Econ 712 Macroeconomic Theory- First Exam. University of Wisconsin

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1 Instructions

- Please answer all questions. If you get stuck in one section move to the next one. Do not waste time on questions that you find hard to solve.
- Partial credit will be awarded if it is clear that you were approaching the problem in an essentially correct manner.
- This is a closed book exam. Students may bring one page (both sides) or two pages (single sides) of notes.
- Please hand in the exam promptly at 12 Noon
- Each question is worth 50 points. The point total for each section is indicated at the beginning of the section. Look at these "prices" when deciding how to allocate your time!
- If you believe that a question is wrong or poorly worded, please make the "minimal" necessary changes to make it "beautiful" and well posed. Of course, unnecessary changes will result in a lower grade.
- Please use one blue book for each question and write only on the "right" page. (The odd numbered page in a newspaper.)
- Please remember to **put your name** in each blue book.
- Try to finish the required questions before you try the extra credit sections. These are typically hard, and carry no 'objective' point value.
- Good luck !

2 Questions

Problem 1 (Endogenous Fertility) *Consider an economy populated by a large number of identical dynasties with utility functions given by*

$$U \equiv \sum_{t=0}^{\infty} \beta^t N_t u(c_t), \qquad 0 < \beta < 1,$$

where $u(c) = c^{1-\theta}/(1-\theta)$, $0 < \theta < 1$. *In this specification,* c_t *is per capita consumption at time* t*, and* N_t *is the number of members of generation* t*. If each individual has* $1 + n_t$ *children, the size of a dynasty satisfies*

$$N_{t+1} \leq N_t(1 + n_t).$$

Having children takes time. Assume that each member of each generation is endowed with e *units of time, and that rearing* $(1+n)$ *children, requires* $b(1+n)$ *units of time. Thus, if a member of generation* t *has* $(1+n_t)$ *children the amount of labor effectively supplied to the market is*

$$\ell_t = e - b(1 + n_t).$$

We assume that this number cannot be negative. We also rule out the possibility of a shrinking population. Thus, we consider the set of feasible $n_t's$ *to be the interval* $[0, e/b - 1)$*. Consider the planner's problem*

$$\max \sum_{t=0}^{\infty} \beta^t N_t u(c_t),$$

subject to

$$\begin{aligned} N_t c_t + N_{t+1} k_{t+1} &\leq zF(k_t, \ell_t) N_t + (1 - \delta) k_t N_t, \\ 0 &< (k_0, N_0), \qquad \textit{given} \end{aligned}$$

where F *is a standard (i.e. twice differentiable, increasing, concave, and homogeneous of degree one) production function, and the depreciation factor satisfies* $0 < \delta < 1$. *The planner maximizes utility choosing sequences* $[\{c_t\}, \{n_t\}, \{k_{t+1}\}, \{N_{t+1}\}]_{t=0}^{\infty}$

1. *(15 points) Describe the first order condition of the planner's problem.*

2. *(15 points) Assume that a steady state exists. Consider economies that vary in the cost of rearing children (i.e.* b*). What does the model say about the impact of differences in* b *upon the capital labor ratio, the wage rate and the interest rate in the steady state?*

3. *(10 points) Go as far as you can providing conditions on* (e, b, β) *that are necessary for the steady state to exist.*

4. *(10 points) Go as far as you can describing under what conditions a steady state exists. If necessary make additional assumptions on the functions involved.*

5. ***Extra Credit:*** *Go as far as you can showing under what conditions the steady state is unique. What does this theory say about the relationship between productivity, z, and population growth, n?*

Solution 2 (Endogenous Fertility) *1. Let the Lagrangian corresponding to the planner's problem be*

$$\sum_{t=0}^{\infty} \beta^t \{N_t u(c_t) + \lambda_t [zF(k_t, e - b(1+n_t))N_t + (1-\delta)k_t N_t$$
$$-N_t c_t + N_{t+1}k_{t+1}] + \mu_t [N_t(1+n_t) - N_{t+1}]\}.$$

