Econ 712 Macroeconomic Theory- First Exam. University of Wisconsin

Instructor: Rody Manuelli

October 20, 2007

1 Instructions

- Please answer all questions. If you get stuck in one section move to the next one. Do not waste time on questions that you find hard to solve.
- Partial credit will be awarded if it is clear that you were approaching the problem in an essentially correct manner.
- This is a closed book exam. Students may bring one page (both sides) or two pages (single sides) of notes.
- Please hand in the exam promptly at 12 Noon
- Each question is worth 50 points. The point total for each section is indicated at the beginning of the section. Look at these "prices" when deciding how to allocate your time!
- If you believe that a question is wrong or poorly worded, please make the "minimal" necessary changes to make it "beautiful" and well posed. Of course, unnecessary changes will result in a lower grade.
- Please use one blue book for each question and write only on the "right" page. (The odd numbered page in a newspaper.)
- Please remember to **put your name** in each blue book.
- Try to finish the required questions before you try the extra credit sections. These are typically hard, and carry no 'objective' point value.
- Good luck !

Problem 1 (Interest Rates and Housing Prices) *Consider an economy populated by a large number of identical dynasties with utility functions given by*

$$U \equiv \sum_{t=0}^{\infty} \beta^t [u(c_t) + v(h_t)], \qquad 0 < \beta < 1,$$

where $u(c)$ and $v(h)$ are strictly increasing and strictly concave functions. In this context, c_t denotes consumption of non-durable (e.g. food) goods, while h_t is the stock of durables (e.g. houses) at time t. Each individual is endowed with one tree. Each tree drops fruit (a non-durable good) according to the process $\{c_t\}$. The supply of houses is fixed (i.e. $h_t = h$, for all t). This economy is open to international trade (and borrowing and lending) in non-durable consumption. Housing is non-tradable. The (gross) world interest rate (in units of non-durable consumption) is denoted $1+r_t^*$. In all sections assume that at $t = 0$ this country's foreign debt (denominated in units of non-durable consumption) is 0.

1. *(10 points) Assume that $c_t = \bar{c}$, for all t, and that $1 + r_t^* = \beta^{-1}$. Determine the price of a house and the equilibrium rent.*

2. *(10 points) Suppose that the domestic tree drops a sequence of dividends $\{c_t\}$ satisfying*

$$(1 - \beta) \sum_{t=0}^{\infty} \beta^t c_t = \bar{c}.$$

   What is the relationship between house prices and domestic output (of non-durables) and the trade balance?

3. *(15 points) Assume now that $c_t = \bar{c}$ but that at $t = 0$ there is a unanticipated shock to world interest rates. After the shock, interest rates are given by*

$$1 + r_t^* = \begin{cases} (1-\phi)\beta^{-1} & t = 1, 2, ...T \\ \beta^{-1} & t = T+1, T+2, ... \end{cases}$$

   Go as far as you can describing the impact of the shock to interest rates on housing prices (i.e. compare housing prices at $t = 0$ before and after the shock to interest rates.

4. *(15 points) Consider the economy from the previous section. Go as far as you can describing the dynamic behavior of housing prices from $t = 0$ to infinity. What does the model say (if anything) about the relationship between housing prices and contemporaneous interest rates? What does the model imply about the relationship between housing prices and the trade balance?*

Note: *In sections 3 and 4 you may assume that u is $c^{1-\theta}/(1-\theta)$ if necessary.*

Solution 2 ((Sketch)) *1. Let λ_t be the Lagrange multiplier associated with the feasibility constraint. The first order conditions of the consumer's problem include*

$$\begin{aligned} u'(c_t) &= \lambda_t, \geq \\ p_{ht}\lambda_t &= \beta[\lambda_{t+1}p_{ht+1} + v'(h_{t+1})]. \end{aligned}$$

