\mathcal{F}_t\}$ adapted?

- Is Y $\mathcal{F}_b$ measurable?

$$\mathcal{F}_b = \{\boldsymbol{\Omega}, \varnothing, \{(H,H)\}, \{(H,T),(T,H),(T,T)\}\}$$

Definition 12 (Stopping Time) *Let $(\Omega, \mathcal{F})$ be a measurable space equipped with the filtration $\{\mathcal{F}_t\}$. A random time T is a stopping time of the filtration if*

$$\{\omega : T(\omega) \leq t\} \in \mathcal{F}_t.$$

It is called **optional time if**

$$\{\omega : T(\omega) < t\} \in \mathcal{F}_t.$$

It follows that all constants are stopping times and if $\{\mathcal{F}_t\}$ is right continuous every stopping time is optional (exercise).

There is a special class of stopping times denoted *hitting times*. Let $\Gamma \in \mathcal{B}(R^d)$ and let

$$H_\Gamma(\omega) = \inf\{t \geq 0; X_t(\omega) \in \Gamma\}$$

Results:

- If Γ is open then H_Γ is optional.

- If Γ is closed and the sample paths of X are continuous, then H_Γ is a stopping time.

1. T optional and $\theta > 0$ implies that $T + \theta$ is a stopping time.

2. T and S stopping times $\longrightarrow$

$$T \wedge S$$
$$T \vee S$$
$$T + S$$

are stopping times

3. T and S optional times $\longrightarrow T + S$ optional. $T + S$ is a stopping time if either

$$\begin{aligned} a) \qquad T &> 0,\ S > 0, \\ b) \qquad T &> 0,\ T \textit{ is a stopping time} \end{aligned}$$

4. $\{T_n\}$ sequence of stopping times so is $\sup_{n \geq 1} T_n$.

Definition 13 *Let T be a stopping time of the filtration $\{\mathcal{F}_t\}$. The σ-field $\mathcal{F}_T$ of events determined prior to the stopping time T consists of those events $A \in \mathcal{F}$ for which*

$$for\ all\ t, \qquad A \cap \{T \le t\} \in \mathcal{F}_t$$

- If $T = t$ then $\mathcal{F}_T = \mathcal{F}_t$.

- Let T be a stopping time and S a random time such that $S \geq T$. If S is $\mathcal{F}_T$-measurable it is also a stopping time.

- Let T and S be stopping times. For any $A \in \mathcal{F}_S$

  $$A \cap \{S \le T\} \in \mathcal{F}_T.$$

  In particular, if $S \le T \rightarrow \mathcal{F}_S \subseteq \mathcal{F}_T$.

- Let T and S be stopping times. Then,

$$\mathcal{F}_{T\wedge S} = \mathcal{F}_T \cap \mathcal{F}_S,$$

and each of the following events

$$\begin{aligned} \{T &< S\}, \\ \{S &< T\}, \\ \{T &\leq S\}, \\ \{T &= S\}, \end{aligned}$$

belongs to $\mathcal{F}_{T\wedge S} = \mathcal{F}_T \cap \mathcal{F}_S$.

Definition 14 (Progressively Measurable) *X is called progressively measurable with respect to $\{\mathcal{F}_t\}$ if for all $t \geq 0$ and $A \in \mathcal{B}(R^d)$ the set*

$$\{(s,\omega) : 0 \leq s \leq t,\ \omega \in \Omega,\ X_s(\omega) \in A\}$$

belongs to the product σ-field $\mathcal{B}([0,t]) \otimes \mathcal{F}_t$.

Definition 15 (Usual Conditions) *A Filtration $\{\mathcal{F}_t\}$ satisfies the usual conditions if it is right continuous and $\mathcal{F}_0$ contains all the $P-null$ sets in $\mathcal{F}$.*

Proposition 16 *If the process X has RCLL paths and is adapted to the filtration $\{\mathcal{F}_t\}$ which satisfies the usual conditions, then there exists a sequence of stopping times $\{T_n\}$ which exhausts the jumps of X, i.e.*

$$\{(t,\omega) : X_t(\omega) \neq X_{t-}(\omega)\} \subseteq \cup_{n=1}^{\infty}\{(t,\omega) : T_n(\omega) = t\}$$

Continuous Time Martingales

- Maintained assumptions:

1. $(\Omega, \mathcal{F}, P)$ is a probability space.

2. $\{\mathcal{F}_t\}$ is a filtration (to which X_t is adapted)

3. $E[\mid X_t \mid] < \infty$, for all $t \geq 0$.

