Econ 712 - Homework # 4
(Version 3 - Issued 10/3/07)

Rody Manuelli

Due on October 11, 2007 [give it to Kyoung Jin]

Problem 1 (Asset Prices and Martingales) *It is sometimes claimed that efficient markets imply that the change in the price of an asset cannot be forecast. The argument goes along the following lines: "If a price change can be anticipated, then this provides an arbitrage opportunity and individuals will take advantage of that until price changes are not forecastable." Let us define a forecast as a conditional expectation. Thus the forecast, at time* t, *of next period's stock price is*

$$\bar{p}_t^s \equiv E_t[p_{t+1}^s],$$

where E_t *stands for the conditional expectation operator. Thus, one interpretation of the popular claim is that*

$$\bar{p}_t^s = p_t^s,$$

or

$$E_t[p_{t+1}^s - p_t^s] = 0.$$

1. *Consider a Lucas tree economy. Discuss whether the assertion is correct in the case that* $\{p_t^s\}$ *is the equilibrium price of an asset which is a claim to a stream of dividends* $\{d_t\}$.

2. *Does your answer to the previous point depend on the nature of the* $\{d_t\}$ *process? In particular, do your findings apply to bonds as well as stocks that display arbitrary patterns of correlation of their dividend payments with equilibrium consumption?*

3. *Let* $\{X_t\}$ *be any sequence of random variables (a stochastic process if you will).* $\{X_t\}$ *is a martingale if*

   $$X_t = E_t[X_{t+s}], \qquad for\ s \geq t.$$

   In terms of this notation, your analysis in 1 determined whether asset prices are martingales. Consider the following random variable:

   $$Y_t = \beta^t u'(c_t) p_t^s + \sum_{j=0}^{t} \beta^j u'(c_j) d_j,$$

where p_t^s is the price of a claim to the stream of dividends $\{d_t\}$. Show that in equilibrium $\{Y_t\}$ is a martingale.

Problem 2 (Future Contracts) *Consider a Lucas tree economy. Assume that markets are complete (i.e. claims to the fruit that drops from the trees and any other assets (e.g. bonds) are traded). Preferences are given by*

$$U = E[\sum_{t=0}^{\infty} \beta^t u(c_t)],$$

where $0 < \beta < 1$, u is strictly concave and "nice".

An individual wants to lock, at time t, the interest rate he will have to pay for a one period (risk free) loan at time $t+s$ (i.e. he borrows at $t+s$ and repays the loan at $t+s+1$).

1. *Go as far as you can determining that rate. [Hint: Use arbitrage arguments]*

2. *Go as far as you can describing the impact on this interest rate of news that affect the distribution of consumption at $t+j$, with $j < s$, but do not change the distribution of equilibrium consumption for $t+j+k$, for $k > 0$.*

Problem 3 (Markets and Inequality) *Consider a Lucas tree economy in which there are two types of trees. Let X be a random variable whose value will be realized at time 1. At $t = 0$, its distribution is known, and it is given by $\Pr[X = 0] = \Pr[X = 1] = 1/2$. Both types of trees drop one unit of consumption at $t = 0$. From then on, the dividends (fruit) is given by*

$$d_t^A = \begin{cases} 2 & \text{if } X = 1, \text{ for all } t \geq 1 \\ 0 & \text{if } X = 0, \text{ for all } t \geq 1 \end{cases}$$

and

$$d_t^B = \begin{cases} 0 & \text{if } X = 1, \text{ for all } t \geq 1 \\ 2 & \text{if } X = 0, \text{ for all } t \geq 1 \end{cases}$$

Assume that there are two types of consumers, I and II. Both types have the same preferences given by

$$U = E[\sum_{t=0}^{\infty} \beta^t u(c_t)],$$

where $0 < \beta < 1$, u is strictly concave with $u(0) = 0$, and $u'(0) < \infty$. Consumers differ in terms of their initial endowment of trees. Type I consumers each owns one type A tree, and type II consumers each own a type B tree.

1. *Assume that at $t = 0$ trees (and any other asset) can be freely traded. Go as far as you can describing equilibrium tree prices, interest rates, and consumption allocations for all t.*

2. *Assume that no assets can be traded at* $t = 0$. *Trading first takes place after the realization of the random variable* X.*Go as far as you can describing equilibrium tree prices, interest rates, and consumption allocations for all* t.

3. *What does the model suggest is the relationship between inequality and market completeness?*