Econ 712 Macroeconomic Theory- First Exam. University of Wisconsin

Instructor: Rody Manuelli

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1 Instructions

- Please answer all questions. If you get stuck in one section move to the next one. Do not waste time on questions that you find hard to solve.
- Partial credit will be awarded if it is clear that you were approaching the problem in an essentially correct manner.
- This is a closed book exam. Students may bring one page (both sides) or two pages (single sides) of notes.
- Please hand in the exam promptly at 12 Noon
- Each question is worth 50 points. The point total for each section is indicated at the beginning of the section. Look at these "prices" when deciding how to allocate your time!
- If you believe that a question is wrong or poorly worded, please make the "minimal" necessary changes to make it "beautiful" and well posed. Of course, unnecessary changes will result in a lower grade.
- Please use one blue book for each question and write only on the "right" page. (The odd numbered page in a newspaper.)
- Please remember to **put your name** in each blue book.
- Try to finish the required questions before you try the extra credit sections. These are typically hard, and carry no 'objective' point value.
- Good luck !

2 Questions

Problem 1 (Productivity and Durable Consumption) *Consider an economy populated by a large number of identical dynasties with utility functions given by*

$$U \equiv \sum_{t=0}^{\infty} \beta^t [u(c_t) + v(z_t)], \qquad 0 < \beta < 1,$$

where $u(c)$ and $v(z)$ are strictly increasing and strictly concave functions. In this context, c_t denotes consumption of non-durable (e.g. food) goods, while z_t is the stock of durables (e.g. houses) at time t. The economy's aggregate feasibility constraint is given by

$$\begin{aligned} c_t + qx_{zt} + x_{kt} &\leq Af(k_t), \\ k_{t+1} &\leq (1-\delta_k)k_t + x_{kt}, \\ z_{t+1} &\leq z_t h(\frac{x_{zt}}{z_t}), \qquad h(0) = 0,\ h(\delta_z) = 1,\ 0 < \delta_z < 1 \\ 0 &< (k_0, z_0), \qquad \textit{given} \end{aligned}$$

where f is a standard (i.e. twice differentiable, increasing, strictly concave, $\lim_{k\to 0} f'(k) = \infty$ and $\lim_{k\to\infty} f'(k) = 0$) production function, and $0 < \delta_k < 1$ is the depreciation rate corresponding to capital. The quantity x_{zt} is interpreted as investment (purchases) of new durables. The function h is assumed to be strictly increasing and strictly concave. In this specification, δ_z corresponds to the depreciation rate of the durable good given that, when $x_{zt} = \delta_z z_t$, the stock of the durable good is unchanged (i.e. $z_{t+1} = z_t$). Let the non-durable good be the numeraire.

1. *(15 points) Define a competitive equilibrium in which households own the stocks of capital and durables and trade (at least) one period bonds*

2. *(10 points) Show that a steady state exists and is unique. What does the model imply about the effect of changing TFP (the constant A) upon the steady state quantities consumed of durables and non-durables? Explain your argument.*

3. *(15 points) Does theory pin down the impact of an increase in TFP (A) on the **ratio** of durables to nondurables (i.e. c^*/z^*, where a * denotes a steady state value). Explain your argument. To answer this section you may assume that the relevant utility functions are*

$$u(c) = \frac{c^{1-\theta}}{1-\theta}, \qquad v(z) = \frac{z^{1-\eta}}{1-\eta}, \qquad (\theta, \eta) > 0.$$

4. *(10 points) Assume now that the economy is open and that individuals can borrow and lend (subject to the standard transversality condition) in the international bond market. Assume that the world interest rate, R^*, satisfies $R^*\beta = 1$.*

*It is claimed that, since the interest rate is constant, the country's level of output and stock of durables will converge in **one period** to their steady state values (not necessarily the same steady state as the closed economy). Discuss this claim.*

***Note**: You are not asked to describe the dynamic behavior (although you would get extra credit for doing this), you are asked to check whether convergence is "fast" (one period) or "slow."*