Definition 17 (Submartingale-Supermatingale-Martingale) X *is a submartingale (supermartingale) is for all* $0 \leq s < t < \infty$ *we have* $P - as$

$$
\begin{aligned}
X_s &\leq E[X_t \mid \mathcal{F}_s] \\
(X_s &\geq E[X_t \mid \mathcal{F}_s]).
\end{aligned}
$$

If X *is both a sub and a supermartingale then it is a martingale*

Inequalities

- Let X be a submartingale and let $\mathcal{F}_\infty = \sigma(\cup_{t\geq 0}\mathcal{F}_t)$, then it follows that if there exists an F_∞-measurable random variable, X_∞, such that

$$X_t \leq E[X_\infty \mid \mathcal{F}_t], \qquad t \geq 0, \ P - as,$$

  then we say that $\{X_t, \{\mathcal{F}_t\}, 0 \leq t \leq \infty\}$ is a submartingale with *last element* X_∞.

- This also applies to martingales

Convergence Results

- Assume that all processes are *right continuous*.

Theorem 18 (Submartingale Convergence) *Let* $\{X_t, \{\mathcal{F}_t\}, 0 \leq t < \infty\}$ *be a right continuous submartingale. If*

$$C \equiv \sup_{t \geq 0} E[X_t^+] < \infty,$$

then

$$\begin{aligned} X_\infty(\omega) &\equiv \lim_{t\to\infty} X_t(\omega) \qquad exists\ P - as, \\ E[\quad | & \quad X_\infty |] < \infty. \end{aligned}$$

where

$$X_t^+ = \max\{X_t, 0\}$$

- If the submartingale is **non-negative** then the existence part is guaranteed without any further assumptions and

$$\{X_t, \{\mathcal{F}_t\}, 0 \leq t \leq \infty\}$$

is a submartingale.

- If instead of a submartingale we have a supermartingale then the analogous conditions/results are

$$C \equiv \sup_{t \geq 0} E[X_t^-] < \infty,$$

then

$$\begin{aligned} X_\infty(\omega) &\equiv \lim_{t \to \infty} X_t(\omega) \qquad exists\ P - as, \\ E[&\mid X_\infty \mid] < \infty. \end{aligned}$$

where

$$X_t^- = \mathsf{max}\{-X_t, 0\}$$

Definition 19 (Uniform Integrability) *A collection of random variables* $\{X_t\}$ *defined on* $(\Omega, \mathcal{F}, P)$ *is uniformly integrable if*

$$\int_{\{|X_t| \geq c\}} | X_t(\omega) | \, dP(\omega)$$

converges to 0, uniformly in t*, as* $c \to \infty$.

Definition 20 (L^1 norm) $\| X_t \|_1 \equiv \int | X_t(\omega) | \, dP(\omega)$*, provided it is finite.*

- If $\{X_t, \{\mathcal{F}_t\}, 0 \leq t < \infty\}$ is a **uniformly integrable** supermartingale then,

$$\sup_t \| X_t \|_1 < \infty,$$

and since $E[X_t^-] \leq \| X_t \|_1$ the supermartingale version of the theorem implies

$$X_\infty(\omega) \equiv \lim_{t \to \infty} X_t(\omega) \qquad exists\ P - as.$$

By uniform integrability convergence is also in $L^1 \to \{X_t, \{\mathcal{F}_t\}, 0 \leq t \leq \infty\}$ is also a supermartingale!

Theorem 21 (Optional Sampling) *Let $\{X_t, \{\mathcal{F}_t\}, 0 \le t \le \infty\}$ be a right continuous submartingale with last element X_∞ and let $S \le T$ be two optional times of the filtration $\{\mathcal{F}_t\}$. Then,*

1.

$$X_S \le E[X_T \mid \mathcal{F}_{S+}], \qquad t \ge 0,\ P - as,$$

2. *If S is a stopping time then*

$$X_S \le E[X_T \mid \mathcal{F}_S], \qquad t \ge 0,\ P - as,$$

and, in particular,

$$E[X_0] \le E[X_T].$$

3. *For a martingale with last element X_∞ the previous inequalities are equalities.*

- **Note**: the theorem assumes that the process has a last element X_∞. This is a strong assumption. See before to check the iff conditions for this!

- Example: Gambling.

- Example:

$$\begin{aligned} X_t &= \sum_{j=1}^{t} Z_j, \qquad Z_j \in \{-1, 1\}, \qquad p = 1/2, \\ X_0 &= 0, \end{aligned}$$

and

$$T = \inf\{t : X_t = 1\}$$

- Is it necessary for the process to have a last element?

