A Stochastic Optimal Control Approach to International Finance and Foreign Debt

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The recent debt crises, especially in Asia, have led to the questions: When is the foreign debt excessive? What are early warning signs of “vulnerability”? In the years prior to the crises, the Asian countries were held up as paragons of economic development. They were characterized by outward-oriented growth which attracted foreign investment, and macroeconomic stability. Inflation was moderate by developing country standards. In the cases of Malaysia and Thailand, the sizeable external current account deficits reflected not public sector budget deficits, but an excess of private investment over private saving. Hence high private saving and capital inflows were financing the growth of capital, which would increase the future productivity of the economy[1].

The literature on debt crises[2] had viewed the vulnerability of countries to debt crises in terms of the concepts “solvency” and “sustainability”. “Solvency” was defined as a condition where the ratio of external liabilities/GDP stabilizes. The long run trade surplus that an indebted country must have to keep the ratio of external liabilities/GDP constant was used as a measure of “solvency”. “Sustainability” was defined as a

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[1] International Monetary Fund, World Economic Outlook (WEO), Crisis in Asia, Interim Assessment, December 1997, Washington, DC, ch. II.

[2] We are drawing upon several sources for a discussion of the literature: Kaminsky and Reinhart (1999), Gian Maria Milesi-Ferretti and Assaf Razin (MF-R), Current Account Sustainability, International Finance Section, Princeton Studies in International Finance, #81, October, 1996; International Monetary Fund, World Economic Outlook (WEO), Financial Crises, May, 1998, Washington, DC; the papers presented at World Bank/International Monetary Fund/World Trade

condition whereby the resulting trade balance will be consistent with "solvency", if current policies are continued.

Recent thinking has questioned the usefulness of these criteria of vulnerability. The limitations of the existing approach can be seen from equation[3] (1). Normalize the population at unity. The time subscripts are important. Let h (t) = L (t)/Y(t) be the ratio of external liabilities L(t) to the GDP denoted Y(t). The rate of change in external liabilities/GDP has three components. The first r(t)h(t) is the interest payments at rate r(t) on the debt/GDP. The second term is minus the growth g(t) of GDP times the ratio of the debt/GDP. The third term is minus B(t) = B1(t)/Y(t) the trade balance as a fraction of GDP.

The GDP is Y(t) = Y[K(t), t], where K(t) is capital and dK(t)/dt is the rate of capital formation. The trade balance is B1(t). The growth of GDP is g(t) = [Y'(K(t),t)] [dK(t)/dt / Y(K(t),t)]. The ratio B(t) = B1(t)/Y(t) of the trade balance/GDP is 1 less absorption/GDP, where absorption is consumption plus investment: Then B(t) = [1 - c(t) - i(t)] where c(t)= C(t)/Y(t) is the <u>ratio</u> of consumption C(t) to GDP and i(t) = [dK(t)/dt] / Y(t) is the <u>ratio</u> of capital formation/GDP. Equation (1a) uses the simpler notation where A(t) = [r(t) - g(t)] is the interest rate less the growth rate, and B(t) = [1 - c(t) -i(t)] is the trade balance/GDP.

(1) $dh(t)/dt = [\, r(t) - g(t)]\, h(t) - [1 - c(t) - i(t)]; \quad g(t) = Y'(K(t),t)\, i(t)$

(1a) $dh(t)/dt = A(t)\, h(t) - B(t)$

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[3] The derivation is as follows: h(t)=L(t)/Y(t), (1/Y(t))dY(t)/dt = g(t) is the growth rate, and the current account deficit dL/dt = rL – B1, where B1 is the trade balance. Y(t)=Y(K(t),t) is the production function. Hence (1/h(t))dh(t)/dt =[(r(t) - g(t)] - B1(t)/L(t); B1(t) = Y(t) - C(t) - dK(t)/dt, where C(t) is total consumption.

The "solvency" criterion is that the steady state debt stabilizes at a finite value $h^* < \infty$. That is, dh(t)/dt converges to zero, so that $A^*h^* = B^*$, where an asterisk indicates a steady state value. This means that the resource transfer B*- the trade balance as a fraction of GDP - equals the interest payments on the foreign debt adjusted for growth A*h*. The "sustainability" condition is that current policies lead to "solvency". Current policies correspond to the B(t) term, which contains the consumption ratio c(t) and i(t), the ratio of capital formation/GDP. In practice, h(t) is compared with B(t)/A(t) to see if the country's debt is "too high" for solvency.

Recent empirical studies of early warning signals (EWS) have not included the solvency-sustainability criteria among the useful signals[4]. There are several reasons why these criteria are not helpful in a world of uncertainty. The first four are technical and the fifth reason is the crux of the problem: the solvency-sustainability criteria have very little to do with the economic question of "vulnerability".

First: It is impossible to know if the debt will stabilize unless one knows A(t) and B(t), for all future time T > t the present. Neither A(t) nor B(t) is a constant, and these values are interrelated. In developing countries in particular, trade imbalances B(t) < 0 are produced by the rate of capital formation as well as by consumption. The rate of capital formation influences both A(t) and B(t). Policies which affect capital formation affect the growth rate in A(t) as well as the trade balance in B(t).

Second: The future values of the interest rate r(t) and the productivity of capital Y'[K(t),t)] are unknown. (a) One does not know at what interest rates foreign investors will be willing to continue to lend to the country. (b) The productivity of capital, the rate of return on investment, is unknown. In many countries, the investment produces tradable goods. The terms of trade, the ratio of export/import prices, are stochastic variables. Hence Y'[K(t),t] is a stochastic variable, affected by the terms of trade as

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[4] Kaminsky and Reinhart, table 5; Berg and Petillo, International Monetary Fund, table 1; Kaminsky, Lizondo and Reinhart, table 5.

well as by the domestic rate of capacity utilization. Formally, $A(t) = [r(t) - g(t)]$ is a stochastic variable, affected by both the world capital markets and goods markets. The evolution of A(t) is not predictable.

Third: Dynamic efficiency requires that, in the steady state, the interest rate equal to the marginal product of capital be at least as great as the growth rate. However insofar as the interest rate exceeds the growth rate, $A > 0$, and the debt will explode, given current policies B.

Fourth: At what level should the debt be stabilized? Even if the debt stabilized at a finite $h^* = B^*/A^*$, there is no presumption that this ratio is optimal in any sense. The value of h(t) may oscillate considerably even if it remains bounded as $t \Rightarrow \infty$.

Fifth: Vulnerability is a situation where the "bad" shocks to the interest rate and the productivity of capital, and their correlation, reduce the consumption per capita C(t) below a "tolerable" level $C_{min}(t)$. Suppose that the "very bad shocks" occur and the country cannot borrow in the world capital market: $dh(t)/dt = 0$. If investment were reduced to zero, then[5] consumption per capita would fall to:

(1b) $C_1(t) = [(g(t) - r(t))\, h(t) + 1]\, Y(t) \geq C_{min}(t)$

The "very bad shocks" mean that $[g(t) - r(t)]$ is negative. The level of the interest payments on the debt r(t)h(t) and the low or negative growth rate g(t) may drive $C_1(t)$ below the tolerable level $C_{min}(t)$. In that case, there will be a social reaction. The government may declare "bankruptcy" rather than allow the consumption $C_1(t)$ to be reduced below $C_{min}(t)$.

The recent literature has concluded that the "solvency - sustainability" approaches are of limited use. Instead, we want to know: how vulnerable is an economy to external shocks or to the "bad" events in the distribution of shocks? The current discussion concerning the redesign of the international financial architecture involves several issues. What are guidelines to anticipate crises? What variables and levels of international debt

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[5] Set dh(t)/dt and i(t) equal to zero, in equation (1) and solve for c(t).

should alert regulators, governments, lenders and international financial institutions to monitor vulnerability?

Our paper is directed to the above issues. We use the techniques of stochastic optimal control-dynamic programming to derive the optimal level of the foreign debt L(t) in a world of uncertainty. Our basic criterion is the maximization of the expectation of the discounted value of utility of consumption over an infinite horizon. There are several noteworthy results, propositions 1 – 4 below.

First: We quantify and formalize the "benchmark" in a stochastic environment of the optimal debt/net worth, denoted f*.

Second: We address ourselves to the vulnerability question, in the following way. Vulnerability is associated with the variance of the utility of consumption at any time. We derive the trade off, at any time, between the expected utility of consumption and the variance of the utility of consumption. We call this the "expected return-risk trade-off". We prove that if the debt/net worth f(t) exceeds the optimal debt/net worth f*, there is inefficiency. By reducing the debt to f*, the expected utility of consumption can be raised and the variance of the utility of consumption - the vulnerability - can be reduced. The monitoring, or early warning signal, concerns seeing if the debt/net worth ratio lies in the inefficient region. This is the key result of this paper.

Third: The current account deficits under optimal policies are considered. Many economists view current account deficits as signs of vulnerability. For example, many are concerned about the continuing US current account deficits, and fear that they are signs of vulnerability. We prove that in appropriate ranges for the parameters in the model, permanent expected current account deficits/net worth are optimal. Only when the actual current account deficit exceeds the optimal level is there a warning sign of vulnerability.

Fourth: the techniques of analysis, optimal stochastic control, are related to the finance literature pioneered by Merton, rather than the economic growth literature with phase diagrams and saddle points. One can view our paper as an extension of the Merton model to the international finance and debt environment. Our theorems imply testable propositions, and implementable policies for monitoring debt and surveillance.

1. A Continuous Time Prototype Model

Our prototype model is a simplification of a complex economy that focuses upon the disturbances that have produced crises, and is analytically tractable. There is a clear economic interpretation of the derived equations. As more realistic assumptions are introduced, both the solution and economic interpretation become less transparent. The prototype model is proposed as a "benchmark" model. We indicate, either in the text or in footnotes, the specific aspects of the prototype model that are simplifications. We consider two sources of uncertainty. One source concerns the value of GDP and the return on capital. The second concerns the interest rate on loans. It is important and realistic to stress that there is a correlation of these two sources of uncertainty.

The model is in real terms and is formulated in terms of the stochastic calculus. Equation (2) defines net worth (wealth) X as capital K owned by the residents of the nation less international debt L, and (2a) is the change in net worth. A negative L represents foreign assets. Equation (3) states that the change in capital dK is the rate of investment per unit of time I(t) times the length of the period dt.

(2) $X(t) = K(t) - L(t)$ (2a) $dX = dK - dL$.

(3) $dK = I(t)\,dt$

The effect of uncertainty upon the state variable real net worth X(t) and real consumption[6] C(t) can be understood by stating the equation (4a) in nominal terms measured in the domestic currency.

Consumption, C(t), investment I(t), and the Gross Domestic Product Y(t) are measured in domestic goods – real – per unit of time, and p(t) is the domestic price index.

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[6] The population is normalized at unity, so that these are per capita magnitudes.

Variable e(t) is the domestic price of foreign exchange. The debt L(t) is denominated in $US. The term i(t)L(t) is the interest payments in $US at the rate of interest i(t) on dollar denominated loans. Assume that all debt is short-term and denominated in US dollars.

(4a) $p(t)C(t)dt = [p(t)Y(t) – e(t)i(t)L(t)]dt – p(t)I(t)dt + e(t)dL$

Divide by p(t), the domestic price index, to obtain consumption C(t) measured in domestic goods. This is the argument in the utility functionU(C(t)).