Solution 2 (Sketch) *1. A Competitive Equilibrium is an allocation, that is, a list of quantities* $[\{c_t\}, \{z_{t+1}\}, \{x_{zt}\}, \{x_{kt}\}, \{k_{t+1}\}, \{n_t\}]_{t=0}^{\infty}$ *and a price system* $[\{w_t\}, \{q_t\}, \{r_t\}, \{R_{t+1}\}]_{t=0}^{\infty}$*,such that*

(a) *(Utility Maximization) the allocation solves the household maximization problem*

$$\max U \equiv \sum_{t=0}^{\infty} \beta^t [u(c_t) + v(z_t)]$$

subject to

$$\begin{aligned} c_t + x_{kt} + q_t x_{zt} + b_{t+1} &\leq w_t + r_t k_t + R_t b_t, \\ k_{t+1} &\leq (1 - \delta_k) k_t + x_{kt}, \\ z_{t+1} &\leq z_t h(\frac{x_{zt}}{z_t}) \end{aligned}$$

with the additional restriction that

$$\lim_{T\to\infty} (\Pi_{j=0}^{T} R_{j+1}^{-1}) b_{T+1} = 0.$$

(b) *(Profit Maximization) The allocation solves the representative firm's maximization problem*

$$\max c_t + q_t x_{zt} + x_{kt} - w_t n_t - r_t k_t$$

subject to

$$c_t + q x_{zt} + x_{kt} \leq AF(k_t, n_t),$$

where we define

$$f(x) \equiv F(x, 1)$$

(c) *(Market Clearing)*

$$\begin{aligned} c_t + q x_{zt} + x_{kt} &\leq Af(k_t), \\ k_{t+1} &\leq (1 - \delta_k) k_t + x_{kt}, \\ z_{t+1} &\leq z_t h(\frac{x_{zt}}{z_t}), \\ n_t &\leq 1. \end{aligned}$$

2. *Let the Lagrangian corresponding to the consumer problem be (ignoring the non-negativity constraints as the focus is on the steady state)*

$$\sum_{t=0}^{\infty} \beta^t \{[u(c_t) + v(z_t)] + \lambda_t [w_t + r_t k_t + R_t b_t - c_t - x_{kt} - q_t x_{zt} - b_{t+1}]$$
$$+\mu_t [z_t h(\frac{x_{zt}}{z_t}) - z_{t+1}] + \xi_t [(1-\delta_k) k_t + x_{kt} - k_{t+1}]$$

The first order conditions in an interior solution (again, since the focus is on the steady state this is without loss of generality) corresponding to the utility maximization problem are

$$\begin{aligned}
u'(c_t) &= \lambda_t, \\
\lambda_t &= \beta \lambda_{t+1} R_{t+1}, \\
\lambda_t &= \beta \lambda_{t+1} [(1-\delta_k) + r_{t+1}], \\
\lambda_t q_t &= \mu_t h'(\frac{x_{zt}}{z_t}), \\
\mu_t &= \beta \{\mu_{t+1} [h(\frac{x_{zt+1}}{z_{t+1}}) - \frac{x_{zt+1}}{z_{t+1}} h'(\frac{x_{zt+1}}{z_{t+1}})] + v'(z_{t+1})\}.
\end{aligned}$$

In an interior equilibrium, prices are such that

$$\begin{aligned}
r_t &= Af'(k_t), \\
w_t &= Af(k_t) - k_t Af'(k_t), \\
R_{t+1} &= (1-\delta_k) + r_{t+1}, \\
q_t &= q.
\end{aligned}$$

In the steady state it follows that

$$\begin{aligned}
\beta[Af'(k^*) + 1 - \delta_k] &= 1, \\
x_z^* &= \delta_z z^*, \\
x_k^* &= \delta_k k^*, \\
Af(k^*) - \delta_k k^* - \delta_z z^* &= c^*, \\
\frac{u'(c^*) q}{h'(\delta_z)} \{1 - \beta[1 - \delta_z h'(\delta_z)\} &= \beta v'(z^*).
\end{aligned}$$

Note that the first equation completely pins down the stock of capital per worker. Using the aggregate feasibility constraint, consumption is a function of this stock and the steady state stock of durables. Thus, the last condition can be written as

$$\frac{u'(Af(k^*) - \delta_k k^* - \delta_z z^*) q}{h'(\delta_z)} \{1 - \beta[1 - \delta_z h'(\delta_z)\} = \beta v'(z^*).$$