- Alternative conditions under which the theorem holds

1. Let $\{X_t, \{\mathcal{F}_t\}, 0 \leq t < \infty\}$ be a right continuous submartingale and let the **optional** times satisfy $S \leq T$, then OST holds if either of these two conditions is satisfied:

   (a) T is bounded (i.e. $\exists a$ such that $T(\omega) \leq a \ P - as$)

   (b) There exists an integrable random variable Y such that$X_t \leq E[Y \mid \mathcal{F}_t], \qquad t \geq 0, \ P - as.$

2. Let $\{X_t, \{\mathcal{F}_t\}, 0 \leq t < \infty\}$ be a right continuous submartingale and let the **stopping** times satisfy $S \leq T$, then

(a) $\{X_{T\wedge t}, \{\mathcal{F}_t\}, 0 \le t < \infty\}$ is a submartingale,

(b)

$$X_{S\wedge t} \le E[X_{T\wedge t} \mid \mathcal{F}_{S\wedge t}], \qquad t \ge 0,\ P - as.$$

3. A submartingale of constant expectation, $E[X_t] = E[X_0]$ is a martingale.

4. Let $\{X_t, \{\mathcal{F}_t\}, 0 \le t < \infty\}$ be a right continuous process with

$$E[\mid X_t \mid] < \infty, \qquad 0 \le t < \infty,$$

then this process is a submartingale iff for every pair of stopping times $S \le T$ of the filtration $\{\mathcal{F}_t\}$ we have

$$E[X_S] \le E[X_T].$$

5. Let T be a bounded stopping time of the filtration $\{\mathcal{F}_t\}$ which satisfies the usual conditions. Let

$$\tilde{\mathcal{F}}_t \equiv \mathcal{F}_{T+t}$$

then $\tilde{\mathcal{F}}_t$ also satisfies the usual conditions and

(a) If $\{X_t, \{\mathcal{F}_t\}, 0 \leq t < \infty\}$ is a right continuous submartingale then so is

$$\tilde{X}_t = X_{T+t} - X_T$$

for the filtration $\{\tilde{\mathcal{F}}_t\}$.

(b) If $\{\tilde{X}_t, \{\mathcal{F}_t\}, 0 \leq t < \infty\}$ is a right continuous submartingale with $\tilde{X}_0 = 0$ $P - as$ then

$$X_t = \tilde{X}_{(t-T)\vee 0}$$

is an $\{\mathcal{F}_t\}$ submartingale

- **Definition (Quadratic Variation)**: Fix $t > 0$ and let $\Pi = \{t_0, t_1, ..., t_m\}$ with $0 = t_0 \leq t_1 \leq ... \leq t_m = t$. Then, the p^{th} variation of X over Π is

$$V_t^{(p)}(\Pi) = \sum_{k=1}^{m} \mid X_{t_k} - X_{t_{k-1}} \mid^p .$$

Let the mesh of Π, denoted $\| \Pi \| = \max_{1 \leq k \leq m} \mid t_k - t_{k-1} \mid$. Then, if $V_t^{(2)}(\Pi)$ converges, in some sense, as $\| \Pi \| \to 0$, then the limit is the quadratic variation. If $X \in \mathcal{M}_2^c$ we have that

$$\lim_{\|\Pi\| \to 0} V_t^{(2)}(\Pi) \xrightarrow{P} \langle X \rangle_t.$$

- Moreover, if $\{X_t, \{\mathcal{F}_t\}, 0 \leq t < \infty\}$ is **continuous** and such that, for each fixed t, and some p

$$\lim_{\|\Pi\| \to 0} V_t^{(p)}(\Pi) \xrightarrow{P} L_t$$

  where L_t is a random variable taking values in $[0, \infty)$, then

- $q > p \to \lim_{\|\Pi\| \to 0} V_t^{(q)}(\Pi) \xrightarrow{P} 0$,

- $q < p \to \lim_{\|\Pi\| \to 0} V_t^{(q)}(\Pi) \xrightarrow{P} \infty$, on the event $\{L_t > 0\}$

Brownian Motion

Definition 22 *A standard one dimensional Brownian Motion (BM) is a continuous adapted process* $W = \{W_t, \{\mathcal{F}_t\}, 0 \leq t < \infty\}$ *defined on some probability space* $(\Omega, \mathcal{F}, P)$ *with the properties that*

1. $W_0 = 0$

2. *For* $0 \leq s < t,\ W_t - W_s$ *is independent of* $\mathcal{F}_s$*.*

3. $W_t - W_s \sim N(0, t - s)$

- Is this definition restrictive? Maybe. Consider the following

- **Result (Levy)**: If $X \in \mathcal{M}^c$ (continuous path martingale) and X is such that its quadratic variation $\langle X \rangle_t = t \; as$, for all intervals $[0, t]$, then X is a Brownian Motion.

- **Result (Breiman/Harrison)**: If X is continuous (i.e. has continuous sample paths) and X has identical independent increments, then X is a Brownian Motion.

- Does a Brownian Motion exist? Yes.