(4b) $C(t)dt = \{Y(t) – [e(t)/p(t)]i(t)L(t)\}dt –I(t)dt + [e(t)/p(t)]dL$

The first term in braces is the real GNP, the GDP less real interest payments. The last term is the amount of domestic goods obtained from the new $US loans.

For simplicity of exposition, we assume "Purchasing Power Parity" (PPP) or the "Law of One Price", that e(t)/p(t) = 1. Explicitly, the PPP hypothesis is that prices are the same at home and abroad, when measured in a common currency. The PPP hypothesis says that the nominal domestic price of foreign exchange rate e(t) adjusts to equalize domestic and foreign prices. Let the foreign price index be p*, then PPP states that e(t)p* = p(t). When p* is normalized at p* = 1, then PPP states that e(t)/p(t) = 1.

Using PPP, we obtain (4c). In this case r(t)L(t) = e(t)i(t)L(t)/p(t) = i(t)L(t). It is the amount of goods that the economy must transfer abroad to pay the interest on L(t) of loans in $US. The r(t) is stochastic. It has two components: the US interest rate on Treasury bonds and the premium charged on short-term foreign debt denominated in US dollars.

(4c) $C(t)dt = \{Y(t) –r(t)L(t)\}dt –I(t)dt + dL$

Solving equation (4c) for the change in the foreign debt dL, we obtain equation (4). Here, the variables are in real terms.

(4) $dL = r(t)L(t)\, dt + [C(t) + I(t)]dt - Y(t)\, dt.$

There are several shocks that affect the value of GDP and the interest payments on the foreign debt. The stochastic variables are not necessarily independent[7], and are denoted

[7] The discussion of financial instability in Mishkin (1999) fits into our analysis.

with time indices.

1.1 Debt payments uncertainty

The first stochastic term in equation (4) is r(t)L(t) the real interest payments on the US dollar denominated short-term debt at interest rate r(t). Term r(t)L(t) is the real equivalent of the nominal term e(t)i(t)L(t)/p(t), using the simplifying PPP assumption that the real exchange rate e(t)/p(t) is constant at unity[8].

The real interest rate $r(t) = r_1(t) + (i(t) - r_1(t))$ has several components. Each component is a stochastic variable. The first component is r_1 is the US real long-term bond yield. This component varies according to the state of the US domestic economy. The second component $(i(t) - r_1(t))$ is the country risk premium charged to foreign firms or countries that borrow US dollars.

The interest rate r(t) charged on short term dollar denominated loans in the is the sum of the two components[9]. Equation (5) is the real value of the interest payments on the debt. The first component is the expectation of the interest payments rL(t)dt. The second component describes the uncertainty: It is $\sigma_1 L(t)dw_1$ which involves Brownian motion $w_1(t)$. The Brownian motion term is $dw_1 = \varepsilon \sqrt{dt}$, where $E(\varepsilon) = 0$ and $E(\varepsilon^2) = 1$. The interest costs on the debt r(t)Ldt are distributed normally with a mean of rL(t) dt and a variance of $(\sigma_1 L(t)\, dw_1)^2 = \sigma^2{}_1 L(t)^2\, dt$.

(5) $r(t)\, L(t)dt = r\, L(t)\, dt + \sigma_1\, L(t)dw_1$

This is a general equation that describes the uncertainty arising from the international financial markets.

[8] In realistic models, the equilibrium real exchange rate is not a constant. Currency crises have been attributed to overvalued exchange rates: the deviation between the actual real exchange rate and an equilibrium value. The latter is consistent with internal and external balance. See Stein and Paladino (1999).

[9] The PPP assumption implies that the exchange rate and price movements cancel each other, so that e(t)/p(t) = 1.

1.2 Uncertainty concerning the real value of the GDP

The real value of GDP is Y(t)dt, in (4), is described by equations (6a), (6b) which imply (6). It is in the spirit of the "endogenous technical change" models[10]. In the endogenous technical change approach, expected output per unit of capital is b. The real value added or real GDP is a stochastic variable. The real GDP per unit of capital varies for several reasons: (a) variations in the rate of capacity utilization due to the business cycle, (b) variations in the terms of trade - the ratio of export/import prices, (c) variations in the profit margin between selling prices and costs, (d) unexpectedly bad/good returns on investments. The distribution of Y(t)dt has a mean bKdt and a variance of $(\sigma_2 K(t))^2\, dt$.

(6a) $Y(t) = b(t)K(t)$

(6b) $b(t)\, dt \sim N(bdt, \sigma_2^2 dt)$

(6) $Y(t)\, dt = bK(t)\, dt + \sigma_2 K(t)\, dw_2$. $Y(t)dt \sim N(bK(t)\, dt, (\sigma_2 K(t))^2\, dt)$,

where $w_2(t)$ is another Brownian motion.

The long run stability of the output/capital ratio b(t) is a well known stylized fact. An estimate[11] of the US productivity of capital b(t) is graphed in appendix B over the period 1959:1 - 1997:2. It is labeled PRODCAP. The Brownian motion assumption in (6b) is an oversimplification, used for analytic purposes, because there is a mean reversion tendency, and the output/capital b(t) is not normally distributed.

1.3 The correlation of the two shocks

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[10] In our prototype model total GDP is derived from a Leontief production function: $Y^* = \min [bK^*(t), a(t)N(t)]$, where N(t) is labor or materials. We assume that capital is the binding constraint. Y*(t) is total GDP and K*(t) is total capital. The capital owned by residents of the country is K(t) and by foreigners it is $K_f(t) = [K^*(t) - K(t)]$. Hence total GDP, denoted by Y*(t) has two parts: bK(t) accrues to residents of the country and $b[K^*(t) - K(t)]$ accrues to foreigners. We define GDP accruing to residents as $Y(t) = Y^*(t) - bK_f(t) = bK(t)$. Coefficient **b** is described in equation (6b), and the increase in a(t) reflects the growth in labor productivity.

[11] See the discussion in appendix B.

The two stochastic terms dw_1, dw_2, are interrelated. We consider the general case, equation (7), where the correlation coefficient ρ could be positive, zero or negative, which varies among countries. We believe that in the advanced (G7) countries the correlation ρ is positive, and it is negative in the emerging market countries.

(7) $E(dw_1\, dw_2) = E(\varepsilon_1\, \varepsilon_2)dt = \rho\, dt$.

In the advanced countries, the real long term rate of interest r(t) is positively correlated with the productivity of capital b(t), due primarily to the business cycle. When investment rises relative to social saving, the economy expands, and there is an excess demand for loans – or an excess supply of bonds. Real interest rates tend to rise.

A very different situation may exist in the emerging market countries, particularly those that have experienced debt crises. The interaction of the real and financial shocks is described by the correlation coefficient ρ. The fragility of the financial system is aggravated by a correlation $\rho < 0$ which has been the case in the Emerging Market countries. The causation between the two shocks dw_1 and dw_2 runs both ways.

When the productivity of capital is shocked below its mean, there are more non-performing loans and bank failures - due to the high debt/equity ratios of banks - and capital flight. The capital outflow drains the reserves. Country risk $i(t) – r_1(t)$ increases. Interest rates are raised further and the correlation ρ becomes even more negative. Similarly, the shock to interest rates deteriorates the balance sheets of firms and households. They are unable to repay their debts to the banks. Bank balance sheets reflect losses. Bank lending declines which, in turn, depresses the economy. Output declines. The productivity of capital b(t) declines, and the real value of the debt payments r(t)L(t) rises. This negative correlation is extremely important in deriving the optimal debt/net worth ratio.

1.4 The dynamic equation for the state variable

Equations (2) - (7) describe the underlying model. Substitute equations (2), (3), (5) and (6) into equation (4). On the basis of these equations, we obtain equation (8) for our state variable X(t) which is wealth or net worth. It states that the change in net worth dX(t) is GNP = [Y(t) - r(t)L(t)]dt less consumption C(t) dt equal to saving S(t)dt.

Equation (8) can be expressed in several ways. Substitute K(t) = X(t) + L(t), and obtain the change in the state variable X(t) in terms of control variables[12] consumption C(t) and the debt L(t). There are two main components of the change in net worth dX(t). The first component is the growth of net worth arising from the productivity of capital b(t)X(t)dt less consumption C(t)dt. This is the source of growth in a closed economy. The second component is the growth of net worth resulting from borrowing in the international market to finance capital formation: [b(t) - r(t)]L(t)dt. Capital has a return b(t) and the cost of the debt is r(t).

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\begin{aligned}
(8)\quad dX(t) &= b(t)K(t)\,dt - r(t)L(t)\,dt - C(t)\,dt \\
&= [b(t)X(t) - C(t)]dt + [b(t) - r(t)]L(t)dt \\
&= S(t)\,dt
\end{aligned}
$$

There are several constraints. First: consumption is positive, or at least non-negative. Second, net worth must be non-negative. The constraint $X(t) > 0$ prevents "Ponzi schemes" where borrowing finances both consumption and interest on the existing debt. If a Ponzi scheme were followed, then debt rises relative to capital and X = K - L will become zero and then negative in finite time. If the net worth is rationally expected to become negative (i.e., the country follows policies that will lead to bankruptcy) the creditors will sell their debt and the interest rates will not be described by (5). Since net worth $X = K - L \geq 0$, this implies that capital $K \geq L$. The capital must be non-negative $K(t) > 0$, but the debt L(t) can be positive, zero or negative. A negative debt implies a creditor position in the international financial markets.

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[12] Saving S(t) is not a control variable since it depends upon the stochastic GNP.

To formulate a stochastic control problem associated with the model, we must specify state and control variables, the dynamics of the state process and the criterion to be optimized.

The controls are consumption C(t) and the debt L(t) =K(t) - X(t). Wealth X(t) is capital K(t) less debt L(t), or capital K(t) = X(t) + L(t) is wealth plus the foreign debt[13]. A simplifying assumption is that the level of the debt can be achieved instantaneously and costlessly[14]. The control on the debt is therefore the same as the control of capital, given the state X(t). If the country has "too much" debt, it sells capital and repays some debt. If the country has "too little" capital, it either sells debt or reduces net foreign assets, and uses the proceeds to acquire capital. Thus one could equivalently take C(t) and K(t) as controls. We impose the state constraint X(t) > 0 and the control constraints C(t) > 0, K(t) $\geq$ 0.

Using the stochastic equations for the productivity of capital and interest rate (eqns. 5-6) we derive stochastic differential equation (8a) for the change in net worth in terms of control variables C(t) and L(t). This is a general description of the dynamics of net worth X(t). The first set of terms in brackets is expected saving, equal to expected GNP less consumption, where expected GNP is E[Y(t) – rL(t)] dt = [bX(t) + (b – r)L(t)] dt and consumption is C(t)dt. The second and third sets of terms in (8a) or (8b) concern the stochastic components of GNP: the stochastic component of the productivity of capital $(X(t)+L(t))\sigma_2 dw_2$ and the stochastic component of interest rates $L(t)\sigma_1 dw_1$.

[13] Equation (8) in terms of control variables consumption and capital K(t) = X(t) + L(t) is dX(t) = [r(t) X(t)dt - C(t)dt] + [b(t) - r(t)]K(t)dt . Both control variables are adjusted instantaneously and costlessly, and the state variable moves differentially

[14] It is more realistic to assume that there are "transactions costs" in varying debt (capital). This assumption changes the dimension of the dynamic system. See Fleming (1998, part 2.4), Constantinedes, Bielecki and Pliska for the use of transactions costs in finance models.