If we ignore the non-negativity constraints, the first order conditions are

$$c_t \ : \ u'(c_t) = \lambda_t, \tag{1a}$$
$$k_{t+1} \ : \ \lambda_t = \beta\lambda_{t+1}[(1-\delta) + zF_k(\kappa_{t+1}, 1), \tag{1b}$$
$$n_t \ : \ \mu_t = \lambda_t z F_\ell(\kappa_t, 1), \tag{1c}$$
$$N_{t+1} \ : \ \mu_t + \lambda_t k_{t+1} = \beta\{u(c_{t+1}) + \lambda_{t+1}[zF(\kappa_{t+1}, 1)k_{t+1} + (1-\delta)k_{t+1} - c_{t+1}] + \mu_{t+1}(1+n_{t+1})\}, \tag{1d}$$

where $\kappa_t \equiv k_t/\ell_t$ is the capital labor ratio. Let $\beta = (1+\rho)^{-1}$

2. *At the steady state (if one exists), per capita quantities are constant, and population is growing at a constant rate. It follows from (1a) and (1b), that*

$$\rho + \delta = zF_k(\kappa, 1). \tag{2}$$

*Under standard conditions on F, this equation is satisfied by a unique capital-labor ratio, κ^*. This ratio is **independent** of the number of children per household. It follows that the wage rate, defined as $w = F_\ell(\kappa^*, 1)$, the interest rate, $r = zF_k(\kappa^*, 1) - \delta$, are also independent of b.*

3. *A necessary condition for existence of a solution (and hence for existence of a steady state) is that the effective discount factor, $\beta^t N_t$, be less than one. At the steady state,*

$$N_{t+1} = N_t(1+n^*).$$

Thus, $\beta^t N_t < 1$, if and only if $\beta(1+n^*) < 1$. Since $(1+n^*) < e/b$, a sufficient condition for existence is

$$\beta\frac{e}{b} < 1.$$

4. *Using (1c) in (1d) and imposing that the Lagrange multipliers be constant, we obtain*

$$\begin{aligned} zF_\ell(\kappa^*,1)+k^* &= \beta\{\frac{u(c^*)}{u'(c^*)}+[zF(\kappa^*,1)+(1-\delta)]k^* \\ &\quad -c^*+zF_\ell(\kappa^*,1)(1+n^*). \end{aligned}$$

Since feasibility implies

$$c^*=[zF(\kappa^*,1)-(\delta+n^*)]k^*,$$

and

$$\frac{u(c^*)}{u'(c^*)}=\frac{c^*}{1-\theta},$$

it follows that the equilibrium n^* solves

$$\begin{aligned} (1-\beta(1+n))zF_\ell(\kappa^*,1) &= \{\frac{\beta}{1-\theta}[zF(\kappa^*,1)-\delta]+ \\ &\quad \beta(1-\frac{\theta}{1-\theta}n)-1\}\kappa^*(e-b(1+n)). \end{aligned}$$

(Check the algebra!). It looks like if

$$(1-\beta)zF_\ell(\kappa^*,1)<\{\frac{\beta}{1-\theta}[zF(\kappa^*,1)-\delta]+\beta-1\}\kappa^*(e-b),$$

there is one steady state with positive n^*.

Problem 3 (Pricing Forward Contracts and Paintings) *Consider an economy populated by a large number of identical households with utility function given by*

$$E_0\{\sum_{t=0}^{\infty}\beta^t u(c_t,a_t)\},\qquad 0<\beta<1,$$

where c_t is consumption of a nondurable good, and a_t is consumption of "art." The function u is twice differentiable, increasing and strictly concave (if necessary you may assume that it satisfies the Inada condition). In this economy, equilibrium per capita consumption is given by the exogenous stochastic process $\{c_t\}$. (If it helps, you may assume that there are a number of trees per household, each dropping dividends $\{d_{it}\}$, and that $c_t=\sum_{i=1}^{N}d_{it}$.) Assume that the per capita supply of art is $\bar{a}$.

1. *(15 points) Let a forward contract with strike price K, be a contract that requires its holder to buy one unit of a particular asset next period (we only consider one period forward contracts) at price K. Consider the forward contract that*

requires its holder to buy one (one period) bond next period at price K. It is claimed that the price of such contract today, $q_t^b(1)$ is given by

$$q_t^b(1) = R_{2t}^{-1} - KR_{1t}^{-1},$$

where R_{jt}^{-1} is the price, at time t, of a j-period bond. It is also claimed that the price of a forward contract to purchase one period bonds at price K, J periods in the future is

$$q_t^b(J) = R_{J+1t}^{-1} - KR_{Jt}^{-1}.$$

Discuss these claims.

2. *(15 points) Consider a tree that drops fruit according to the stochastic process $\{d_t\}$. It is claimed that the price of a $J-$period forward contract that will deliver a share of that tree J periods into the future at price K is*

$$q_t^s(J) = p_t - KR_{Jt}^{-1},$$

where p_t is the current price of a share.