Constructing a Brownian Motion

- Random walk interpretation of a BM. Consider a random walk Z_t such that

$$\Delta Z = Z^h_{j+1} - Z^h_j = \begin{cases} \Delta h & with\ probability\ p \\ -\Delta h & with\ probability\ 1-p. \end{cases}$$

It follows that

$$\begin{aligned} E[Z^h_{j+1} - Z^h_j] &= \Delta h(2p-1), \\ Var[Z^h_{j+1} - Z^h_j] &= 4p(1-p)(\Delta h)^2. \end{aligned}$$

- Time interval of length T, and let $T = n\Delta t$.

- The cumulative change in $Z_{j+1}^h - Z_j^h$ over the whole interval is

$$\Delta_T Z = \underbrace{\Delta Z_1 + ... + \Delta Z_n}_{n\ times}.$$

- It follows that

$$\begin{aligned} E[\Delta_T Z] &= n\Delta h(2p-1) = T\frac{\Delta h}{\Delta t}(2p-1), \\ Var[\Delta_T Z] &= n4(1-p)\Delta h^2 = T\frac{\Delta h^2}{\Delta t}4p(1-p). \end{aligned}$$

- Pick parameters (p and Δh) so that:
  - mean is μT,
  - variance is $\sigma^2 T$

$$
\begin{aligned}
T\frac{\Delta h}{\Delta t}(2p-1) &= \mu T \\
T\frac{\Delta h^2}{\Delta t}4p(1-p) &= \sigma^2 T.
\end{aligned}
$$

or,

$$
\begin{aligned}
\Delta h &= \frac{\mu \Delta t}{2p-1} \\
(\Delta h)^2 &= \left(\frac{\sigma^2 \Delta t}{4p(1-p)}\right)^{1/2}.
\end{aligned}
$$

- The solution is

$$\begin{aligned}\Delta h &= \sigma\sqrt{\Delta t}\left[\sqrt{1+\left(\frac{\mu}{\sigma}\right)\Delta t}\right],\\ p_i &= \frac{1}{2}\{1\pm\left(\frac{\mu}{\sigma}\right)\frac{\sqrt{\Delta t}}{\sqrt{1+\left(\frac{\mu}{\sigma}\right)\Delta t}}\}.\end{aligned}$$

- Since this is an approximation that is valid when $\Delta t \longrightarrow 0$, the solution can be described as

$$\begin{aligned}\Delta h &= \sigma\sqrt{\Delta t},\\ p_i &= \frac{1}{2}\{1\pm\left(\frac{\mu}{\sigma}\right)\frac{\sqrt{\Delta t}}{\sqrt{1+\left(\frac{\mu}{\sigma}\right)\Delta t}}\}.\end{aligned}$$

- Special Case: Weiner process or standard Brownian motion (SBM), we have $\mu = 0$ and $\sigma = 1$.

$$\begin{aligned}\Delta h &= \sigma\sqrt{\Delta t},\\ p_i &= \frac{1}{2}.\end{aligned}$$

- Thus, one way of thinking about Brownian Motion is as the sum of a sequence of Bernoulli $i.i.d.$ random variables, with the size of the jump of the order of the square root of dt

- What is the *quadratic variation* of this process?

$$\langle \Delta Z \rangle = \sum_{j=1}^{n} [\underbrace{\Delta Z_j}_{\Delta h}]^2 = \sum_{j=1}^{n} [\sqrt{\Delta t}]^2 = n\Delta t = T.$$

- Random walk has the property that its quadratic variation is t in any interval of length t.

- Another derivation of a Brownian Motion.

- Intuition: Sum of white noise "scaled" by $\sqrt{dt}$

- Consider the difference between a $Z(T+s)$ and $Z(s)$. The length of this interval is T. Partition this interval into n equally spaced segments of length Δt. Thus, $T = n\Delta t$. Let's define

$$Z(T+s) - Z(s) = \sum_{i=1}^{n} \varepsilon_i \sqrt{\Delta t},$$

where ε_i is a collection of independent random variables, each distributed $N(0,1)$. It follows that

$$\begin{aligned} E[Z(T+s) - Z(s)] &= 0, \\ var[Z(T+s) - Z(s)] &= n\Delta t = T. \end{aligned}$$

- Given the properties of the normal distribution,

$$Z(T+s) - Z(s) \sim N(0, T).$$

- "Push" the notation and define

$$dZ_t = Z(t+dt) - Z(t),$$

then,

$$dZ_t \approx \varepsilon_t \sqrt{dt},$$

from which it follows that

$$\begin{aligned} E[dZ_t] &= 0, \\ var[dZ_t] &= dt. \end{aligned}$$

- If Z_t^1 and Z_t^2 are two standard Brownian Motion, then

$$E[dZ_t^1 dZ_t^2] \approx \varepsilon_t^1 \varepsilon_t^2 dt = \rho_{12} dt,$$

where ρ_{12} is the correlation coefficient between the two normal random variables.