We shall be using a HARA utility function, which allows us to use as controls the ratios of: debt/net worth f = L/X, capital/net worth k = K/X and consumption/net worth c = C/X. Equation (8b) is (8a) in terms of the control ratios.

stochastic differential equation for net worth

(8a) $dX(t) = [bX(t) + (b-r)L(t)) - C(t)]\, dt - L(t)\sigma_1 dw_1 + (X(t)+L(t))\sigma_2 dw_2$.

(8b) $dX(t) = [(b - c) + (b-r)f]\, X(t)\, dt - f\, X(t)\sigma_1 dw_1 + (1+f)X(t)\sigma_2 dw_2$.

There are several differences between our model and the usual economic growth models. First: Our model of an economy is a stochastic system, so that there are many paths that the state variable X(t) can take given the controls and the initial data. The stochastic disturbances have been underemphasized in the studies of optimal control because the authors generally use deterministic models, which assume unique (saddle point) paths to the optimal steady state[15]. Optimal stochastic control theory attempts to deal with models - such as the one described above - in which the Brownian motion disturbances are important. This means that the future is unpredictable. All decisions will have to be made based upon the currently observed state. Dynamic programming is the required tool of analysis.

Second: Our approach can be considered as the extension of the Merton finance model to the realm of international finance and debt. The productivity of capital b(t) = Y(t)/K(t) and the real interest rate r(t) are exogenous stochastic variables, and there is

[15] Merton's analysis (1990) is based upon stochastic optimal control, and marked a change from the deterministic control approach which used the Maximum Principle of Pontryagin. The article by Infante and Stein (1973) on optimal growth considered a deterministic model where there was only qualitative but not quantitative knowledge of the parameters. Using dynamic programming, they derived feedback control laws - just based upon current observations - which would drive the economy to the optimal trajectory that would exist if there were perfect knowledge. Their feedback control laws are robust to perturbations, whereas the open loopcontrols based upon the Maximum Principle lead to saddle point instability.

no mechanism of convergence of the two for a given country. For the world as a whole, there is indeed an equation that links the two[16].

Third: Instead of the convergence mechanism of b(t) and r(t), we have an alternative "equilibrium" concept. The variables in the optimization decision concern the debt/net worth f* or capital net/worth k* = 1 + f*, and the consumption/net worth c*. The asterisk denotes the optimum value. Based upon controls (f*, c*), we derive the following "equilibrium" relations between the following sets of variables: (a) The expected growth of net worth and its variance; (b) The expected consumption and its variance; (c) The expected current account deficit and its variance. From (b) we derive an expected return-risk tradeoff, which will define regions of vulnerability to shocks. This can be considered to be the crux of our paper.

Fourth: The Dynamic Programming approach implies a system that is stable. In the next part, we use Dynamic Programming to derive the optimal controls.

2. Derivation of Optimal Consumption, Capital, Debt and the Growth of Net Worth in Continuous Time: over an Infinite Horizon: Prototype Model

We derive the optimal foreign debt, capital and consumption in the prototype stochastic growth model described in part 1. There are many criteria of optimality. In this paper[17] we use the expected present value of a HARA utility function $U(C(t)) = (1/\gamma)C(t)^{\gamma}$ as the criterion of optimality, where $\gamma < 1$. The use of HARA, or the logarithmic function which is a special case of HARA, reduces the dimension of the problem and allows us to solve the model analytically.

[16] This is done in section (2.2) below.
[17] In subsequent papers we use risk sensitive and robust control criteria.

We use the dynamic programming method[18]. Equation (9) is our value function. The initial value of wealth $X = X(0)$. The discount rate is δ. Relative risk aversion[19] is $(1 - \gamma) > 0$. The logarithmic utility function is derived when $\gamma = 0$. The optimization is over an infinite horizon. The expectations are taken over the
dw_i, $i = 1,2$ where w_i is Brownian motion. The maximum is taken over a set Γ of admissible controls. The controls $C(t)$, $L(t)$ admitted must be such that the state and control constraints are satisfied

Constraints: $X(t) > 0$, $C(t) > 0$, $K(t) = X(t) + L(t) \geq 0$.

Moreover, in selecting controls $C(t)$ and $L(t)$ it is not possible to anticipate future values of the Brownian motion terms $w_1(s)$, $w_2(s)$ for time $s > t$.

(9) $V(X) = \max_\Gamma E \{\int_0^\infty (1/\gamma)\, C(t)^\gamma e^{-\delta t} dt \}$, $\gamma < 1$ and subject to equation. (8).

The HARA utility function implies that the value function $V(X)$ is homogeneous of degree γ and is equation (10). The proof is as follows. If the state X, and controls C and L are multiplied by a value $\lambda > 0$, then dynamic equation (8) is satisfied. The new value function $V(\lambda X)$ is:

$$V(\lambda X) = \max_\Gamma E \{\int_0^\infty (1/\gamma)\, [\lambda C(t)]^\gamma e^{-\delta t} dt \} = \lambda^\gamma V(X).$$

The value function of X is also homogeneous of degree γ. Hence we may write the value function as (9a) where constant $A > 0$ is to be determined. The first two derivatives are (9b) and (9c).

(9a) $V(X) = (A/\gamma)\, X^\gamma$

(9b) $V_x = A\, X^{(\gamma-1)}$

(9c) $V_{xx} = A\,(\gamma-1)\, X^{(\gamma-2)}$

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[18] For a more precise mathematical description, see Fleming and Rishel (1975, ch. 6) or Fleming and Soner (1992, ch. 3-4).

From equations (8a) and (9), the derived Bellman stochastic dynamic programming (DP) equation is (10a). The expectations have been taken into account in its derivation. In appendix A, we (a) give the conditions on the model parameters such that equation (10) has a solution with A > 0, and (b) sketch a formal derivation of equation (10).

$$(10a)\ \delta V(X) = \text{Max}_{C,L} \{(1/\gamma)C^{\gamma} + V_x[(b-r)(X+L) + rX - C] + (V_{xx}/2)\,[(L^2\sigma_1^2) + (X+L)^2\sigma_2^2 - 2L(X+L)\,\rho\sigma_1\sigma_2]\}$$

The HARA function permits us to measure the variables: consumption, capital and debt as fractions of net worth: C/X = c, L/X = f, k = K/X, where lower case letters refer to the ratios. Instead of C and L, we can equivalently take c and f as the control variables. The control constraints are then

$c > 0,\ f = k - 1 \geq -1, \qquad (k = K/X)$

Use equations (9a) - (9c) in (10a) to derive dynamic programming equation (10).

<u>DP equation for prototype model</u>

$$(10)\ \delta/\gamma = b + \max_c\ [(1/\gamma)c^{\gamma}/A - c] + \max_f [\,(b-r)f + (\gamma-1)/2\ (f^2\sigma_1^2) + (\gamma-1)/2\ (1+f)^2\sigma_2^2 - (\gamma-1)(1+f)f\,\rho\sigma_1\sigma_2\,]$$

where: c = C/X, f = L/X.

In equation (9a), we must have A > 0. In case $\gamma > 0$, there must be a restriction that the discount factor δ is not "too small"[20]. The restriction is required for V(X) to be finite. In the case where $\gamma \leq 0$, the restriction is satisfied and A > 0. For simplicity, we often consider a logarithmic utility function, $\gamma = 0$; and then the restriction A > 0 is satisfied.

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[19] Relative risk aversion RRA = - d (ln U') / d (ln C) = (- U''/ U') C. When $U(C(t)) = (1/\gamma)C^{\gamma}(t)$, then RRA = $(1-\gamma)$.

[20] See appendix A.

In part 2.1, we derive the optimal consumption, capital and debt ratios. In part 2.3 we derive the growth of wealth.

2.1. Optimal Consumption, Capital and Debt

Our continuous time - infinite horizon model is quite similar to the Merton model, with different emphases. In Merton's model, there is a safe asset with a fixed return r and a risky asset. The only source of income is the interest payments on the portfolio. The returns on the two assets are exogenous[21]. There is no human income. Our model is the open growing economy extension of the Merton model, with emphases upon the debt, current account deficit, and vulnerability. (a) There is growth in the GDP resulting from capital formation, which is financed by both domestic saving and foreign borrowing. The current account deficit is net foreign borrowing. (b) There is no safe asset. Both the productivity of capital and, for any country, the interest rate are stochastic variables. (c) The correlation between the productivity of capital and the interest rate vary by type of country. (d) Capital and wealth are constrained to be non-negative. The latter ensures that capital k(t) exceeds debt f(t), and prevents Ponzi schemes. (e) In the world economy, the productivity of capital and world interest rates are linked.

The optimal consumption/net worth ratio c* in equation (11) is determined by taking the maximum over c in equation (10), where $A > 0$ is a constant determined by equation (A1) in appendix A. We consider $\gamma \leq 0$. Similarly, the optimal debt/net worth ratio f* is obtained by taking the maximum over f in equation (10), constrained by $f \geq -1$. If $f^* > -1$, it is given by equation (12), simplified as (12a). Otherwise, $f^* = -1$, endpoint

[21] Fleming and Zariphopoulou (1991) introduced three securities: a risky asset, a safe asset and a debt instrument with a fixed interest rate. The expected return on the risky asset exceeds the fixed interest rate on loans which exceeds the fixed interest rate on the safe asset. There will not be simultaneous investment in the safe asset and borrowing, since the borrowing rate exceeds the interest rate on the safe asset.

maximum. Optimal capital/net worth k* is (13) simplified as (13a), provided that k* = 1 + f* > 0.

Term (b-r) is the <u>expected</u> net return: the expected productivity of capital less the expected interest rate. The stochastic aspect of the problem is contained in (ρ, θ).The correlation between the two shocks is ρ. The ratio $\theta = \sigma_1/\sigma_2$ is the ratio of the standard deviations associated with the two shocks. Equations (11)-(13) are discussed in great detail below.

(11) $c^* = C(t)/X(t) = A^{1/(\gamma-1)}$

(12) $f^* = L(t)/X(t) = [(b-r)/(1-\gamma)\ \sigma_2^2(1 + \theta^2 - 2\rho\theta)] - [(1 - \theta\rho)\ \}/(1 + \theta^2 - 2\rho\theta)]$

if $f^* > -1$

(13) $k^* = [(b-r)/(1-\gamma)\ \sigma_2^2(1 + \theta^2 - 2\rho\theta)] + [\theta\ (\theta - \rho)\ \}/\ (1 + \theta^2 - 2\rho\theta)\ ]$,

if $k^* = 1 + f^* > 0$.

Note that c*, f* and k* are constants. They do not vary with time t and do not depend on the initial wealth X.

In the case of $\gamma = 0$, the logarithmic utility function[22], the value of A is $(1/\delta)$ the reciprocal of the discount rate. In that case, the optimal consumption/net worth is equation (11a). Optimal consumption is a fixed proportion of net worth, where the factor of proportionality is the discount rate or time preference.[23]

(11a) $c^* = C(t)/X(t) = \delta$, when $\gamma = 0$

The economic determinants of the optimal debt ratio f* in (12), and optimal capital/net worth in (13), are understood by writing them as (12a) and (13a) respectively which contain three crucial terms: k_m, λ and $\rho\theta$. See Box 1.

Equations (12a) and (13a) are graphed in figure 1 for two cases, corresponding to two different sets of countries. In country I, the correlation $\rho > 0$, and in country II the

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[22] See appendix A.