It follows, given that $u'(c_t) = u'(\bar{c})$, that the price of the average house satisfies

$$p_h = \frac{\beta}{1-\beta}\frac{v'(h)}{u'(\bar{c})}.$$

In order to determine the rental price, q, observe that an investor has two options: he can buy a bond or he can buy a house, rent it for a period and then sell it. The two projects have to offer the same rate of return. Let $\beta^{-1} = 1 + \rho$. Thus

$$1 + \rho = \frac{p_h + q}{p_h},$$

or

$$\begin{aligned} \rho p_h &= q, \\ (1+\rho)\frac{v'(h)}{u'(\bar{c})} &= q. \end{aligned}$$

2. *Denote the endowment sequence by e_t. Since the relevant budget constraint is given by*

$$\sum_{t=0}^{\infty} \beta^t c_t + \sum_{t=0}^{\infty} p_{ht}(h_{t+1} - h_t) = \sum_{t=0}^{\infty} \beta^t e_t,$$

and

$$\sum_{t=0}^{\infty} \beta^t e_t = \sum_{t=0}^{\infty} \beta^t \bar{c},$$

then the optimal choice of the consumer is $h_t = h$ and $c_t = \bar{c}$. The equilibrium is as in the previous case. In this case there is no connection between the trade balance —given by the difference between $\bar{c} - e_t$— and house prices.

3. *Let's first determine the equilibrium consumption sequence. The first order condition corresponding to the optimal choice of bonds is*

$$\begin{aligned} u'(c_t) &= \beta(1-\phi)\beta^{-1}u'(c_{t+1}), \qquad t = 0, 1, ...T, \\ u'(c_t) &= u'(c_{t+1}) \qquad t \geq T+1. \end{aligned}$$

It follows that the sequence $\{c_t\}$ is decreasing and it converges after T periods. In equilibrium the budget constraint implies that

$$\sum_{t=0}^{T} \left(\frac{\beta}{1-\phi}\right)^t c_t + \sum_{t=T+1}^{\infty} \beta^t c_L = \sum_{t=0}^{T} \left(\frac{\beta}{1-\phi}\right)^t \bar{c} + \sum_{t=T+1}^{\infty} \beta^t \bar{c}.$$

Satisfaction of the budget constraint implies that $c_0 > \bar{c}$, and $c_{T+1} = c_L < \bar{c}$. Form the Euler equation corresponding to the equilibrium choice of housing services, we get that

$$p_{ht} = \sum_{j=1}^{\infty} \beta^j \frac{v'(h_{t+j})}{u'(c_t)} = \frac{1}{\rho} \frac{v'(h)}{u'(c_t)}.$$

The housing price formula is just a special case of the general asset pricing formula for a stock. The general formula (in this non-stochastic world) is

$$p_{st} = \sum_{j=1}^{\infty} \beta^j \frac{u'(c_{t+j})}{u'(c_t)} d_{t+j},$$

where d_{t+j} are dividends measured in units of consumption. In the case of the house, the "dividend" is just the marginal utility, $v'(h)$, which evaluated in terms of consumption good (not in utility) is just $v'(h)/u'(c_{t+j})$ at time $t+j$. The result then follows.

Since we argued that $c_0 > \bar{c}$,then the time zero price after the shock is higher than before, that is,

$$\frac{1}{\rho} \frac{v'(h)}{u'(c_0)} > \frac{1}{\rho} \frac{v'(h)}{u'(\bar{c})}.$$

The effect of a decrease in interest rates is to induce a (maybe large) jump in house prices.