- Let's consider now a (μ, σ) Brownian Motion. given by,

$$X_t = X_0 + \mu t + \sigma Z_t.$$

It follows that

$$X_{t+k} = X_t + \mu k + \sigma (Z_{t+k} - Z_t).$$

- Then, the best predictor of X_{t+k} given X_t is

$$E[X_{t+k} \mid X_t] = X_t + \mu k,$$

while the one standard deviation bands around this estimate are:

$$X_t + \mu k \pm \sigma\sqrt{k.}$$

Remark 23 *What dominates the behavior of the best estimate,* μ *or* σ*? Since* $X_{t+k} - X_t$ *has mean* μk *and standard deviation* $\sigma\sqrt{k}$*, then*

$$\frac{\mu k}{\sigma\sqrt{k}} \to \infty \text{ as } k \to \infty, \qquad \text{[mean dominates]}$$

while

$$\frac{\mu k}{\sigma\sqrt{k}} \to 0 \text{ as } k \to 0, \qquad \text{[standard deviation dominates]}$$

- What do we know about Brownian Motion? There are a number of results that hold for all martingales (and some for martingales with continuous paths) that directly apply to a Brownian Motion.

Claim 24 *Let* $W = \{W_t, \{\mathcal{F}_t\}, 0 \leq t < \infty\}$ *be a Brownian Motion then,*

1. W *is a Markov process.*

2. W *is a martingale.*

3. $\langle W \rangle_t = t$ *[Quadratic Variation]*

Proof. *Using the definition of quadratic variation*

$$\langle W \rangle_t \equiv \lim_{n\to\infty} \sum_{i=1}^{2^n-1} [W_{t_{i+1}^n} - W_{t_i^n}]^2,$$

where

$$t_i^n = \frac{it}{2^n}, \qquad i = 0, 1, ..., 2^n.$$

Since $W_{t_{i+1}^n} - W_{t_i^n} \sim N(0, t_{i+1}^n - t_i^n) = N(0, \frac{t}{2^n})$, *it follows that*

$$Z_i \equiv \sqrt{2^n}[W_{t_{i+1}^n} - W_{t_i^n}] \sim N(0, t).$$

Thus,

$$\sum_{i=1}^{2^n-1} [W_{t_{i+1}^n} - W_{t_i^n}]^2 = \langle W \rangle_t \equiv \sum_{i=1}^{2^n-1} \frac{1}{2^n} Z_i^2.$$

By the law of large numbers this term converges $(a.s.)$ *to* $E[Z_i^2] = t$ ■

4. $W^2 - \langle W \rangle_t$ *is a martingale*

Claim 25 *If a function is differentiable (with bounded derivative) then its quadratic variation is zero*

Proof. *Let $f(t)$ be a function. Then, given a partition $\Pi = \{0 = t_0 < t_1 < ... < t_n = t\}$, the mean Value Theorem implies that*

$$f(t_{i+1}) - f(t_i) = f'(\tau_i)(t_{i+1} - t_i).$$

The quadratic variation satisfies

$$\begin{aligned}
\left|\sum_{i=1}^{n}[f(t_{i+1}) - f(t_i)]^2\right| &= \left|\sum_{i=1}^{n}[f'(\tau_i)(t_{i+1} - t_i)]^2\right| \\
&\leq \max\left|f'(\tau)\right| \sum_{i=1}^{n}(t_{i+1} - t_i)^2 \\
&\leq \max_i\left|f'(\tau_i)\right| \max_i(t_{i+1} - t_i)t.
\end{aligned}$$

However, as the mesh of the partition goes to 0, $\max(t_{i+1} - t_i) \longrightarrow 0$ *and, hence, the quadratic variation is zero.* ■

Claim 26 *Let* $W = \{W_t, \{\mathcal{F}_t\}, 0 \leq t < \infty\}$ *be a Brownian Motion then its total variation is* ∞

Proof. *Let* $\Pi = \{0 = t_0 < t_1 < ... < t_n = t\}$ *be a partition of the interval* $[0, t]$*. Then,*

$$\sum_{i=1}^{n} [W_{t_{i+1}} - W_{t_i}]^2 \leq \max_i (W_{t_{i+1}} - W_{t_i}) \sum_{i=1}^{n} \left| W_{t_{i+1}} - W_{t_i} \right|.$$

$\max_i(W_{t_{i+1}} - W_{t_i}) \to 0$ *(by continuity). Since* $\langle W \rangle_t = \sum_{i=1}^{n} [W_{t_{i+1}} - W_{t_i}]^2 = t$ *is positive (and finite), it follows that the term* $\sum_{i=1}^{n} \left| W_{t_{i+1}} - W_{t_i} \right|$ *must converge to* ∞ ■

Starting from the standard BM we can create what we will denote a (μ, σ)-BM. This is a process X_t given by

$$X_t(\omega) = X_0 + \mu t + \sigma W_t(\omega).$$

In the following claims the X_t process is the one given above.