[23] This is the consumption function used in the NATREX dynamic model of the real exchange rate and international debt. This consumption function guarantees that the debt will converge and is our intertemporal budget constraint. See Stein in Stein, Allen et al (1995).

correlation $\rho < 0$. We relate equations (12) and (13) to the Merton model, the literature of the economics of futures markets, and international debt characteristics.

BOX 1 Summary of optimal controls

(11a) $c^* = C(t)/X(t) = \delta$ when $U[C(t)] = \log C(t), \gamma = 0$

(12a) $f^* = \lambda\, k_m + \lambda\, (\rho\theta - 1)$

(13a) $k^* = \lambda k_m + \lambda\theta(\theta - \rho) \geq 0$

Equations (12a), (13a) hold if $k_m + \theta(\theta - \rho) \geq 0$. Otherwise $f^* = -1$, $k^* = 0$.

Merton point: $k_m = (b-r)/(1-\gamma)\sigma_2^2$ Parameters: $\theta = \sigma_1/\sigma_2$ = standard deviation of interest rate/ standard deviation productivity of capital; ρ = correlation between interest rate and productivity of capital;

$r(t) \sim N(r, \sigma_1^2)$; $\sigma_1^2\, dt = \text{var}\, r(t)$ **;** $b(t) \sim N(b, \sigma_2^2)$; $\sigma_2^2\, dt = \text{var}\, b(t)$;

$\rho = \sigma_{12}/\sigma_1\sigma_2$, $\text{var}\,(b(t)-r(t)) = \sigma^2\, dt = (\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2)dt$

slope $\lambda = (\sigma_2^2/\sigma^2) = 1/(1 + \theta^2 - 2\rho\theta)$

intercept $f(0) = \lambda\,(\rho\theta - 1) = \text{argmin}_f\, \text{var}\,[dX(t)/X(t)]$

In Merton's model, there is a safe asset whose interest rate is constant, and a risky asset. The uncertainty concerns the return on the risky asset. There is no debt. Term $k_m = (b-r)/(1-\gamma)\sigma_2^2$ is Merton's solution for the ratio of risky assets to net worth, when the interest rate on the safe asset is constant In his model, when risky assets are held, $k_m \geq 0$. Refer to k_m as the "Merton value" of the ratio of capital/net worth. The value of k_m is the expected value of the difference between the productivity of capital and the interest rate (b-r), divided by the variance (σ_2^2) of the productivity of capital times relative risk aversion $(1-\gamma) > 0$. It is the risk adjusted expected net return. The ratio of risky assets/net worth rises with the risk adjusted expected net return $(b - r)/(1-\gamma)\sigma_2^2$.

If the investor is risk neutral $\gamma = 1$, then there is a "bang-bang" solution. For positive expected net return the ratio $k_m = \infty$; otherwise, it is zero.

In our model, there is borrowing as well as lending at an uncertain interest rate. Borrowing can finance either consumption or investment. Borrowing to finance capital involves an asset and a liability. The net return is (b(t)-r(t)), the difference between the productivity of capital and the real rate of interest. Each component is a stochastic variable. Brownian motion terms dw_1 in the interest rate, and dw_2 in the productivity of capital, affect the variance of the net return. The two shocks dw_1 and dw_2 may be correlated positively or negatively, or may be independent of each other.

Whereas the k_m term stresses the expectations, the other two terms slope λ and intercept $f(0) = (\rho\theta - 1)$ stress the magnitude of, and correlation, between the two shocks which involve deviations from the respective means. The values of these terms are related to the question: will the financing of capital by debt lower or raise the riskiness of the net return? This is similar to the issue in futures markets[24]: Given a position in the spot market, will the sale of a futures contract - the assumption of a liability against an asset - lower or raise the riskiness of the total position[25]? We now explain the economic determinants of the slope λ, and intercept $f(0) = \lambda\,(\rho\theta - 1)$ of the optimal debt/net worth functions.

[24] See Stein (1986, ch.2) for an analysis of hedging of risk in futures markets.

[25] In the economics of futures market, the firm decides to produce quantity s and must decide on the quantity of futures to be sold. The price of output p is uncertain, with a variance called var p. The price of the commodity specified in the futures contract is also uncertain. (The production and hedging decisions are simultaneously determined). If there were no sales of futures, the variance of profits is: s^2 var p. The ratio y of the variance of profits on the total spot and futures position/variance of the price as a function of the short or long position in futures, denoted x, is a parabola with a minimum at x = s, where s is total output. The correlation between the price of output p relevant to the firm and the price of the commodity specified in the futures contract is r. At the minimum risk point, the variance of the profits is $(1-r^2)\,s^2$var p. Hence the sale of debt to finance capital corresponds to the sale of a futures contract. The hedging substitutes a basis risk, the spread between the futures and spot price, for the total risk of variations in the spot price. See Stein (1976: 35-36).

2.1.1 The slope term λ.

In equation (12) or (12a), a unit change in the Merton point k_m changes both the optimum ratio of debt/net worth and capital/net worth by λ. Term $\lambda = 1/(1 + \theta^2 - 2\rho\theta) = \text{var}(b(t)) / \text{var}(b(t)- r(t)) = \sigma_2^2/\sigma^2$ is the ratio of the variance of the productivity of capital ($\sigma_2^2 = \text{var}\ b(t)$) to the variance of the net return, (denoted by $\text{var}(b(t)-r(t)) = \sigma^2$ without any subscript).

If the correlation $\rho > \theta/2$, where $\theta = \sigma_1/\sigma_2$, parameter λ exceeds unity and risky borrowing reduces the riskiness of the net return. In that case, borrowing at an uncertain interest rate is a hedging of the risk of investing in risky capital, just as the sale of a futures contract against some fraction of the spot position reduces the riskiness of the total position. When $\lambda > 1$, a unit increase in the Merton point k_m induces a greater than unit rise in f* the optimal ratio of debt/net worth, and in k capital/net worth.

If $\rho < \theta/2$ then the financing of capital, by borrowing at a risky rate, increases the riskiness of the total position. When $\lambda < 1$, a unit rise in the Merton point induces a smaller rise in both f* and k*.

2.1.2 The intercept term f(0) and risk minimization

When the Merton point $k_m = 0$, such that the expected net return b-r = 0, is it optimal to have some risky assets and be a debtor? This is equivalent to asking if the intercept term $f(0)= \lambda (\rho\theta - 1)$ in the debt equation (12a) is positive. We show that the intercept term is the ratio of debt/net worth that minimizes the variance of the percentage change in net worth, var [dX(t)/X(t)]. From equation (8b), derive equation (14a) for the variance of the percentage change in net worth. The value of f that minimizes the variance of the percentage change of net worth is f(0) in equation (14b).

(14a) $\text{var}[dX(t)/X(t)] = [f^2\sigma_1^2 + (1+f)^2\sigma_2^2 - 2f(1+f)\rho\sigma_1\sigma_2]dt$

(14b) $f(0) = \lambda(\rho\theta - 1) = \text{argmin}_f\ \text{var}[dX(t)/X(t)]$

In part 3, the section on "vulnerability", we relate the intercept term to the efficient frontier between the expected growth of consumption (expected "return") and the variance of the growth of consumption (risk).

2.1.3 Summary

We summarize the results so far as follows concerning the optimal ratio of debt/net worth. The rate of interest and the productivity of capital are stochastic variables with a correlation ρ, which is positive, zero or negative.

(a) The optimal debt/net worth f* is a constant which is a linear function of the Merton point. The latter is the expected net return adjusted for the risk concerning the productivity of capital. The situation is described in figure 1 for countries I and II, which have different correlation coefficients.

(b) The intercept term f(0) is the ratio of optimal debt/net worth, when the expected net return (b-r) is zero. Ratio f(0) may be positive (debt) or negative (credit), depending upon the sign of $(\rho\theta - 1)$. At ratio f(0), the total risk is minimized.

(c) Countries which have Merton points $k_m > (1 - \rho\theta)$ should be debtor countries: $f^* > 0$ such as country I. Countries such as country II which have Merton points $k_m < (1 - \rho\theta)$ should be creditor countries: $f^* < 0$.

(d) A unit rise in the Merton point raises the optimum ratio f* by more (less) than unity if $\rho > \theta/2$ ($\rho < \theta/2$).

(e) The optimum ratio of debt/net worth f* is independent of the discount rate[26]. The discount rate determines solely the optimal consumption ratio.

In deterministic economic growth theory, the variables of interest are the growth rate of capital and of GDP, the real interest rate, and characteristics of the utility of consumption. Section 2.2 derives the world rate of interest and section 2.3 derives growth rates. Proposition 1 links the optimal debt/net worth to the growth rate. Part 3 concerns vulnerability and utility, and the basic analysis is contained in propositions 2 and 3.

|

2.2 The World mean rate of interest

For a given country, the mean rate of interest[27] r is exogenous and has not as yet been linked to the mean productivity of capital. In the world as a whole, the (mean) interest rate r is determined by an equilibrium condition, equation (15a). It states that the world rate of interest r equilibrates the supply of debt by debtor countries to the demand for debt by creditor countries. Alternatively, the Merton point for each country adjusts to achieve the equality in (15a). Figure 1 illustrates the reasoning behind (15a).

(15a) $f^*(1)X_1(t) + f^*(2)X_2(t) = 0$

Let there be two counties I and II where the lines in figure 1 correspond to equation (12a) for the optimal debt f*. Let the Merton point be $k_m(1)$ for country I and k_m (2) for country II, and the respective net worth is $X_1(t)$ and $X_2(t)$. Country I is the debtor and supplies $f^*(1)X_1(t)$ of debt; and country II is the creditor and demands quantity $f^*(2)X_2(t)$ of foreign assets. The rate of interest r adjusts to equilibrate the international market for debt. The world rate of interest r is determined as follows.

Define : $s^2(i) = \sigma^2(i)\ (1-\gamma_i)$, risk aversion times the variance[28] of $[b_i(t) - r_i(t)]$, and $x = X(1)/[X(1) + X(2)]$ the positive fraction of world net worth going to country I. Weight $w(x) = [x/s^2(1)] / [x/s^2(1) + (1-x)/s^2(2)]$ is a positive fraction.

The optimal debt/net worth is:

(12b) $f^* = (b-r)/s^2 + f(0)$

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[26] The independence of the optimal debt and the discount rate does not occur in all models. In particular, if the debt must be repaid at a terminal date then the discount rate does affect the optimal debt. The independence issue is examined in our subsequent papers.

[27] See equation (5).

From (15a) and (12b), derive (15.1). The mean interest rate r is a weighted average of the productivities of capital and the f(0), where $1 \geq w(x) \geq 0$ represents the weight, defined above. The intercept term f(0) was shown in equation (14b) below to be the ratio of debt/net worth that minimize risk.

(15.1) $r(x) = [b(1)w(x) + b(2)(1-w(x))] + [f_1(0)\, s^2(1)w(x) + f_2(0)s^2(2)(1-w(x))]$,

$1 > w(x) = [x/s^2(1)] / [x/s^2(1) + (1-x)/s^2(2)] > 0$, for $1 > x > 0$.

Alternatively, write (15.1) as (15.2).

(15.2) $r(x) = \alpha + \beta\, w(x)$, where

$\alpha = b(2) + f_2(0)s^2(2)$

$\beta = [b(1)-b(2)] + s^2(1)[f_1(0) - f_2(0)s^2(2)/s^2(1)]$

The mean interest rate r(x) is a smooth monotonic function of $1 > x > 0$ and (α,β) are constants. In general, the ratio of debt/net worth f*(1) of country 1 is a smooth continuous function of x, equation (15.3).