4. *The analysis follows easily from the asset pricing formula in the previous section. Since*

$$p_{ht} = \frac{1}{\rho} \frac{v'(h)}{u'(c_t)}$$

and $\{c_t\}$ is decreasing and converges after T periods, then house prices are decreasing as well until period T. After that period, they stabilize at

$$p_{hL} = \frac{1}{\rho} \frac{v'(h)}{u'(c_L)},$$

which is lower than the "before shock" prices. The shock to interest rates make house prices initially overshoot and then they settle at a permanently lower level. The effect of interest rates is ambiguous. In the first period a lower interest rate created an appreciation of the stock of houses. In subsequent periods, house prices decrease even though interest rates remain low. Finally, when interest rates stabilize at the "old" level, house prices stabilize as well but at a lower level. In the first few periods this economy is running a trade deficit (as $c_0 > \bar{c}$) and house prices increase (in period 0) and then decrease (from period one on). Since in the long run the country has a trade surplus (i.e. $c_L < \bar{c}$) it must be the *case that there is some $t^* < T$ such that after t^* the trade balance is positive. However, from t^* to T, house prices are still decreasing. In summary, the model does not predict any simple relationship between interest rates, the trade balance and house prices.*

Problem 3 (Public Goods, Taxes and Congestion) *Consider an economy populated by a large number of identical dynasties with utility functions given by*

$$U \equiv \sum_{t=0}^{\infty} \beta^t u(c_t), \qquad 0 < \beta < 1,$$

where $u(c)$ is a strictly increasing and strictly concave function. Production in this economy requires capital and labor as well as a government provided good. This good, denoted g, can be interpreted as infrastructure and is subject to congestion. If a firm employs k units of capital and n units of labor, its output is

$$y = F(k,n)\left(\frac{g}{F(\bar{k},\bar{n})}\right)^{\phi}, \qquad 0 < \phi < 1,$$

*where $F(\bar{k},\bar{n})$ is interpreted as the average level of output per firm. Thus, holding g constant, an increase in output by "other" firms reduces this firm's productivity due to congestion. From the point of view of an **individual** firm, there are constant return to scale: doubling the amount of private inputs (holding g and the output of other firms constant) doubles private output.*

Capital evolves according to

$$k_{t+1} \leq (1-\delta_k)k_t + x_{kt},$$

and feasibility requires that

$$c_t + g_t + x_{kt} \leq y_t,$$

and, as usual, the level of initial capital is given.

1. *(15 points) Consider the planner's problem. Note that the planner understands that the "true" production function is given by*

   $$y = F(k,n)^{1-\phi} g^{\phi}$$

   and chooses g (as well as other inputs) optimally. Describe the condition that determines the optimal steady state capital per worker.

2. *(15 points) Consider now a competitive equilibrium in which the planner chooses g optimally and finances g with lump-sum taxes. Is the steady state capital per worker optimal in the sense of coinciding with the planner's choice in the previous section?*

3. *(20 points) Consider now a competitive equilibrium in which the planner uses a tax on capital income to finance g. Assume that tax revenue equals government spending (no public debt). To be precise, let r be the rental price of capital, then tax revenue is*

$$\tau r k,$$

where τ is the tax rate chosen by the government. Let $\hat{k}(\tau)$ and $\hat{g}(\tau)$ be the equilibrium levels of the capital stock per worker and the publicly provided good. Go as far as you can finding the tax rate (if any) that has the property that the steady state equilibrium values of capital per worker and the public good coincide with the planner's choice? More precisely, is there a tax rate, τ, such that

$$\begin{aligned} \hat{k}(\tau) &= k^*, \\ \hat{g}(\tau) &= g^*, \end{aligned}$$

where (k^*, g^*) are the steady state values from the solution to the planner's problem?

Solution 4 ((Sketch)) 1. *The planner's first order conditions include*

$$1 = \phi F(k_t, n_t)^{1-\phi} g_t^{\phi-1},$$

which implies that, for all t,

$$g_t = \phi^{1/(1-\phi)} F(k_t, n_t).$$

The Euler equation for capital is

$$u'(c_t) = \beta u'(c_{t+1})[(1-\delta_k) + (1-\phi)F(k_{t+1}, n_{t+1})^{-\phi} g_{t+1}^{\phi} F_k(k_{t+1}, n_{t+1})].$$

Using the optimal choice of g in the previous equation we get that

$$u'(c_t) = \beta u'(c_{t+1})[(1-\delta_k) + (1-\phi)\phi^{\phi/(1-\phi)} F_k(k_{t+1}, n_{t+1})].$$

Thus, if we let $\beta = (1+\rho)^{-1}$, the steady state capital labor ratio satisfies

$$\rho + \delta_k = (1-\phi)\phi^{\phi/(1-\phi)} f'(k^*),$$

where $f(k) \equiv F(k, 1)$.

