Econ 712 Problem Set #4

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1 Asset Prices and Martingales

1.1 Lucas Economy

The price is given by

$$p_t^s = \sum_{j=1}^{\infty} E_t \left[\beta^j \frac{u'(c_{t+j})}{u'(c_t)} d_{t+j} \right],$$

where c_t is the equilibrium consumption. By the law of iterated expectation, we have

$$\bar{p}_t^s = E_t[p_{t+1}^s] = \sum_{j=1}^{\infty} E_t \left[\beta^j \frac{u'(c_{t+j+1})}{u'(c_{t+1})} d_{t+j+1} \right],$$

It is easily seen that the assertion is not true. This can be seen for even the simple case such as a linear utility function $u(c) = u_1 c + u_0$ with constant u_0 and u_1. In general,

$$p_t^s = \sum_{j=1}^{\infty} \beta^j E_t \left[d_{t+j} \right] \neq \sum_{j=1}^{\infty} \beta^j E_t \left[d_{t+j+1} \right] = E_t[p_{t+1}^s].$$

unless we impose a certain conditions to the dividend stream.

1.2 Arbitrary $\{d_t\}$?

Rewrite the price as

$$p_t^s = \sum_{j=0}^{\infty} \beta^j E_t \left[\frac{u'(c_{t+j})}{u'(c_t)} \right] E_t[d_{t+j}] + \beta^j cov_t \left(\frac{u'(c_{t+j})}{u'(c_t)}, d_{t+j} \right).$$

Assume that for each integer $i \geq 0, j \geq 1$,

$$E_t\left[\frac{u'(c_{t+i+j})}{u'(c_{t+i})}\right] = k \text{ (constant)} \tag{1.1}$$

$$E_t[d_{t+j}] = d \text{ (constant)} \tag{1.2}$$

$$d_{t+j} \text{ is independent of } c_{t+j} \tag{1.3}$$

Under assumptions (1.1), (1.2) and (1.3), we can say that the assertion holds since

$$\begin{aligned} E_t[\bar{p}^s_{t+1}] &= \sum_{j=0}^{\infty} \beta^j E_t\left[\frac{u'(c_{t+j+1})}{u'(c_{t+1})}\right] E_t[d_{t+j+1}] + \beta^j cov_t\left(\frac{u'(c_{t+j+1})}{u'(c_{t+1})}, d_{t+j+1}\right) \\ &= \sum_{j=1}^{\infty} \beta^j k E_t[d_{t+j+1}] = \frac{dk\beta}{1-\beta} = p^s_t \end{aligned}$$

If any of the assumption is viloated, then the assertion is not satisfied at all. But, if the utility function is linear, (1.2) is automatically satisfied. Moreveore, we can drop (1.3), i.e.,

$$cov_t\left(\frac{u'(c_{t+j+1})}{u'(c_{t+1})}, d_{t+j+1}\right) = cov_t\left(1, d_{t+j+1}\right) = 0.$$

In conclusion, the assertion never holds in general. But, even if the dividend process has an arbitrary correlation with the equilibrium consumption, the forcastibility is satisfied for the risk-neutral representative agent world with constant mean dividend stream. Note (1.1) is the sufficient condition in order that the price of bond satisfies the assertion.

1.3 Martingale

$\{X_t\}$ is a martingale is equivalent to that $X_t = E_t[X_{t+1}]$ for all t. The proof is done by induction and the law of iterated expectation. (Why?) Thus, it is sufficient to show

$$E_t[Y_{t+1}] = Y_t.$$

Then,

$$
\begin{aligned}
E_t[Y_{t+1}] &= E_t\left[\beta^{t+1}u'(c_{t+1})p^s_{t+1} + \sum_{j=0}^{t+1}\beta^j u'(c_j)d_j\right] \\
&= E_t\left[\beta^{t+1}u'(c_{t+1})(p^s_{t+1} + d_{t+1}) + \sum_{j=0}^{t}\beta^j u'(c_j)d_j\right] \\
&= E_t\left[\beta^t u'(c_t)p^s_t + \sum_{j=0}^{t}\beta^j u'(c_j)d_j\right] = Y_t
\end{aligned}
$$

2 Futures Contracts

2.1 Price of the Futures Contracts

Let the interest rate of the bond at time t with maturity $t+j$ be $R_{t,j}$. The no arbitrage argument says that at time t, the return from holding $t+s+1$ period bonding must be equivalent to the return from holding $t+s$ period bond together with the one period futures contract between $t+s$ and $t+s+1$ given in the problem. Let R^F_t be that interest rate. Then, it follows

$$R_{t,s+1} = R_{t,s}R^F_t$$

or

$$R^F_t = \frac{R_{t,s+1}}{R_{t,s}} = \frac{E_t[u'(c_{t+s})]}{\beta E_t[u'(c_{t+s+1})]}. \tag{2.4}$$

2.2 Impact of the Consumption Shock

(2.4) shows that the future price is only dependent on the $t+s$- and $t+s+1$-consumption. So, there is no change in return.