(15.3) $f^*(1) = [b(1) - r(x)]/s^2(1) + f_1(0) = [b(1) - \alpha - \beta w(x)]/s^2(1) + f_1(0)$

There are several implications of equations (15.2) and (15.3). First: the discount rate does not directly affect the world rate of interest. The discount rate affects consumption, which affects the growth of net worth. The growth of net worth is affected by the discount rate, as shown in section (2.3) below. Hence, the effect of the discount rate upon the rate of interest is indirect via the fraction of net worth owned by each country.

Second: The country will not change from a debtor to a creditor, and vice-versa, given the mean productivities of capital. The f*(x) is continuous on [0,1]. The monotonicity of the debt f*(x) follows from the monotonicity of r(x) in (15.2).

Consider f* of country 1, denoted f*(1), for x = 0 and x = 1.

$[f^*(1)|x=0] = [b(1) - b(2)]/s^2(1) + [f_1(0) - f_2(0)\, s^2(2)/s^2(1)]$

$[f^*(1)|x = 1] = 0$

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[28] See the definition in Box 1.

Obviously if $x = 1$, there will be no foreign debt: there would be no other side to the transaction. If $x = 0$, the country is very small, then the debt will depend upon the relative productivities of capital [b(1)-b(2)] and the relative risk minimizing debt/net worth, the second term. "Once a debtor, always a debtor" follows from the assumption that the productivity of capital does not decline as k rises.

2.3 Optimal growth of net worth, capital and debt

In the deterministic economic growth models: the output/capital ratio converges to a constant, and the growth rates of capital and output converge to the exogenous growth of effective labor. In our model, the mean output /capital ratio b is a constant, and the control variables are the ratios of capital/net worth or debt/net worth, and consumption/net worth. Therefore, capital and debt will grow at the rate of growth of net worth.

We proved[29] that the controls c^* and f^* which optimize the expected discounted HARA utility criterion (9) are constants. Since $k^* = 1 + f^*$, equivalently c^* and k^* are optimal controls. We now prove an important Proposition 1: The value f^* also optimizes the expected growth of net worth, among all controls such that consumption/net worth and debt/net worth are constant: $c(t) = C(t)/X(t) = c$, $f(t) = L(t)/X(t) = f$ for all t. We also verify in proposition 2 that bankruptcy cannot occur. For simplicity of exposition, continue to work with the case where $\gamma = 0$, the logarithmic utility function.

PROPOSITION 1: For any constant ratio c of consumption/net worth, $c > 0$, the ratio of debt/net worth which maximizes the expected growth rate of net worth is the same as the optimal ratio of debt/net worth in equation (12a) which maximizes the expected discounted value of utility. Both are independent of the discount rate.

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[29] See Box 1 above.

proof: Let the consumption/net worth ratio $c = C(t)/X(t)$, $c > 0$ and debt/net worth ratio $f = k - 1 > -1$ be constant. The growth of net worth equation (8b) is repeated as (16) abbreviated as (16a), where no optimality conditions are imposed.

(16) $dX(t) = [(b - c) + (b-r)f]\, X(t)\, dt - f\, X(t)\sigma_1 dw_1 + (1+f)X(t)\sigma_2 dw_2$.

(16a) $dX(t) = AX(t)\, dt + B_1 X(t) dw_1 + B_2 X(t) dw_2$.

$A = [(b - c) + (b-r)f]$; $B_1 = - f\sigma_1$; $B_2 = (1+f)\sigma_2$

$dw_i = \varepsilon_i \sqrt{dt}$ $i = 1,2$ $\varepsilon \sim N(0,1)$

The growth in the deterministic case is Adt. It arises from the productivity of capital less the consumption ratio (b-c) plus the net return from borrowing (b-r)f. The stochastic elements add the second and third terms in (16) or (16a). Using the stochastic calculus – the Ito equation - equation (16) or (16a) implies equation (17). It is abbreviated as (17a), using the definitions for A and B in (16a) and for g(f) in (18b).

(17) $d \ln X(t) = [(b-c) + (b-r)f]\, dt - f\, \sigma_1 dw_1 + (1+ f)\sigma_2 dw_2 - (1/2)[f^2\sigma_1^2\varepsilon_1^2 + (1+ f)\sigma_2^2\varepsilon_2^2 - 2f(1+f)\sigma_1\sigma_2\varepsilon_1\varepsilon_2]dt$

(17a) $d \ln X(t) = g(f)\, dt + (B_1\, dw_1 + B_2\, dw_2)$

The expected growth of net worth over an interval dt is $E(d \ln X(t)) = g(f)\, dt$ defined in (18a)- (18b).

(18a) $E(d \ln X(t)) = g(f)\, dt$

(18b) $g(f) = [(b-c) + (b-r)f] - (\sigma_2^2/2)[f^2\theta^2 + (1+f)^2 - 2f(1+f)\, \rho\theta]$

The expected growth rate of net worth $E(d \ln X(t)) = g\, dt$ in (18b) is the sum of two terms. The first term in brackets is the deterministic growth rate and the second term in brackets is a quadratic function[30] of constant debt/net worth.

[30] We always assume that there is a positive variance of the returns on capital and on the net return.

The debt/net worth which maximizes the expected growth rate g in (18b), is f** in equation (19).

(19) $f^{**} = \text{argmax}_f\, g(f) = \lambda k_m + \lambda(\rho\theta - 1)$

The values f** and k** = 1+f** are exactly the optimal debt/net worth f* and capital /net worth k* derived respectively in (12a) (13a). Hence, the optimal capital or debt that maximizes the expected discounted value of utility also maximizes, the expected growth rate of net worth, given a constant consumption ratio. **QED**

3. Vulnerability: Bankruptcy, Risk-Expected Return Tradeoff and Optimality

An economy is bankrupt if net worth X(t) is negative. An economy is vulnerable to shocks if the negative shocks to the productivity of capital and the positive shocks to the interest rate are likely to force consumption to decline below a tolerable level[31]. Social unrest would then occur. In such cases, the country would declare bankruptcy – even if it were not bankrupt – to avoid the serious decline in consumption.

In this most important part of our paper we prove several propositions concerning bankruptcy and vulnerability. First: Proposition 2 states that if, at any time, consumption is C(t) = cX(t) a constant fraction c > 0 of net worth X(t) and the debt L(t) = fX(t) is also a constant fraction f (positive or negative) of current net worth, then bankruptcy cannot occur. This feedback control mechanism does not require that the policies be optimal: f = f* and c = c*. Second: We derive the expected return-risk trade-off associated with a foreign debt/net worth f > -1, independent of time. There is an efficient region where increases in debt increase both expected return and risk, and there are inefficient regions where increases in the debt decrease expected return and

[31] In equation (1b), $C_1(t) < C_{min}(t)$.

increase risk. An economy is vulnerable to shocks in the inefficient regions. These concepts are stated in Proposition 3 below, the crux of the paper.

3.1 Avoidance of Bankruptcy

The dynamics of the wealth process dX(t) expressed in equation (16) or (16a) contain the deterministic part AX(t)dt and a stochastic part $[B_1X(t)\varepsilon_1 + B_2X(t)\varepsilon_2]\sqrt{dt}$, where the ε terms are N(0,1). The shocks, the variance of the stochastic parts, are proportional to dt, the length of the time interval. Consequently, the controls allow a continuous adjustment to bounded shocks per unit of time; and the economy will not be suddenly plunged into bankruptcy by a cataclysmic event over a short time interval. The shocks are continuous over time interval dt and are not like earthquakes. We now prove the "bankruptcy" proposition.

PROPOSITION 2. Let the dynamic process be described by equation (16) or (16a). Given an initial positive net worth X(0) > 0, if c and f are constant, the net worth at any subsequent time can never be negative, bankruptcy can never occur regardless of the bounded shocks to the productivity of capital or interest rate.

proof:

The dynamics of wealth equation (16) or (16a) is an Ito process and implies equation (17). The solution of (17) or (17a) is equation (20) for logarithm X(t), and (21) for X(t). Insofar as the initial net worth X(0) > 0, the net worth X(t) at any time t > 0 will be positive, because each exponential is non-negative.

(20) $\ln X(t)/X(0) = g\,t + [B_1w_1(t) + B_2w_2(t)]$

(21) $X(t) = X(0)\,[\exp(gt)]\{\exp[B_1w_1(t) + B_2w_2(t)]\} > 0$, for $X(0) > 0$

The shocks are continuous - and not jump processes such as earthquakes – over a small interval. Given an initial positive net worth, if c = C(t)/X(t) and f = L(t)/X(t) are constant, the net worth at any subsequent time can never be negative, bankruptcy can

never occur regardless of the bounded shocks to the productivity of capital or interest rate. Any shock that affects wealth will lead to an immediate proportionate adjustment of
consumption, capital and debt that will preclude net worth from becoming negative.
QED

3.2 Vulnerability: Historical Experience

An economy is vulnerable to shocks if they are likely to force consumption to decline below a tolerable level. Social unrest then occurs. In this part, we give an intuitive description of the risk-expected return trade-off, by drawing upon the historical experience. This will motivate the significance of proposition 3 concerning the relation between the ratio of debt/net worth and the expected return-risk tradeoff in section 3.3 below.

Consumption over a time interval dt is equation (22), based upon (4b), in real terms[32]. It is GNP plus new borrowing [e(t)/p(t)]dL(t) less capital formation I(t)dt. The GNP is the GDP, equal to productivity of capital times capital b(t)K(t)dt, less the real interest payments on the debt [e(t)/p(t)]i(t)L(t) = r(t)L(t)dt, where b(t) and r(t) = [e(t)/p(t)]i(t) are stochastic variables.

(22) $C(t)dt = \{b(t)K(t)dt - [e(t)/p(t)]i(t)]\, L(t)\}dt + \{[e(t)/p(t)]dL - I(t)dt\}$

Vulnerability and crises concern the effects of shocks to the productivity of capital b(t), the real interest rate r(t) and their correlation ρ, upon consumption. The recent literature[33] retrospectively describes the chronology and origins of currency (balance of

[32] We are not using the PPP assumption here.

[33]See Kaminsky and Reinhart (1999) for an excellent analysis of the empirical regularities and sources and scope of problems concerning the onset of currency and banking crises. See also International Monetary Fund: WEO, May 1998: 81-82, Kaminsky, Lizondo and Reinhart (1997); World Bank/International Monetary Fund/World Trade Organization Conference on Capital Flows, Financial Crises and Policies, World Bank, Washington DC April 15-16, 1999.

payments) and banking (financial) crises[34]. Both types of crises have been preceded by a multitude of weak and deteriorating objective economic fundamentals. There were very few crises where the economic fundamentals were sound[35]. In the period after financial markets were liberalized, there was an interaction between currency and banking crises. Both are associated with a decline in b(t) and a rise in r(t) = [e(t)/p(t)]i(t), produced by both internal and external factors.

There are unsustainable internal macroeconomic policies. Expansionary monetary and fiscal policies increase absorption C(t)dt + I(t)dt relative to GNP and increase the external debt. There is "over-investment" in real assets, which drives equity and real estate prices to "unsustainable" levels: lowers the non-speculative rate of return on investment.