Claim 27 *The process $X_t(\omega) = X_0 + \mu t + \sigma W_t(\omega)$ is a submartingale (supermartingale) if $\mu \geq (\leq) 0$.*

Claim 28 *$Y_t \equiv X_t - \mu t$ is a martingale,*

Claim 29 *$\langle Y \rangle_t = \sigma^2 t$.*

Claim 30 $Y_t^2 - \sigma^2 t$ *is a martingale.*

Claim 31 $\left(\frac{Y_t}{\sigma}\right)^2 - t$ *is a martingale.*

Claim 32 (Wald Martingale) *Let* $Z_t \equiv e^{[\beta X_t - q(\beta)t]}$, *with* $q(\beta) \equiv \beta\mu + \frac{1}{2}\beta^2\sigma^2$. *Then,* Z_t *is a martingale.*

Proof. *To see this, note that*

$$e^{[\beta X_t - q(\beta)t]} = e^{[\beta(X_t - X_s) - q(\beta)(t-s)]} e^{[\beta X_s - q(\beta)s]}.$$

Taking expectations conditional on $\{\mathcal{F}_s\}$ *on both sides we get*

$$E[Z_t \mid \mathcal{F}_s] = Z_s E[e^{[\beta(X_t - X_s) - q(\beta)(t-s)]} \mid \mathcal{F}_s].$$

Thus, it suffices to show that

$$E[e^{[\beta(X_t - X_s) - q(\beta)(t-s)]} \mid \mathcal{F}_s] = 1.$$

However, $\beta(X_t - X_s) \sim N(\beta\mu(t-s), \beta^2\sigma^2(t-s))$, *and the expectation of any such log-normal random variable is*

$$E[e^{\beta(X_t - X_s)} \mid \mathcal{F}_s] = e^{\beta\mu(t-s) + \frac{1}{2}\beta^2\sigma^2(t-s)} = e^{q(\beta)(t-s)},$$

which completes the proof ■

Claim 33 $(X_t - \mu t)^2 - \sigma^2 t$ *is a martingale*

- An important property of Brownian Motion is that it is a Strong Markov process.

Definition 34 (Strong Markov) *A process X with initial distribution λ (and associated probability measure P^{λ}) is said to be a strong Markov process if*

1. $P^{\lambda}[X_0 \in \Gamma] = \mu(\Gamma)$, *for all* $\Gamma \in \mathcal{B}(\Re^d)$

2. *For any optional time* S, $t \geq 0$ *and* $\Gamma \in \mathcal{B}(\Re^d)$

$$P^{\lambda}[X_{S+t} \in \Gamma \mid \mathcal{F}_{S+}] = P^{\lambda}[X_{S+t} \in \Gamma \mid X_S],$$

$P^{\lambda} - as$ *on* $\{S < \infty\}$

- There is another way of describing the strong Markov property which reveals more clearly its implications.

- Let $W = \{W_t, \{\mathcal{F}_t\}, 0 \leq t < \infty\}$ be a Brownian Motion and let S be a stopping time of the filtration $\{\mathcal{F}_t\}$. Let

$$W_t^* \equiv W_{S+t} - W_S.$$

- Then

1. W_t^* is a Brownian motion.

2. W_t^* is independent of S.

- *Implication* of the strong Markov property. Let T be an arbitrary stopping time. Define the random variable Z_t^* by

$$Z_t^*(w) \equiv W_{T(\omega)+t}(\omega), \qquad t \geq 0, \qquad on\ \{\omega : T(\omega) < \infty\}.$$

- Note that the process Z_t^* need not be defined at all on the set $\{\omega : T(\omega) = \infty\}$.

- Let the function F be a measurable mapping such that

$$E^x[\mid F(Z) \mid] < \infty, \qquad \forall x \in \Re.$$

- Strong Markov property implies that

$$E^x[F(Z_t^*) \mid \mathcal{F}_T] = F(Z_0^*) = F(W_T), \qquad on\ \{\omega : T(\omega) < \infty\}.$$

- This result comes in handy when one needs to compute functions of a stopped BM.

The Reflection Principle

- Let $W = \{W_t, \{\mathcal{F}_t\}, 0 \leq t < \infty\}$ be a standard BM, and let

$$T_b(\omega) \equiv \inf\{t \geq 0 : W_t(\omega) = b\}.$$

It follows that T_b is a stopping time of the filtration $\{\mathcal{F}_t\}$.

Proposition 35 $P^0[T_b < t] = 2P^0[W_t > b] = \frac{2}{\sqrt{2\pi}} \int_{bt^{-1/2}}^{\infty} e^{-\frac{x^2}{2}} dx$

Proof. *(sketch)*

1. *First claim that* $P^0[T_b < t] = P^0[T_b < t, W_t > b] + P^0[T_b < t, W_t < b]$.