3 Markets and Inequalities

Before solving the problem mathematically it is worth while to intuitively guess the solution. At period 1 one of tree will die and the other will provide 2 unit of dividends forever from then on. So, if the agents are risk-averse, then they want to avoid all or nothing situation. From symmetry of the dividend structure, there is no reason that the price of one tree is different from the other at time. Thus, It is not hard to imagine that they trade half of shares of each tree at period 0 if the market is complete.

3.1 Complete Markets

Let $q_t(X)$ be the price of the Arrow-Debreu security that claims on time t-consumption and is contingent on state $X \in \{X_0 = 0, X_1 = 1\}$. Note q_0 is not dependent on the state. Then, we can rewrite the household i's problem as

$$\max\, u(c_0^i) + \sum_{t=1}^{\infty} \beta^t \frac{1}{2}[u(c^i(X_0) + u(c^i(X_1))], \quad i = A, B$$

subject to

$$q_0 c_0^A + \sum_{t=1}^{\infty}[q_t(X_0)c_t^A(X_0) + q_t(X_1)c_t^A(X_1)] = q_0 + \sum_{t=1}^{\infty} q_t(X_1)2 \tag{3.5}$$

$$q_0 c_0^B + \sum_{t=1}^{\infty}[q_t(X_0)c_t^B(X_0) + q_t(X_1)c_t^B(X_1)] = q_0 + \sum_{t=1}^{\infty} q_t(X_0)2 \tag{3.6}$$

(3.5) is the budget constraint for agent A and (3.6) is the budget constraint for agent B. The next step is to define a competitive equilibrium of the economy.

The competitive equilibrium (or Arrow-Debreu equilibrium) is defined by an allocation $(c_0, \{c_t^i(X)\}_{t=1}^{\infty})$ for $i = A, B$ and the prices $(q_0, \{q_t(X)\}_{t=1}^{\infty})$ such that

(1) Given prices, the allocation solves each household problem

(2) Market clears:

$$c_0^A + c_0^B = 2$$
$$c_t^A(X_0) + c_t^B(X_0) = 2, \quad \forall t \geq 1$$
$$c_t^A(X_1) + c_t^B(X_1) = 2, \quad \forall t \geq 1.$$

Put λ and μ be the Lagrange multiplier for problem of agent A and agent B, respectively. Then, FOCs are

$$u'(c_0^A) = \lambda q_0, \quad u'(c_0^B) = \mu q_0$$
$$\frac{1}{2}\beta^t u'(c_t^A(X_0)) = \lambda q_t(X_0), \quad \frac{1}{2}\beta^t u'(c_t^B(X_0)) = \mu q_t(X_0)$$
$$\frac{1}{2}\beta^t u'(c_t^A(X_1)) = \lambda q_t(X_1), \quad \frac{1}{2}\beta^t u'(c_t^B(X_1)) = \mu q_t(X_1)$$

By dividing the left ones by the right ones respectively, we have

$$\frac{u'(c_0^A)}{u'(c_0^B)} = \frac{\lambda}{\mu}$$
$$\frac{u'(c_t^A(X_0))}{u'(c_t^B(X_0))} = \frac{\lambda}{\mu}$$
$$\frac{u'(c_t^A(X_1))}{u'(c_t^B(X_1))} = \frac{\lambda}{\mu}$$

Note that the right hand sides are all constants. By the market clearing condition, we have[1]

$$c_0^A = c_t^A(X_0) = c_t^A(X_1) := c^A \text{ (constant)}, \quad \forall t \geq 1 \text{ and}$$
$$c_0^B = c_t^B(X_0) = c_t^B(X_1) := c^B \text{ (constant)}, \quad \forall t \geq 1.$$