The external factors that produce crises arise from: overvalued exchange rates[36], declines in the terms of trade and rises in world interest rates. An overvalued exchange rate or decline in the terms of trade (ratio of export/import prices) decreases the growth of exports which decreases the return on capital b(t) relative to the interest rate on debt. These factors increase the ratio of non-performing loans: i.e., deteriorate the quality of the loan portfolios of banks. The banks increase the risk premium on loans and raise interest rates.

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[34] Currency crises were measured as exchange market pressure: a weighted average of the percentage decline in reserves and the percentage depreciation of a currency. Banking crises were measured in terms of events.

[35] Kaminsky and Reinhart (1999:491)

[36] The NATREX model - Stein, Allen, (1997), Stein (1999) demonstrates that exchange rate movements and currency crises result from movements in objectively measured economic real fundamentals. The NATREX model derives a moving equilibrium real exchange rate that is compatible with internal and external equilibrium. This rate is a function of objectively measured real "fundamentals" denoted Z(t), a vector of social consumption/GDP and the productivity of capital at home and abroad. The NATREX is denoted as R[Z(t)]. This gives economic content to the "trend" used in the empirical literature. We define misalignment as the deviation of the actual real exchange rate R(t) from the NATREX. We show that misalignment R(t) – R[Z(t)] leads to changes in the nominal exchange rate. Crises are expected to be successful only if there is misalignment. For a discussion of what is known concerning equilibrium exchange rates see MacDonald and Stein (1999, ch.1).

With of the integration of world capital markets, movements in interest rates in the major industrialized countries are important to the emerging economies. Sustained declines in world interest rates initially induced capital flows to the emerging markets, as world investors sought higher yields. A subsequent abrupt rised the interest rates in industrial countries raised the cost of borrowing, and impaired the ability of the emerging market countries to service the shorter-term dollar denominated debt. Investors, domestic as well as foreign, become apprehensive about the value of the currency and attempted to convert domestic assets into foreign assets. The exchange rate tended to depreciate and/or international reserves decline. The rise in interest rates i(t) and depreciation of the exchange rate e(t) led to a rise in $r(t)L(t) = e(t)i(t)L(t)/p(t)$ the real interest payments on the foreign debt.

Initially, the lower growth rate generated a banking crisis, which raised real interest rates. In turn, the higher interest rates also adversely affected economic activity and reduced the growth rate. The negative correlation $\rho(t)$ between b(t) and $r(t) = [e(t)/p(t)]i(t)$, ties together the impact that external and internal shocks, the banking and currency crises exerted upon the level of consumption in equation (22). When the crisis occurs, the quality of loans is downgraded. Foreigners are reluctant to increase the dollar denominated debt $dL(t) = 0$. Even if all capital formation ceased $dK(t) = 0$, the level of consumption C(t) in equation (22) would have to be reduced to:

$C_1(t)dt = \{b(t)K(t)dt - [e(t)/p(t)]i(t)]\, L(t)\}\, dt$.

If $C_1(t)$ is less than a minimum socially acceptable level, "bankruptcy" will be declared, or there will be a social crisis.

Fourth: Contagion effects[37] may occur when a core of countries have a common lender or very close trade relationships with each other. If the common lender bank is confronted with a marked rise in non-performing loans in several countries, then it may attempt to reduce overall risk by lending less, and on less favorable terms, to the other

[37] See Reinhart and Kaminsky (1999) in World Bank/IMF/World Trade Organization .

countries. Close trade relationships produce another possible channel for "contagion".

3.3 The trade-off between Expected Return and Risk

We derive an "expected return-risk" trade-off of the utility of consumption associated with a foreign debt/net worth ratio. There is an efficient region where expected return and risk are positively related. The optimal debt is on the boundary of the efficient region. There is an inefficient region where the expected return and risk are negatively related. An economy is vulnerable to shocks when the debt is in the inefficient region. By decreasing the debt, the expected return can be increased and the risk can be decreased. We now make these concepts precise.

As before, utility U(t) is the logarithm of consumption C(t), which is a given proportion c of net worth X(t). Hence utility is equation (23), relative to the values in the initial period. Use equation (20) for the value of log X(t)/X(0).

(23) $U(t) - U(0) = \log C(t)/C(0) = \log X(t)/X(0) = g(f)\, t + B_1 w_1(t) + B_2 w_2(t)$

Expected utility relative to the initial period E[U(t)] - U(0) is the g(f)t term. The expected growth of utility - equation (24)- is growth rate g(f) defined in equation (18b) above. Call g(f) the expected return per unit of time.

(24) **expected return** $= E[U(t) - U(0)] / t = (1/t)E[\log C(t)/C(0)]$
$= (1/t)\, E[\log X(t)/X(0)] = g(f).$

The expected return (equation 24) is plotted in figure 2 for an arbitrary[38] c > 0. Term g(f) is a concave function of the debt/net worth ratio, and reaches a maximum at the optimal debt ratio f*. This was proved in proposition 1 above.

The variance of utility is equation (25), based upon (23)(24) and the definition of B's in equation (16a). Call this variance, the "risk" per unit of time.

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[38] With the logarithmic utility function, the optimum $c = \delta$, the discount rate.

(25) **risk** = (1/t) var [U(t) - U(0)] = (1/t) var log C(t)/C(0) = (1/t)var log X(t)/X(0)

$$= (1/t)\, E\, [B_1 w_1(t) + B_2 w_2(t)]^2 = [f^2\sigma_1^2 + (1+f)^2\sigma_2^2 - 2f(1+f)\rho\sigma_1\sigma_2]$$

Note that the risk depends upon f but not on c. The variance of the utility, the risk, is also plotted in figure 2. It is a convex function of the debt.

The minimum value of the risk is obtained when $f = f(0) = \lambda(\rho\theta-1)$ in equation (26), or (14b). The debt associated with the minimum value of the risk f(0) is precisely the intercept term in the optimal debt/net worth $f^* = \lambda k_m + \lambda(\rho\theta-1)$, in equation (12a).

(26) $f(0) = \text{argmin}_f\, [f^2\sigma_1^2 + (1+f)^2\sigma_2^2 - 2f(1+f)\rho\sigma_1\sigma_2] = \lambda\, (\rho\theta - 1)$

We interpret risk as a measure of vulnerability of an economy to shocks. According to equation (14) a large unfavorable shock in either the interest rate or the productivity of capital produces a large decline in wealth X(t). For a fixed ratio c = C(t)/X(t) there is a corresponding large drop in consumption C(t). As the debt/net worth f deviates from its minimum at f(0), the variance of the stochastic terms in equation (14) increases, hence also the vulnerability.

We now state the crucial proposition in our paper. It relates optimality to expected return and risk - vulnerability. Figure 2 illustrates the results.

PROPOSITION 3. The efficient region for the debt f (positive for debtor, negative for creditor) lies between f(0) and f*, in the following manner. (i) When the Merton point $k_m > 0$, then the optimal debt/net worth f* exceeds the value f(0)
(ii) When the Merton point $k_m < 0$, then the optimal debt/net worth f* is less than the value f(0). (iii) In the efficient region: as the debt is varied, there is a positive relation between expected return and risk. (iv) In the inefficient region: as the debt is varied, there is a negative relation between expected return and risk.

Proof. The expected utility of consumption, the "expected return", is maximal at the optimal debt $f^* = \lambda k_m + f(0)$, because $g(f)$ is maximal at $f = f^* = f^{**}$ shown in equation (19). The risk or vulnerability is minimal at f(0). For positive Merton points $k_m > 0$, the optimal debt f* exceeds f(0). Therefore for $k_m > 0$, it follows that when $f > f^*$, then $f > f_o$. The region $f > f^*$ is inefficient. See figure 2. The rise in f above f* reduces expected utility and raises the risk-vulnerability. In the region where $f(0) < f < f^*$, the debt is below optimal, there is a trade-off. Expected return and risk will both rise as the debt is brought to the optimal level. Hence for $k_m > 0$, the region $f^* > f > f(0)$ is efficient, insofar as increased expected return is obtained at the expense of more risk or vulnerability. Regions $f < f(0)$ and $f > f^*$ are inefficient, insofar as a rise in debt can increase expected return and reduce risk or vulnerability.

The argument is symmetrical for negative Merton points, $k_m < 0$. The minimum risk value f(0) is the debtor ($f > 0$) or creditor ($f < 0$) position that is optimal if the Merton point is zero, that is when the expected net return $(b-r) = 0$. When $k_m < 0$, the optimal debt f* is less than f(0), the minimum risk value. Suppose that $f(0) = \lambda (\rho\theta - 1) < 0$, the minimum risk occurs if the country is a creditor. Then the optimal position is that the country should become more of a creditor and take on more risk. There are two inefficient regions and one efficient region. The efficient region is: $f^* < f < f(0)$, and the two inefficient regions are outside that range. **QED**

The efficient regions are summarized.

When $k_m > 0$, $f(0) < f < f^*$. When $k_m < 0$, $f^* < f < f(0)$

f* = optimal debt/net worth f(0)= minimum risk debt/net worth

4 The Optimal Expected current account deficit

It is frequently argued that continued current account deficits are unsustainable and increase the probability of a crisis. For example, the US is a debtor country and the current deficit/GDP has been increasing during the decade of the 1990s. Should this be construed as a sign of vulnerability? In the current section we use our analysis based upon stochastic optimal control to answer the following questions: When optimal policies are followed, what is the expected current account? Should the richest country in the world be a debtor? What is a sustainable current account deficit? Our answer is implied by proposition 4.

The current account deficit is the change in the debt dL(t) over a given time interval. From equation (4), the actual current account deficit as a fraction of net worth is equation (27). It is equal to absorption/net worth less GNP/net worth over time interval dt. Absorption is equal to consumption C(t) plus investment I(t). The second term is GNP, equal to GDP less interest payments on the debt: Y(t) – r(t)L(t). Shocks to GNP are produced by the Brownian motion terms in the interest rate dw_1 and the productivity of capital dw_2. A current account deficit results when absorption exceeds GNP. Equation (27) is definitional: it does not specify whether or not optimal policies are being followed.

(27) $dL(t)/X(t) = \{[C(t) + I(t)] - [Y(t) - r(t)L(t)]\}\, dt / X(t)$

We now derive the expected current account deficit when the consumption ratio and debt ratio are optimal. It is stated as Proposition 4.

PROPOSITION 4: The optimal expected current account deficit/net worth ratio is a quadratic function of the optimal debt/net worth ratio. If the Merton point $k_m > (1 - \rho\theta) > 0$, and the expected interest rate exceeds the discount rate $r > \delta$, then a permanent expected current account deficit/net worth is optimal. If the Merton point $k_m < (1 - \rho\theta)$, and $r > \delta$, then a permanent expected current account surplus is optimal.

proof: In the optimality case, the change in the debt/net worth is $dL(t)/X(t) = d(f^*X(t))/X(t)$ in equation (28). It is the product of the optimal debt f* and the

+percent change of net worth.

(28) $dL(t)/X(t) = f^* \, dX(t)/X(t)$

The optimal debt/net worth $f^* = \lambda k_m + \lambda(\rho\theta - 1) \geq -1$ from equation (12) or (12a), graphed in figure 1 above. The optimal growth $dX(t)/X(t)$ is derived from equation (16). As expected net worth grows when optimal policies are followed, the expected optimal debt should grow at the same rate. Then the expected current account deficit/net worth denoted by Z over a period dt, is equation (29). Let asterisks denote the quantities when optimal policies are followed.