2. *The set* $\{T_b < t, W_t > b\} = \{W_t > b\}$. *Why? If* $W_t > b$ *it must be the case that* $T_b < t$, *as* $W_0 = 0$. *Thus, the extra conditioning does not change the set. What about the other set,* $\{T_b < t, W_t < b\}$? *On this set, the Brownian motion hit* b *at some point, and then drifted to a point* $c < b$. *Given that the distribution of a Brownian motion is symmetric (because the Normal is a symmetric distribution), the probability of traveling* down *a distance* $b - c$ *is the same as the probability of traveling* up *the same distance. Thus,*

$$P^0[T_b < t, W_t < b] = P^0[T_b < t, W_t > b].$$

Thus,

$$\begin{aligned} P^0[T_b &< t] = 2P^0[T_b < t, W_t > b] = 2P^0[W_t > b] \\ &= \frac{2}{\sqrt{2\pi}} \int_{bt^{-1/2}}^{\infty} e^{-\frac{x^2}{2}} dx \end{aligned}$$

■

- The above proof is somewhat heuristic (but close to being right) but it relies heavily on the idea that a Brownian motion has the strong Markov property. Where was this used? We basically used to argue that the process

$$W_{T_b+t} - W_{T_b}$$

  is a Brownian motion that is independent of $\mathcal{F}_{T_b}$, the σ-field of events determined prior to T_b. Of course, to claim the strong Markov property, T_b has to be a stopping time as we showed (it is a hitting time).

- Here is another application of the strong Markov property

Proposition 36 *Let* $W = \{W_t, \{\mathcal{F}_t\}, 0 \le t < \infty\}$ *be a standard BM, with* $X_0 = 0$*. Let*

$$M_t \equiv \sup\{W_s, 0 \le s \le t\},$$

then

$$P^0[W_t \le x, M_t \le y] = \Phi(\frac{x}{\sqrt{t}}) - \Phi(\frac{x-2y}{\sqrt{t}})$$

where Φ *is the* cdf *of a standard normal random variable, i.e. a* $N(0,1)$

Proof.

1. *Note that*

$$P^0[W_t \le x, M_t \le y] = P^0[W_t \le x] - P^0[W_t \le x, M_t > y].$$

2. *First term is*

$$P^0[W_t \le x] = \Phi(\frac{x}{\sqrt{t}})$$

given that W_t *is an* $N(0,t)$ *random variable.*

3. $P^0[W_t \leq x, M_t > y]$? *This event corresponds to the case in which the process W_t hit the value y, and then traveled to a point below x.*

4. *Conditional on being at y, traveling* down *below x (a distance $y - x$) has the same probability (given symmetry of the Normal distribution) of traveling* up *a distance $y - x$.*

5. *Let T be the first time that the process W_t hits y. Then, define $W^*_{t-T} \equiv W_t - W_T = W_t - y$.*

6. *Need to compute the probability of W^*_{t-T} increasing by at least $(y - x)$.*

7. *This is the probability of the event* $\{W^*_{t-T} \geq y - x\} = \{W_t \geq 2y - x\} = \{W_t \leq x - 2y\} = \Phi(\frac{x-2y}{\sqrt{t}})$.

$$
\begin{aligned}
P^0[W_t &\leq x, M_t > y] = P^0[T < t, W^*_{t-T} > y - x] \\
&= P^0[T < t, W^*_{t-T} > y - x] = P^0[W_t \leq x - 2y] = \Phi(\frac{x - 2y}{\sqrt{t}}).
\end{aligned}
$$

Thus, putting together these two pieces we get the desired result.

■

Some Results

- Assume that $W = \{W_t, \{\mathcal{F}_t\}, 0 \le t < \infty\}$ is a standard BM. Start the process at $W_0 = x$ (use P^x on $(\Omega, \mathcal{F})$.

- Let
$$T_b(\omega) \equiv \inf\{t \ge 0 : W_t(\omega) = b\}$$
and
$$M_t \equiv \sup\{W_s, 0 \le s \le t\}.$$

Proposition 37 *For $t > 0$ and $a \le b, b > 0$*

$$P^0[W_t \in da,\ M_t \in db] = \frac{2(2b-a)}{\sqrt{2\pi t^3}} e^{-\frac{(2b-a)^2}{2t}} da\ db.$$

Proof. *See previous results.* ■

Proposition 38 *Let* $t > 0$, $b > 0$

1. $P^0[\ M_t \in db] = P^0[|\ W_t\ | \in db] = P^0[M_t - W_t \in db] = \frac{2}{\sqrt{2\pi t}} e^{-\frac{b^2}{2t}} db.$

2. $P^0[\max_{0 \le u \le t} \mid W_u \mid \ge b] \le 4 P^0[W_t \ge b] \le \sqrt{\frac{t}{2\pi}} \frac{4}{b} e^{-\frac{b^2}{2t}}.$