2. *In a competitive equilibrium, firms choose their capital stock so that the marginal product equals the rental price. Thus, letting q_t denote the rental price at time t, the first order condition is*

$$q_t = F_k(k_t, n_t) \left(\frac{g_t}{F(\bar{k}_t, \bar{n}_t)} \right)^{\phi},$$

while the equality of rates of return condition requires that

$$R_t = \beta[(1-\delta_k) + q_t].$$

Since taxes are lump-sum, no margin is distorted. Given that all firms are identical and that the government uses the efficient rule to determine government spending, i.e. it sets

$$g_t = \phi^{1/(1-\phi)} F(\bar{k}_t, \bar{n}_t),$$

it follows that the equilibrium marginal product of capital satisfies

$$q_t = F_k(k_t, n_t)\phi^{\phi/(1-\phi)}.$$

In the steady state the interest rate equals the discount factor and the equilibrium capital-labor ratio, denoted $\hat{k}$*, satisfies*

$$\rho + \delta_k = \phi^{\phi/(1-\phi)} f'(\hat{k}).$$

Comparing this condition with the planner's version, it is clear that $\hat{k} > k^*$*. The problem is that the private sector fails to internalize the fact that an expansion of private output requires more public goods and this is costly. Even though taxes raise sufficient revenue, this is not enough as they do not signal the true cost of additional capital.*

3. *In an equilibrium with taxes, the appropriate version of the condition that determines the optimal choice of capital is*

$$q_t = F_k(k_t, n_t)\left(\frac{\tau F_k(\bar{k}_t, \bar{n}_t)\bar{k}_t}{F(\bar{k}_t, \bar{n}_t)}\right)^{\phi}.$$

In the steady state, the no arbitrage condition requires

$$1 = \beta[(1-\delta_k) + (1-\tau)q]$$

or,

$$\rho + \delta_k = (1-\tau)f'(k)\left(\frac{\tau f'(k)k}{f(k)}\right)^{\phi}.$$

Thus, the optimal tax rate (if it exists) solves

$$\rho + \delta_k = (1-\tau)f'(k^*)\left(\frac{\tau f'(k^*)k^*}{f(k^*)}\right)^{\phi},$$

and it must be such that the level of government spending is efficient. This requires

$$\tau f'(k^*)k^* = g^* = \phi^{1/(1-\phi)} f(k^*).$$

There is no τ that can satisfy both equations (after all we have two equations in one unknown!). It is possible to show (modulo some algrbraic mistake) that a necessary condition for a tax that satisfies the budget constraint to satisfy the Euler equation is that k^* is consistent with

$$\frac{f'(k^*)k^*}{f(k^*)} = \phi^{\phi/(1-\phi)},$$

but this, in turn, implies that the tax on capital is 100%! In this case, consumers will choose to invest zero, and this cannot be an equilibrium.

If the government has access to a mix of lump sum and capital taxes then it is possible to support the first best. To see this, note that the government can choose the ratio of the publicly provided good to private output, $f(k)$, optimally, i.e.

$$\frac{g^*}{f(k^*)} = \phi^{1/(1-\phi)},$$

and then pick the tax rate so that

$$\rho + \delta_k = (1-\tau)f'(k^*)\phi^{\phi/(1-\phi)},$$

which requires that $\tau = \phi$. Thus the right Pigouvian tax internalizes the extra cost of an additiona unity of capital but lump sum taxes are still required to raise the appropriate amount of resources.