The expected current account deficit/net worth will be positive if both the optimal debt f^* and expected optimal growth of net worth $E(dX^*(t)/X^*(t))$ are positive. The optimal f^* is positive (equation 12a) if the Merton point $k_m > (1 - \rho\theta)$. The expected growth is positive if $A^* = (b - \delta) + (b - r)f^* > 0$.

$$
\begin{aligned}
(29)\ Z(f^*) &= [E(dL^*(t))/X(t)] = f^* \, E(dX^*(t)/X^*(t)) \\
&= f^* A^* \, dt = [(b-\delta)f^* + (b-r)f^{*2}]dt.
\end{aligned}
$$

We have shown that the expected optimal current account deficit Z(f) is a quadratic function of the optimal debt f^*. The graph of equation (29) is a parabola: figure 3. There are two roots. One is the origin $f_1 = 0$. The second is $f_2 = -(b-\delta)/(b-r)$.

We now prove the rest of proposition 4. When the Merton point $k_m = (b-r)/(1-\gamma)\sigma_2^{\,2}$ is positive[39] then $(b-r) > 0$, the expected productivity of capital exceeds the expected interest rate. When both $(b > r)$ and $(r > \delta)$, then $(b > r > \delta)$, the second root $f_2 = -(b-\delta)/(b-r)$ is negative and less than -1. Figure 3 is drawn for this case, but the algebraic treatment is general.

The optimal debt f^* is positive - the country is a debtor - when the Merton point exceeds $(1 - \rho\theta)$. At $f^* = 0$, the slope of Z is $(b-\delta) > 0$. The expected current account deficit/net worth is a quadratic function rising with $f > 0$ as graphed in figure 3.

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[39] We always consider the case where there is risk aversion, $\gamma < 1$, and risk $\sigma^2 > 0$.

Therefore for all $k_m > (1 - \rho\theta)$ and $(r - \delta) > 0$, it is optimal to have expected current account deficits. A debtor country will be at point $f^* = 0C > 0$ and have expected current account deficit net worth $Z = [E(dL)/dt]/ X = CC' > 0$. In creditor countries, $k_m < (1 - \rho\theta)$ and the ratio of optimal debt/net worth $f < 0$ such as point 0D. The optimum expected current account surplus/net worth is DD'. **QED**

Figure 3 or equation (29) is our benchmark for the expected ratio of the current account /net worth when optimal policies are followed. It follows that permanent current account deficits/net worth do not imply that non-optimal policies are being followed.

5. Conclusions: The Implications of the Stochastic Optimal Control Approach for International Debt and Current Account Deficits

Our approach towards the foreign debt and is an extension of the finance models of Merton and Samuelson to an open growing economy with free capital movements. We summarize the implications of our contribution for both the debt of emerging market countries and for the US current account deficits.

First, the stochastic optimal control approach is predicated upon a system where we are unable to anticipate the future due to the Brownian motion terms, and is based upon dynamic programming. There is a stabilizing feedback control mechanism. Bankruptcy cannot occur, regardless of the shocks[40]. Our optimality criterion is the expected discounted value utility of consumption over an infinite horizon, subject to the constrained laws of motion. Our optimal debt/net worth f* and consumption/net worth are derived from the expected utility maximization using stochastic optimal control.

Second: Our derived equation for the optimal foreign debt/net worth, denoted f*, has the following important properties. (a) It is the debt that maximizes the expected value of the utility of consumption over an infinite horizon, given the constrained law of

[40] Proposition 2 states these conditions.

motion of net worth. (b) For any given consumption/net worth ratio $c > 0$, the optimal debt/net worth f* also maximizes the expected growth of net worth.

Third: We view the probability of a crisis in terms of vulnerability. Crises occur when the shocks force a reduction in consumption below a minimum tolerable level. Our measure of vulnerability is the variance of the utility of consumption per unit of time. Since we use a logarithmic utility function, vulnerability - or risk - is measured as the variance per unit of time of the logarithm of consumption.

Fourth: We derive a frontier between the expected return and vulnerability. “Expected return” is measured as the expected growth rate of utility. A foreign debt is called “excessive” if a reduction can reduce vulnerability without sacrificing expected return. We prove that when the debt exceeds the optimum f*, the country is in an inefficient region: the debt is “excessive”.

Many studies[41] claim that weaknesses in the financial sector were at the root of the Asian crises. With underdeveloped private securities markets until the 1990’s, corporations relied heavily upon the banking system for financing. External borrowing by the corporate sector, intermediated mainly through the banking system, was the main vehicle by which foreign funds were mobilized. One of the features of the Southeast Asian crisis was the large size and critical role-played by the corporate sector foreign debt. The very high and rapidly growing debt/equity ratio in the Southeast Asian economies indicated that both the banking and corporate sectors were becoming increasingly vulnerable to adverse shocks.

What should be an Early Warning Signal (EWS) that the short-term dollar denominated debt is excessive? An EWS of vulnerability is directly related to the difference between the actual f(t) and the optimal debt/net worth $f^* = \lambda k_m + \lambda(\rho\theta - 1)$.

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[41] See for example: International Monetary Fund, World Economic Outlook (WEO) May 1998, pp. 6, 85-105.

This approach contains all of the elements featured in the description of a debt crisis, and integrates the real shocks with the fragility of the banking system.

There are several crucial concepts and terms in our analysis. Term $k_m = (b - r)/(1-\gamma)\sigma_2^2$ is referred to as the Merton point. It is the mean productivity of capital less the mean real interest rate (b-r), and $\sigma_2^2 (1-\gamma)$ is the variance of the productivity of capital times[42] relative risk aversion. The intercept term $\lambda(\rho\theta - 1)$ denoted f(0) is the ratio of debt/net worth that minimizes the risk measured as the variance of utility per unit of time. The ρ is the correlation of the two risks and θ is the ratio of the standard deviation of the interest/productivity risks. The interaction of the real and financial shocks is the correlation coefficient ρ.

The fragility of the financial system is aggravated by a correlation $\rho < 0$ which has been the case in the Emerging Market countries. When the productivity of capital is shocked below its mean, there are more non-performing loans, bank failures due to the high debt/equity ratios of banks, reserve losses, capital flight and currency depreciation. These factors increase the cost of servicing the foreign debt and aggravate the decline in consumption. The conclusion is that: given the Merton point, the optimal debt/net worth of an economy with a $\rho < 0$ is low, compared to one where interest rates are positively correlated $\rho > 0$ with the productivity of capital as in the US. We have used the techniques of stochastic optimal control to respond to the question: when is it rational for market participants to anticipate a crisis?

Our analysis also has significant implications for the evaluation of the US current account deficit. One asks: is it rational for the richest country in the world to be a debtor? Are the sustained current account deficits sustainable? Our analysis implies the following.

It is optimal for the US to be a debtor country $f^* > 0$, if the <u>mean net</u> return (b-r) exceeds the <u>risk</u> terms $(1-\gamma)\sigma_2^2(1-\rho\theta)$. The mean net return (b-r) in the US need not

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[42] We generally have worked with the case where $\gamma = 0$, the log utility function.

be higher than exists in the emerging market (EM) countries for it to be rational for the US to be a debtor and the (EM) countries to be creditors. The correlation ρ between the productivity and interest rate shocks is positive in the US and negative in the emerging markets. Debt reduces risk in the US and increases risk in the EM. Insofar as the risk terms $\sigma_2^2(1-\rho\theta)$ are much lower in the US than in the EM, so that f* is higher in the US, it is optimal for the US to be a debtor.

Permanent current account deficits can be optimal and sustainable for the US. The optimal expected current account deficit/net worth is the optimal debt/net worth f* times the optimal expected growth in net worth. Insofar as optimal net worth is growing, and growth is maximal when the debt/net worth is optimal, permanent current account deficits can be optimal and sustainable.

There is an important area of research that has been neglected here. It concerns the relation of the optimal debt f* to "risk aversion" $(1-\gamma)$ that is contained in the Merton point k_m .In Merton's analysis the agent must select the ratio of risky assets/net worth and safe assets/net worth. The k_m is the optimum ratio of risky assets/net worth. The portfolio chosen by the agent has no effect upon the interest rate on the safe asset.

In our case, there is no safe asset. The debt is financed at an uncertain interest rate over its lifetime, and the return on capital is also stochastic[43]. Suppose that there are two countries A and B with (b-r) > 0: the expected productivity of capital exceeds the expected interest rate. Assume that country A has a low coefficient of risk aversion $(1-\gamma)$ close to zero and country B has a higher coefficient $(1-\gamma)$ close to unity. Country A would optimally incur an infinite amount of debt to finance capital formation, and B would incur a finite amount. Would the international capital lenders be willing to finance country A's proposed capital formation? We would expect that the international lenders

[43] A creditor country faces the risk of variations in the interest rate on foreign loans. Hence there is risk on any debt f > 0 or creditor position f < 0.

have their own coefficients of risk aversion ($1-\gamma^*$) and that they would charge countries A and B different interest rates depending upon the amount of debt they have incurred.

Therefore, the "optimal debt" f^* and the interest rate at which the country can borrow must be interrelated. In the present paper, we have shown the power of the stochastic optimal control approach on the basis of a prototype model. The more realistic complications will be covered in a subsequent paper.

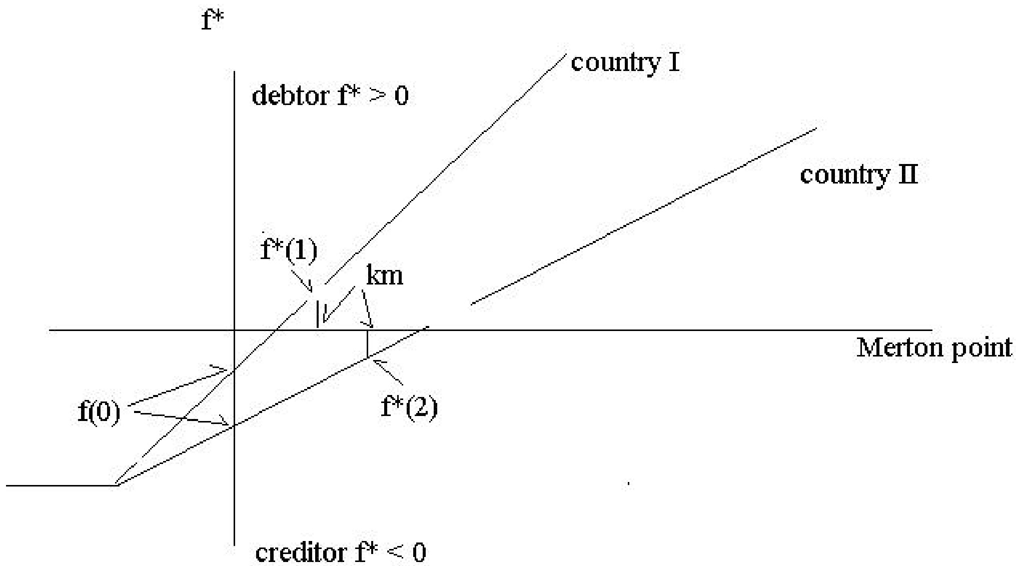

Figure 1. Optimal debt/net worth f*, countries I, II. f(0) is min risk point.
Merton point expected net return adjusted for productivity risk

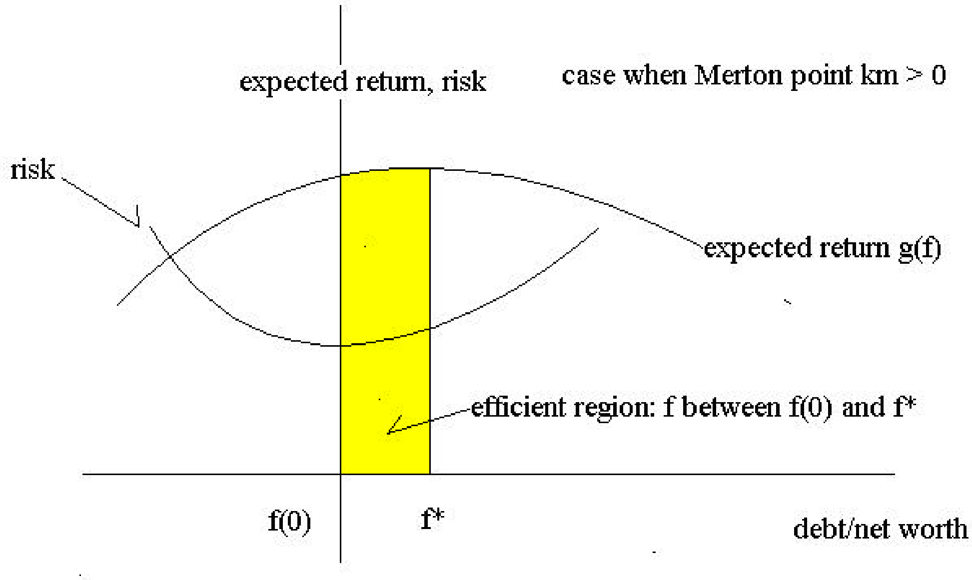

Figure 2. Expected return-risk tradeoff, when km > 0. Risk minimal at f= f(0).