3. $1 \rightarrow P^0[\ T_b \le t] = P^0[\ M_t \ge b]$ with density

$$P^0[\ T_b \in dt] = \frac{b}{\sqrt{2\pi t^3}} e^{-\frac{b^2}{2t}} dt,$$

has Laplace transform

$$E^0[e^{-\alpha T_b}] = e^{-b\sqrt{\alpha}}.$$

Proof. (of the Laplace transform result) Define the martingale

$$X_t = e^{\lambda W_t - \frac{1}{2}\lambda^2 t}.$$

Since, $W_{T_b} = b \; as$ and $X_0 = 1$. Then, the OST theorem implies

$$\begin{aligned} E^0[X_{T_b}] &= E^0[X_0] = 1, \\ E^0[e^{\lambda b - \frac{1}{2}\lambda^2 T_b}] &= 1, \\ E^0[e^{-\frac{1}{2}\lambda^2 T_b}] &= e^{-\lambda b}, \end{aligned}$$

then, setting $\lambda = \sqrt{2\alpha}$, the result follows. ■

Claim 39 $\lim_{t\to\infty} P^0[T_b \le t] = 1$ *but* $E^0[T_b] = \infty$.

Proof. To see this, use the formula

$$P^0[\, T_b \in dt] = \frac{b}{\sqrt{2\pi t^3}} e^{-\frac{b^2}{2t}} dt,$$

to compute

$$P^0[\, T_b \le t] = \int_0^t \frac{b}{\sqrt{2\pi s^3}} e^{-\frac{b^2}{2s}} ds.$$

It is convenient to use the change of variable formula $z = \frac{b}{\sqrt{s}}$. Note that

$$dz = -\frac{1}{2b^2} \frac{b}{\sqrt{s}} \frac{b^2}{s} ds = -\frac{1}{2b^2} z^3 ds$$

or,

$$ds = -\frac{2b^2}{z^3} dz.$$

Using the same change of variables in the integral, it follows that

$$\int_0^t \frac{b}{\sqrt{2\pi s^3}} e^{-\frac{b^2}{2s}} ds = \int_{\infty}^{\frac{b}{\sqrt{t}}} -\frac{2}{\sqrt{2\pi}} z^3 e^{-\frac{1}{2}z^2} \frac{1}{z^3} dz = 2\int_{\frac{b}{\sqrt{t}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$$
$$= 2[1 - \Phi(\frac{b}{\sqrt{t}})]$$

where, as before, $\Phi(x)$ is the cdf of a standard normal random variable. Thus,

$$P^0[\, T_b \leq t] = 2[1 - \Phi(\frac{b}{\sqrt{t}})] \to_{t\to\infty} 1.$$

Now, to show that $E^0[T_b] = \infty$ we use a similar computation. Note that

$$E^0[T_b] = \int_0^{\infty} \frac{bs}{\sqrt{2\pi s^3}} e^{-\frac{b^2}{2s}} ds.$$

Thus, using the same change of variables as before, we get

$$E^0[T_b] = 2\int_0^{\infty} \frac{1}{\sqrt{2\pi}} z e^{-\frac{1}{2}z^2} \frac{1}{z^3} dz = 2\int_0^{\infty} \frac{1}{\sqrt{2\pi}} \frac{1}{z^2} e^{-\frac{1}{2}z^2} dz.$$

However, this is just the expectation of $1/X^2$ where $X \sim N(0,1)$. It is a standard result that this expectation is infinite (or just compute the integral) ■

Claim 40 *Let T be defined by (for $a, b > 0$)*

$$T = \inf\{t : W_t = -a, \;\; or \; W_t = b\}$$

then show that $E^0[T] < \infty$

1. **Proof.** *To see this, let*

$$T_n \equiv \min\{T, n\}.$$

Since the stopping times T_n are bounded, the OST theorem can be applied. Since, as we showed before, the process

$$Z_t = W_t^2 - t$$

is a martingale, the OST theorem implies that

$$E^0[Z_{T_n}] = E^0[Z_0] = 0,$$

However, $W_{T_n}^2 \leq \min\{a, b\}$*. Thus,*

$$0 = E^0[Z_{T_n}] = E^0[W_{T_n}^2 - T_n] \leq \min\{a, b\} - E^0[T_n].$$

It follows that

$$E^0[T_n] \leq \min\{a, b\}.$$

But now the sequence $\{T_n\}$ *is converging to* T*, and is bounded below by* 0*. Thus, the monotone convergence theorem (version I with* $Y = 0$*) implies the desired result.*

Remark 41 *Note that in the previous case, when the stopping time is* T_b*, this bound does not bind (of course, since, in that case we have to take* $a \longrightarrow \infty$*.*

■