This is an equation in one unknown, z^*. Since h is strictly concave (and $h(0) = 0$), the term

$$\frac{q}{\beta h'(\delta_z)}\{1-\beta[1-\delta_z h'(\delta_z)\} \equiv M$$

is a positive constant. Thus, the steady state stock of durables solves

$$u'(Af(k^*)-\delta_k k^* - \delta_z z^*)M = v'(z^*),$$

with k^* given by

$$\beta[Af'(k^*)+1-\delta_k] = 1.$$

Let

$$L(z;A,k^*) \equiv u'(Af(k^*)-\delta_k k^* - \delta_z z^*)M,$$

then the equilibrium steady state stock of durables is the solution to

$$L(z;A,k^*) = v'(z^*).$$

Under Inada type conditions, we have

$$\begin{aligned}
\lim_{z\to 0} L(z;A,k^*) &= u'(Af(k^*)-\delta_k k^*)M > 0,\\
\lim_{z\to[Af(k^*)-\delta_k k^*]\delta_z^{-1}} L(z;A,k^*) &= \infty,\\
\lim_{z\to 0} v'(z) &= \infty,\\
\lim_{z\to[Af(k^*)-\delta_k k^*]\delta_z^{-1}} v'(z) &= v'([Af(k^*)-\delta_k k^*]\delta_z^{-1}) > 0,
\end{aligned}$$

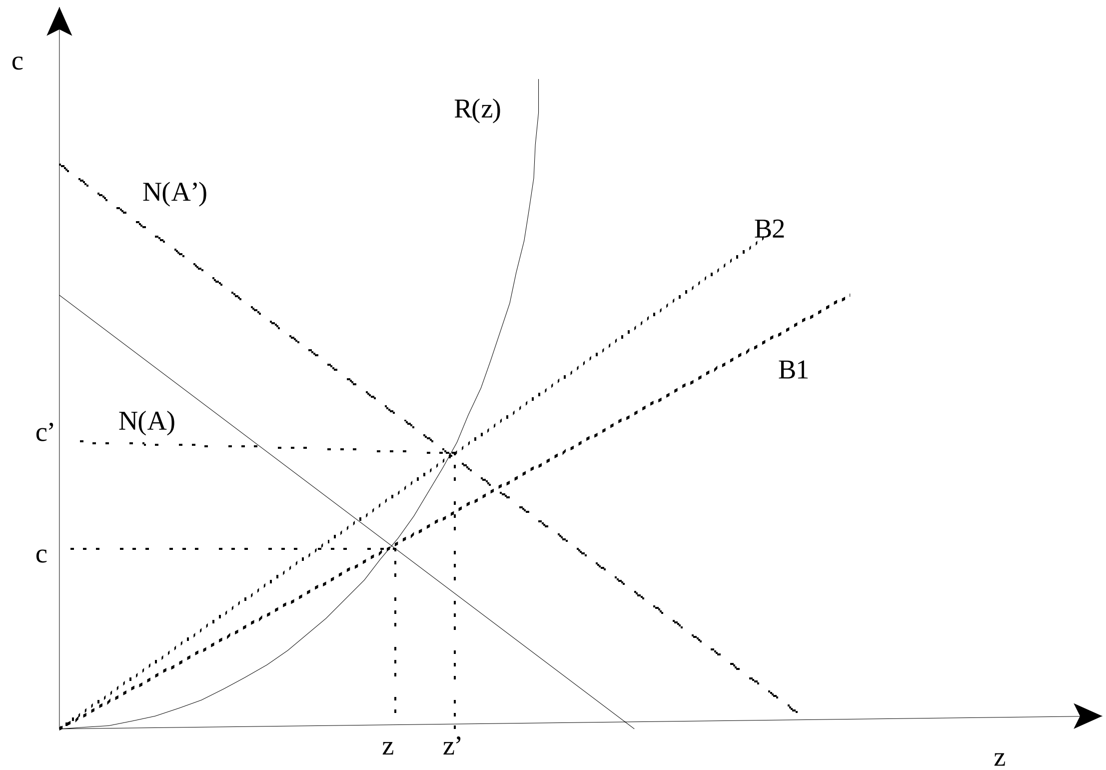
Given that the functions $L(z;A,k^*)$ and $v'(z)$ are continuous this establishes existence. Moreover, simple differentiation shows that $\partial L(z;A,k^*)/\partial z > 0$, while $v''(z) < 0$. This implies that the steady state is unique. An increase in A increases k^* and $Af(k^*) - \delta_k k^*$. This shifts the function $L(z;A,k^*)$ down and, hence, it establishes that that the solution $z^*(A)$ is increasing in A. Moreover, since $v'(z^*(A))$ is decreasing in A, so must be $u'(c^*(A))$ (which is proportional to $L(z;A,k^*)$). This establishes that $c^*(A)$ is increasing in A as well.