Now, the prices of Arrow-Debreu securities are

$$q_0 = \frac{u'(c^A)}{\lambda} = \frac{u'(c^B)}{\mu}$$
$$q_t(X_0) = \frac{u'(c^A)}{2\lambda}\beta^t = \frac{u'(c^B)}{2\mu}\beta^t$$
$$q_t(X_1) = \frac{u'(c^A)}{2\lambda}\beta^t = \frac{u'(c^B)}{2\mu}\beta^t.$$

[1] The result comes from the fact that each equation has the same form:

$$u'(x) = \frac{\lambda}{\mu}u'(2-x)$$

Putting these into the budget constraints (3.5) and (3.6), we can obtain

$$c_0^A = c_t^A(X_0) = c_t^A(X_1) = 1 \text{ and } c_0^B = c_t^B(X_0) = c_t^B(X_1) = 1,$$

which means that each agent has the same amount of consumption every period. Then the prices are given by invoking usual Euler equations

$$R_{t+1}^{-1} = E_0\left[\frac{\beta u'(c_{t+1}^A)}{u'(c_t^A)}\right] = E_0\left[\frac{\beta u'(c_{t+1}^B)}{u'(c_t^B)}\right] = \beta, \quad \forall t \tag{3.7}$$

$$s_{-0}^A = s_{-0}^B = 1 + \sum_{j=1}^{\infty}\beta^j E_0\left[\frac{u'(c_j^A)}{u'(c_0^A)}d_j^A(X)\right] = 1 + \frac{\beta}{1-\beta} = \frac{1}{1-\beta} \tag{3.8}$$

$$s_0^A = s_0^B = \sum_{j=1}^{\infty}\beta^j E_0\left[\frac{u'(c_j^A)}{u'(c_0^A)}d_j^A(X)\right] = \frac{\beta}{1-\beta}. \tag{3.9}$$

Note that R_{t+1}^{-1} is the price of 1-period bond and s_{-0}^A and s_{-0}^B are the time-0 prices of stock A and B before the time-0 dividend is given and s_0^A and s_0^B are the prices after the time-0 dividends are given.

Once X is realized, the price of the tree with no dividend is 0 and the price p_t^o of the tree with dividend of 2 is

$$p_t^o = \sum_{j=1}^{\infty}\beta^j E_t\left[\frac{u'(c_{t+j}^A)}{u'(c_t^A)}2\right] = \frac{2\beta}{1-\beta}, \quad t \geq 1, \tag{3.10}$$

which is constant.

3.2 Incomplete Markets

After the realization of X, the price of the dead tree will be 0 afterward. The owner of dead tree consumed 1 at period 0 and never be able to consume after that since he doen't afford to buy and sell the goods. After the realization there is only one agent who has constant endowment 2 at each period, which means the marginal rate of substitution of the owner of the existing tree keeps constant. Thus, the price of the existing tree is $p_t^o = \frac{2\beta}{1-\beta}$ same as (3.10). The price of the bond is β just as in (3.7).

The prices at time 0 are given by

$$R_1^{-1} = E_0 \left[\beta \frac{u'(c_1(X)}{u'(c_0)} \right] = \frac{\beta}{2} \left(\frac{u'(0)}{u'(1)} \times 0 + \frac{u'(2)}{u'(1)} \times 2 \right) = \frac{\beta u'(2)}{u'(1)}. \quad (3.11)$$

and

$$\begin{aligned} p_0^A = p_0^B &= E_0 \left[\beta \frac{u'(c_1(X))}{u'(c_0)} (d_1(X) + p_1^o) \right] \\ &= \frac{\beta}{2} \frac{u'(2)}{u'(1)} \left(2 + \frac{2\beta}{1-\beta} \right) = \left(\frac{\beta}{1-\beta} \right) \frac{u'(2)}{u'(1)} \end{aligned} \quad (3.12)$$

Note that the incomplete market stock price (3.12) is lower than the complete market stock price (3.9). This is because the risk averse agent prefers 1 unit of consumption with probability 1 to consumption with mean 1. The incomplete market interest rate R_1 of (3.11) is higher than the complete market interest rate R_1 of (3.7). This is because the bond is more valuable under this environment.

3.3 Inequality and Complete Markets

This model deals with the effect of future income shock. Notice that there is no aggregate uncertainty in this model. The aggregate endowment is always 2 (constant). If the market is complete, then agents can keep their consumption constant no matter how the uncertainty realized by buying and selling their future endowment (or Arrow-Debreu securities) since there is no market risk.

However, it does not work in the incomplete market. Even if there is no aggregate risk in the market, a certain inequality derived by a shock can last forever without proper financial instruments.