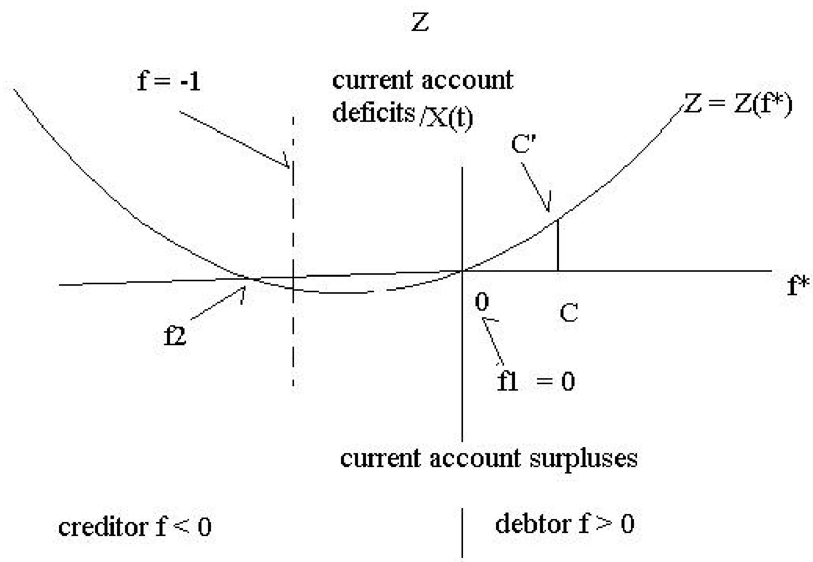

Figure 3. Optimum current account deficit(+) or surplus(-)/net worth Z = Z(f)

APPENDIX A

In this appendix, we give conditions on the model parameters such that equation (10) has a solution with $A > 0$. Then we sketch a formal derivation of equation (10) using the dynamic programming principle. In case $\sigma_1 = 0$ (where only the productivity is random) our problem is equivalent to the classical Merton optimal portfolio model with a single risky asset. An argument which proves that $V(X) = (A/\gamma)X^\gamma$ is indeed the value function, and that the controls are indeed optimal, is given in Fleming-Soner (1992: 174). See also Fleming-Rishel (1975: 160) for the corresponding finite time horizon result. The verification argument when both σ_1 and σ_2 can be non-zero in our paper is entirely similar.

Conditions for $A > 0$

In equation (10)

$\max_c [(1/\gamma A)c^\gamma - c] = [(1-\gamma)/\gamma] A^{1/(\gamma-1)}$

The maximum occurs at $c = C/X = c^* = A^{1/(\gamma-1)}$ as in equation (11). The max over $f \geq -1$ in equation (10) is at $f = f^*$, which satisfies (12) if $f^* > 1$. When $f^* > -1$, a calculation making use of expression (12a) for f^*, gives

$(b - r) f^* + (1/2)(\gamma - 1)[f^{*2}\sigma_1^2 + (1+f^*)^2\sigma_2^2 - 2(1+f^*)f^*\rho\sigma_1\sigma_2]$

$= \lambda \sigma_2^2(1-\gamma)f^{*2} + \sigma^2 f^{*2} + \sigma_2^2$

where λ and σ^2 are defined in Box 1.

Thus, equation (10) becomes (after multiplying by γ)

(A1) $\delta = b\gamma + (1-\gamma) A^{1/(\gamma-1)} + \gamma[\lambda\sigma_2^2(1-\gamma) + \sigma^2)f^{*2} + \sigma_2^2]$

To have a solution with $A > 0$, we must have

(A2) $\delta > b\gamma + \gamma[\lambda\sigma_2^2(1-\gamma) + \sigma^2)f^{*2} + \sigma_2^2]$

When $\gamma < 0$, this imposes no restriction on the discount factor $\delta > 0$. However, for $0 < \gamma < 1$, the inequality (A2) imposes a positive lower bound on δ. (The case $f^* = -1$ is similar).

When $\gamma = 0$, $U(C) = \ln C$ and $V(X) = A \ln X + B$. From equation (10), $A = 1/\delta$. Moreover, $c^* = 1/A = \delta$ as in equation (11a).

Derivation of equation (10)

The dynamic programming principle states that, for each finite time $S > 0$ and where $S \geq t \geq 0$

(A3) $V(X) = \max E\{ \int_{S>t>0} (1/\gamma)C(t)^\gamma e^{-\delta t} dt + V(X(S))e^{-\delta S}\}$, $S \geq t \geq 0$

where the max is taken among the admissible controls on the time interval 0 to S. We recall that $X(0) = X$. Then over the interval $S > t > 0$

(A4) $V(X(S))e^{-\delta S} - V(X) = \int_{S>t>0} d[V(X(t))e^{-\delta t}] dt = \int [-\delta V(X(t)) + d V(X(t))]e^{-\delta t} dt$

From equation (8a) and the Ito differential rule

(A5) $dV(X(t)) = V_x(X(t))dX(t) + (1/2)V_{xx}(X(t)[L(t)^2\sigma_1^2 + (X(t)+L(t))^2\sigma_2^2 - 2L(t)(X(t)+L(t))\rho\sigma_1\sigma_2]dt$

We take expectations in (A4) and recall that expectations of the stochastic integral terms are zero.

(A6) $E\,[V(X(S))]e^{-\delta S} - V(X) = E\,\{\int_{0<t<S} [-\delta V(X(t)) + V_x(X(t))((b-r)(X(t) + L(t)) + rX(t) - C(t)) + (1/2)V_{xx}(X(t))(L(t)^2\sigma_1^2 + (X(t)+L(t))^2\sigma_2^2 - 2L(t)(X(t)+L(t))\rho\sigma_1\sigma_2]e^{-\delta t}dt$

We add to each side of (A6) $E[\int_{S>t>0} (1/\gamma)C(t)^\gamma e^{-\delta t}\, dt]$.

According to (A3), the result is non-positive, and is zero when optimal controls are chosen. We then let S => 0, and consider only controls which are nearly constant on the interval 0 to S.

This completes the formal derivation of equation (10a). By taking the maximum over C, L in (10a), optimal policies C*(X), L*(X) as functions of wealth X are obtained.

When $V(X) = (A/\gamma)X^\gamma$ is substituted in (10a) with c = C/X, f = L/X, equation (10a) is obtained after dividing by X. The optimal policies are C*(X) = c*X, L*(X) = f*X where constants c* and f* are as in equations (11), (12), provided that f* > -1.

APPENDIX B

THE PRODUCTIVITY OF CAPITAL AND REAL LONG TERM RATE OF INTEREST

There are two issues: theoretical and empirical. The first concerns a way to rationalize the production function (6a)-(6c). Assume that there is a Leontief production function: $Y(t) = \min [b(t)K(t), a(t)N(t)]$, where K(t) is capital and N(t) is labor. The technical change is labor augmenting: $da(t)/dt > 0$, and b(t) is given by (6c). Let capital be the constraining input, or assume that $b(t)K(t) = a(t)N(t)$. This gives us $Y(t) = b(t)K(t)$. Capital productivity is b(t). Labor productivity is $Y(t)/L(t) = a(t)$ which grows at rate (1/a(t)da(t)/dt.

The second issue is empirical. What does b(t) look like in the US? The productivity of capital $Y(t)/K(t) = b(t)$ and the real long term interest rate are crucial stochastic variables. We assumed that $b(t)\,dt \sim N(b\,dt, \sigma_2^2 dt)$. The US data is presented here to show the strength and weaknesses of this assumption and to give the reader an intuitive feel for the variables.

We cannot measure the average product of capital b(t) in the model, because we cannot measure capital. It is the market value of all of the existing capital goods by vintage $0 < v < t$ and the sum is deflated by a price index P(t).Call it $J(t) = \int p(v)K(v)/P(t)\,dv$. We expect that the prices of old capital goods are positively related to their vintages, due to technical change: In any year $t \geq 1999$, the price of a 1999 "computer" p(1999) is higher than p(1996) that of a 1996 model. However, since there is a very limited market in old capital goods we do not know what is J(t) .

We can measure the productivity of capital in marginal terms: $b(t) = [dY(t)/dt] / [dK(t)/dt] = g(t)/z(t)$, where $g(t) = (1/Y(t))\, dY(t)/dt$ = growth rate of GDP, and $z(t) = (1/Y(t))dK(t)/dt$ is the ratio of investment (capital formation) to GDP. The graph below plots this measure of b(t) and refers to it as PRODCAP (output growth /investment ratio). The data cover the period 1959:1 – 1997:2.

PRODCAP measure is not the same as the average product of capital $b(t) = Y(t)/K(t)$ in the text for two reasons. First, the model ignored obsolescence and depreciation. Whereas PRODCAP in the figure is a gross, b(t) in the model is a net, measure. Second: since b(t) varies with time $dY(t)/dK(t) = b(t) + db(t)/[dK(t)/K(t)]$.

We assumed that the productivity of capital b(t) dt has a mean b dt and a variance $\sigma_2^2 dt$. The figure below shows that the situation is somewhat more complicated. The productivity of capital is stationary: it is mean reverting.

The real long term rate of interest r(t) in equation (5) is also a stochastic variable. We measure it as the long term US government bond yield less the previous year's rate of inflation. This variable, denoted USRLT, is not mean reverting; and is consistent with the hypothesis.

Some pertinent estimates are as follows. To obtain (b-r), we would have to subtract obsolescence and depreciation from the mean PRODCAP.

	Mean	Stand. Dev.
PRODCAP = gross b(t)	0.189	0.179
USRLT = r(t)	0.038	0.024
Correlation ρ 0.22		

As explained in the text, we use the Brownian motion assumption for the productivity of capital instead of the mean reversion assumption, because the latter raises the order of the system and we cannot solve the system without a computer. At this stage we want to understand the basic analytical processes, before we simulate.

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