3. *No, theory does not pin down the impact of changes in TFP on the ratio of durables and non-durables. Whether this ratio is increasing or decreasing, depends on the curvature of the functions u and v. To see this, use the functions suggested in the question to rewrite the steady state condition as*

$$\begin{aligned}
c^* &= M^{1/\theta} z^{*\eta/\theta} \equiv R(z),\\
c^* &= Af(k^*) - \delta_k k^* - \delta_z z^* \equiv N(A) - \delta_z z.
\end{aligned}$$

To make the case, let $\eta > \theta$. This implies $R(z)$ is a strictly convex function of z. As argued before, an increase in A (say from A to A') shifts the function $N(A)$

up. The function $R(z)$ does not depend on A. The ratio c/z is given by the slope of the linear function from the origin to the equilibrium point. In the following graph, the original (low A) equilibrium has a ratio of non-durables to durables given by the slope of $B1$. After the increase in A (to A' in this example), the new equilibrium point has a higher ratio of non-durables to durables corresponding to the slope of the line labeled $B2$.



The intuition is that since the utility function corresponding to the durable good exhibits more curvature than the utility function of the non-durable, it is less responsive to changes in wealth. Since, in this case, changes in TFP correspond to changes in wealth (prices are constant; there is no substitution effect) the equilibrium level of c increases more than the equilibrium level of z. If $\eta < \theta$, the function $R(z)$ is concave. In this case increases in TFP increase the durable-nondurable ratio. If $\eta = \theta$, the function $R(z)$ is linear and growth has no effect on c/z.

4. *Given the open economy assumption, it follows that $\lambda_t = \lambda$ for every t. This implies that*

$$\beta[Af'(k^*_{t+1}) + 1 - \delta_k] = 1, \qquad t \geq 1.$$

Thus, output converges to the steady state after one period. Moreover, it follows that

$$u'(c) = \lambda$$

If the stock of durables converged as well in one period, investment in the durable is $q\hat{x}_z$*, where*

$$z^* = z_0 h(\hat{x}_z/z_0),$$

and z^* *denotes the steady state level of consumption of the durable good. The (first two) budget constraints are*

$$\begin{aligned} c + k^* - (1-\delta_k)k_0 + q\hat{x}_z + b &= Af(k_0), \\ c + \delta_k k^* + q\delta_z z_0 h(\hat{x}_z/z_0) + b &= Af(k^*) + R^* b, \end{aligned}$$

where, without loss of generality, we have assumed that the initial amount of foreign assets is 0. The variable b *denotes the net asset position from time 1 onwards. In the steady state, assets must be constant (otherwise the transversality condition is violated). It is convenient to combine these two constraints to eliminate* b*. Simple algebra shows that under the guess that the level of durables converges after one period, the combined budget constraint is*

$$\begin{aligned} (1+\rho)c + q\delta_z z_0 h(\hat{x}_z/z_0) + \rho q\hat{x}_z &= Af(k^*) - (\rho+\delta_k)k^* + \\ & \rho[Af(k_0) + (1-\delta_k)k_0] \end{aligned}$$

where, to simplify notation, we set $\beta = (1+\rho)^{-1}$*. In addition to the budget constraints, there are other optimality conditions that the choice of durable good consumption must satisfy. The optimal choice of investment in the durable is such that*