3. *(15 points) It is claimed that if the utility function is separable, i.e. if we assume that*

$$u(c, a) = u_c(c) + u_a(a),$$

for some "nice" functions $u_c(c)$ and $u_a(a)$, that art prices will be independent of consumption because the marginal utility of art is independent of the marginal utility of consumption and the supply or art is fixed.

4. *(5 points) Assume that, at time t, it is learned that at time $t + J$, the per capita supply of art will increase to $(1 + \gamma)\bar{a}$, $\gamma > 0$. Go as far as you can analyzing the impact of such an announcement on art prices today. Do not assume that the utility function is separable.*

5. ***Extra Credit:*** *Go as far as you can analyzing the impact of the announcement described in section 4 upon short term interest rates.*

Solution 4 (Pricing Forward Contracts and Paintings) *The derivation of the first order conditions is standard.*

1. *The derivation of the first order conditions is standard. There are many ways of pricing these forward contracts. One can use the price kernel of Arrow securities or directly treat them as (one period) shares with random return. In either case, it follows that*

$$q_t^b(1) = \beta E_t[\frac{u'(c_{t+1})}{u'(c_t)}(R_{1t+1}^{-1} - K)].$$

Since

$$R_{jt}^{-1} = \beta^j E_t[\frac{u'(c_{t+j})}{u'(c_t)}]$$

the result follows. A similar argument shows that

$$q_t^b(J) = R_{J+1t}^{-1} - KR_{Jt}^{-1}.$$

2. *Using the standard pricing formulas, the price of the forward contract* $q_t^s(J)$ *satisfies*

$$q_t^s(J) = \beta^j E_t[\frac{u'(c_{t+j})}{u'(c_t)}(p_{t+j} - K)] = \beta^j E_t[\frac{u'(c_{t+j})}{u'(c_t)}p_{t+j}] - KR_{Jt}^{-1}.$$

Since the price of a share satisfies

$$u'(c_t)p_t = \beta E_t[u'(c_{t+1})p_{t+1}] + \beta E_t[u'(c_{t+1})d_{t+1}],$$

repeated substitution implies that

$$u'(c_t)p_t = \beta^j E_t[u'(c_{t+j})p_{t+j}] + \sum_{k=1}^{j} \beta^k E_t[u'(c_{t+k})d_{t+k}].$$

Using this formula in the formula for the price of the forward contract implies

$$q_t^s(J) = p_t - KR_{Jt}^{-1} - \sum_{k=1}^{j} \beta^k E_t[u'(c_{t+k})d_{t+k}].$$

It follows that the claim is not true.

3. *A standard derivation (treating the payoff of art,* $u_a^{'}(a)/u_c^{'}(a)$ *as its dividend and pricing it as a regular stock, or simply writing down the budget constraint) shows that the price of art satisfies*

$$u_c'(c_t)p_{at} = \beta E_t[u_c'(c_{t+1})p_{at+1}] + \beta E_t[u_a'(a_{t+1})].$$

It follows (by repeated substitution) that

$$p_{at} = \sum_{j=1}^{\infty} \beta^j E_t[\frac{u_a'(a_{t+j})}{u_c'(c_t)}].$$

If $u_a'(a_{t+j}) = u_a'(a)$, *this formula simplifies to*

$$p_{at} = \frac{\beta}{1-\beta}\frac{u_a'(a)}{u_c'(c_t)},$$

which implies that in "good times" art prices increase: art prices and consumption are positively correlated. Note that the model does not have clear implications about the relationship between art prices and interest rates.

4. The "general" asset pricing formula (with non-separable preferences) is

$$p_{at} = \sum_{j=1}^{\infty} \beta^j E_t[\frac{\partial u(c_{t+j}, a_{t+j})/\partial a_{t+j}}{\partial u(c_t, a_t)/\partial c_t}].$$

If u is strictly concave, then an increase in any future a_{t+j} will decrease $\partial u(c_{t+j}, a_{t+j})/\partial a_{t+j}$ and, hence, lower art prices. Note that any changes in the stock of art that will take place at $t + J$ will leave the process $\{\partial u(c_{t+k}, a_{t+k})/\partial c_{t+k}\}_{k=0}^{J-1}$ unchanged. It follows that (see section 1) R_{kt}^{-1} will not change for $k = 1, 2, ...J - 1$.