$$\lambda q = \mu_t h'(\frac{x_{zt}}{z_t}).$$

Using this condition, the first order condition corresponding to the optimal choice of the stock of durable is

$$\begin{aligned} \left[h'(\frac{x_{zt}}{z_t})\right]^{-1} &= \beta\{\left[h'(\frac{x_{zt+1}}{z_{t+1}})\right]^{-1}[h(\frac{x_{zt+1}}{z_{t+1}}) - \frac{x_{zt+1}}{z_{t+1}}h'(\frac{x_{zt+1}}{z_{t+1}})] + \frac{v'(z_{t+1})}{\lambda q}\}, \\ z_{t+1} &\leq z_t h(\frac{x_{zt}}{z_t}) \end{aligned}$$

Under the steady state assumption the first condition is just

$$\left[h'(\frac{\hat{x}_z}{z_0})\right]^{-1} = \beta\{[h'(\delta_z)]^{-1}[1 - \delta_z h'(\delta_z)] + \frac{v'(z_0 h(\frac{\hat{x}_z}{z_0}))}{u'(c)q}.$$

Thus, if there is a solution to this equation and

$$\begin{aligned} (1+\rho)c + q\delta_z z_0 h(\hat{x}_z/z_0) + \rho q\hat{x}_z &= Af(k^*) - (\rho+\delta_k)k^* + \\ & \rho[Af(k_0) + (1-\delta_k)k_0] \end{aligned}$$

then there is convergence is one period. Under standard conditions this will be satisfied.

Problem 3 (Equilibrium Migration) *Consider a two island economy in which individuals care about the expected present discounted value of income. Island A is a "safe" island. If an individual chooses this island his income is $y_t = y > 0$, for all t. Island B has large number of trees. Each tree drops fruit according to a cdf that depends on the number of people in the island. If N individuals decide to migrate to island B, the cdf for the wage (fruit) distribution is*

$$F(w;N) = \begin{cases} \frac{w}{B(N)} & w \leq B(N) \\ 1 & w > B(N) \end{cases}.$$

Assume that $B(N)$ is a decreasing function of the total number of individuals in the island (this includes both employed and unemployed individuals). A person employed faces a probability $1-\eta$ of losing his job. There is no on the job search. Unemployed individuals receive unemployment compensation given by $b > 0$. An individual who decides to move to Island B is initially unemployed. Normalize the population to 1.

1. *(10 points) Fix $N < 1$. Show that the optimal policy is of the reservation wage variety and go as far as you can characterizing the reservation wage. Let the reservation wage be denoted $\hat{w}(N)$.*
2. *(15 points) Provide conditions on the primitives of the model so that there exists a unique N^* such that every individual is indifferent between staying in Island A or migrating to B. [Note: Assume that when individuals compare the payoffs associated with each island they take the (equilibrium) value of N as given; i.e. they assume that they are "small" and cannot affect the distribution of wages.]*
3. *(20 points) Consider the (steady state) effect of a permanent decrease in y. Go as far as you can determining if this shock induces migration and/or changes in the unemployment rate (not the total number of unemployed individuals) in Island B.*
4. *(5 points) Go as far as you can analyzing the impact of an increase in the level of the unemployment benefit (b), on the number of people who choose Island B. Is it possible to determine what happens to the unemployment rate?*

Solution 4 (Sketch) *For a fixed N this is a standard search problem. The details of the solution can be found in Ljunqvist and Sargent.*

1. *The reservation wage, $\hat{w}$, solves*

$$\hat{w} - b = \frac{\beta\eta}{1-\beta\eta} \int_{\hat{w}}^{B(N)} (w - \hat{w}) \frac{dw}{B(N)}.$$

Standard calculations show that the expected value of being unemployed (Q) satisfies

$$Q(1-\beta) = \hat{w} + \int_{\hat{w}}^{B(N)} (w - \hat{w}) \frac{dw}{B(N)}.$$

2. *The no-migration condition is that the expected value of migrating (Q) be equal to the expected value of staying, which is just $y/(1-\beta)$. This condition requires that*

$$y = \hat{w} + \int_{\hat{w}}^{B(N)} (w - \hat{w}) \frac{dw}{B(N)}.$$

Using the equilibrium value of the reservation wage, this condition requires

$$\hat{w} = \beta\eta y + (1 - \beta\eta) b \equiv \hat{w}(y, b)$$

Thus, the equilibrium reservation wage does not depend on the number of people in each island. The 'force' that pins down the reservation wage is the requirement that the utilities be the same in both locations. An interior equilibrium requires that $\hat{w}(y, b) > 0$ which, in turn, involves restrictions across the parameters. Given $\hat{w}(y, b)$, the equality of expected utilities in the two islands is such that the following equilibrium condition holds:

$$(1 - \beta\eta)(y - b) = \int_{\hat{w}(y,b)}^{B(N)} (w - \hat{w}(y, b)) \frac{dw}{B(N)}.$$

We would like to argue that, under some condition on the function $B(N)$, the previous equation has a unique solution. First, since the left side is independent of N uniqueness is guaranteed if the right side is monotone in N. If we assume that $B(N)$ is differentiable, the derivative of the right hand side with respect to N is

$$\left[\frac{B(N) - \hat{w}(y, b)}{B(N)} - \int_{\hat{w}(y,b)}^{B(N)} \frac{w - \hat{w}(y, b)}{B(N)} \frac{dw}{B(N)}\right] B'(N) < 0.$$

Thus, if there is a solution, it must be unique. The right side of the existence condition is a continuous function of N. It follows that a (unique) interior equilibrium exists if (this is a sufficient condition)

$$\int_{\hat{w}(y,b)}^{B(0)} (w - \hat{w}(y, b)) \frac{dw}{B(0)} > (1 - \beta\eta)(y - b) > \int_{\hat{w}(y,b)}^{B(1)} (w - \hat{w}(y, b)) \frac{dw}{B(1)}.$$

Since $\hat{w}(y, b)$ is independent of the function $B(N)$, it is straightforward to simply assume that $B(N)$ satisfies the previous inequalities. Let the solution to the equilibrium condition be denoted $N^*(y, b)$

3. *The steady state unemployment rate in island B is*

$$u = \frac{1-\eta}{1-\eta+(1-\frac{\hat{w}(y)}{B(N)})}.$$

To discuss the impact of changes in y (and b), it is convenient to view the equilibrium condition as determining the level of B. Thus, let B^* be the solution to

$$(1-\beta\eta)(y-b) = \int_{\hat{w}(y,b)}^{B^*} (w-\hat{w}(y,b))\frac{dw}{B^*} = \frac{1}{2B^*(y,b)}\left(B^*(y,b)-\hat{w}(y,b)\right)^2 \quad (1)$$

It can be verified that

$$\frac{\partial B^*(y,b)}{\partial y} > 0, \qquad \textit{and } \frac{\partial B^*(y,b)}{\partial b} < 0.$$

From this, it follows that

$$\frac{\partial N^*(y,b)}{\partial y} = \frac{\partial B^*(y,b)/\partial y}{B'(N^*(y,b))} < 0.$$

Thus, a decrease in y increases the fraction of the population in the risky island. Let $z = \hat{w}(y,b)/B^*(y,b)$. The equilibrium condition (1) can be written (after some manipulation) as,

$$2(1-\beta\eta)\frac{y-b}{\beta\eta(y-b)+b} = \frac{(1-z)^2}{z}. \quad (2)$$

The right hand side of equation (2) is decreasing in z (recall that $z = \hat{w}(y,b)/B^*(y,b)$ implies that $z \in [0,1]$). Since the left side of equation (2) is increasing in y; it follows that productivity decreases in the safe island increase the wage-to-upper-bound ratio (z). Since the unemployment rate is

$$u = \frac{1-\eta}{1-\eta+(1-z)},$$

it follows that unemployment is Island B increases when productivity in Island A decreases.

4. *An increase in b has (qualitatively) the opposite effect of an increase in y in terms of its impact on the population of Island B. Since $\partial B^*(y,b)/\partial b < 0$ and $B'(N) < 0$, it follows that*

$$\frac{\partial N^*(y,b)}{\partial b} > 0.$$

As in the previous case, it is immediate that the left hand side of equation (2) is decreasing in b. Thus increases in b increase the wage to upper bound ratio and this increases the unemployment rate.