Econ 712 Problem Set #3

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Problem 1 - Capital Utilization

The firm's problem is simply defined by

$$\max zF(k_t, n_t) - n_t w_t - r_t k_t$$

for some $F(k,n)$ constant returns to scale. As usual we assume that the labor is inelastically supplied and so $n^* = 1$. Put $zf(k) = zF(k,1)$. Then we have

$$r_t = zf'(k_t) \quad \text{and} \quad w_t = zf(k_t) - zk_t f'(k_t). \tag{1}$$

We assume INADA conditions for the production function:

$$\lim_{k\to 0} f'(k) = \infty \quad \text{and} \quad \lim_{k\to 0} kf''(k) = -\infty$$

Now going back to the representative agent's problem, we set up the following Lagrangean

$$\mathcal{L}(\mathbf{c}, \mathbf{x}, \mathbf{k}, \mathbf{b}, \lambda, \mu) = \sum_{t=0}^{\infty} \beta^t \{u(c_t) + \lambda_t(w_t + r_t k_t + R_t^* b_t - c_t - x_{kt} + \frac{\phi}{2}\frac{x_{kt}^2}{k_t} - b_{t+1})$$
$$+ \mu_t((1-\delta_k)k_t + x_{kt} - k_{t+1})\}$$

Assume there is an interior solution and FOCs are given by

$$\begin{aligned}
[c_t]: &\quad u'(c_t) = \lambda_t \\
[x_{kt}]: &\quad \mu_t = \lambda_t \left(1 + \frac{\phi x_{kt}}{k_t}\right) \\
[k_{t+1}]: &\quad -\mu_t + \beta(1-\delta_k)\mu_{t+1} + \beta\lambda_{t+1}\left(r_{t+1} + \frac{\phi}{2}\left(\frac{x_{kt+1}}{k_{t+1}}\right)^2\right) = 0 \\
[b_{t+1}]: &\quad \lambda_{t+1}\beta R_t^* = \lambda_t
\end{aligned}$$

with TVCs

$$\begin{aligned}
[Feasibility]: &\quad c_t + x_{kt} + \frac{\phi}{2}\frac{x_{kt}^2}{k_t} + b_{t+1} = w_t + r_t k_t + R_t^* b_t \\
&\quad k_{t+1} = (1-\delta_k)k_t + x_{kt} \\
[TVC1]: &\quad \lim_{T\to\infty} \beta^T u'(c_T) k_{T+1} = 0 \\
[TVC2]: &\quad \lim_{T\to\infty} \beta^T u'(c_T) b_{T+1} = 0
\end{aligned}$$

Now it is easy to check that the steady state is pinned down by

$$zf'(k^*) = (1+\delta_k\phi)(\beta^{-1} - 1 + \delta_k) - \frac{\phi\delta_k^2}{2} \tag{2}$$

Note that the right hand side is positive and if $\phi = 0$ (or no adjusting cost) it is collapsed to $\beta^{-1} - 1 + \delta_k$, which is the exactly same as that of the model in the lecture note. Also, note $x_k^* = \delta k^*$.

1-1. Dynamics of output $f(k_t)$

From the FOCs it is easily seen that $\lambda_t = \lambda_{t+1}$ and so $c_t^* = c^*$ (constant) for all $t \geq 0$. By some proper manipulation, we obtain

$$\frac{\phi\beta}{2}a_{t+1}^2 + \phi\beta(1-\delta_k)a_{t+1} = \phi a_t - \beta(r_{t+1} + 1 - \delta_k) - 1 \tag{3}$$

$$k_{t+1} = k_t(1 - \delta_k + a_t) \tag{4}$$

where a_t is defined by the investment-capital ratio

$$a_t = \frac{x_{kt}}{k_t}.$$

Recall $r_{t+1} = zf'(k_{t+1}) = zf'(k_t(1-\delta_k + a_t))$. Thus, although system (3) and (4) are not explicitly defined, they tell us how to get (k_{t+1}, a_{t+1}) from (a_t, k_t). Hence, we will describe their behavior on (k, a)-plane and we will be able to observe the dynamics of this economy. Note that the equilibrium path must satisfy $(k_t, a_t) \to (k^*, \delta_k)$ as $t \to \infty$ where k^* is defined by (2).

First consider a locus $a = a_1(k)$ consisting of the pairs (k, a) satisfying $a_t = a_{t+1}$. From (3)

$$\frac{\phi\beta}{2}a_1^2 - \phi(1-\beta+\beta\delta_k)a_1 + \beta zf'(k(\delta_k - 1 + a_1)) - (1-\beta-\beta\delta_k) = 0. \quad (5)$$

Taking a derivative with respect to k, we have

$$(\phi\beta a_1(k) - \phi(1-\beta+\beta\delta_k) + \beta zkf'')a_1'(k) = -\beta(1-\delta_k + a_1(k))zf''$$

The right hand side is strictly positive. Here we can't exactly infer how big $a_1(k)$ is near $k \approx 0$. It is, however, easy to see that $a_1(k) \approx \delta_k$ when k is not far from k^*. Thus, the coefficient of $a_1'(k)$ is negative for a sufficiently large initial capital k_0. Hence, $a_1'(k) < 0$. Now if $a > a_1(k)$, then $a_{t+1} > a_t$ from (3) and vice versa.

Secondly, we consider the case when $k_t = k_{t+1}$. This locus corresponds to function $a_2(k) = \delta_k$. Then, it is also easy to see that if $a > a_2(k)$, then $k_{t+1} > k_t$ from (4) and vice versa.

As seen in the figure we can check the equilibrium manifold converging to (k^*, δ_k). Similarly to the previous problems, we can argue that the path in region II is not the solution since it eventually across $a = 0$ and gets the negative capital stock. Also, we can show the path in region IV is not the solution. At region IV we have $a_t \to \infty$, which implies $x_{kt} >> k_{t+1}$, which is impossible.

In conclusion, the capital stock is strictly increasing, so is the output along the equilibrium path. Then, it is the case that the capital interest rate r_t is decreasing and the real wage rate w_t is increasing by (1).[1]

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[1] Notice that $\frac{dw_t}{dk_t} = -zf''(k_t) > 0$.

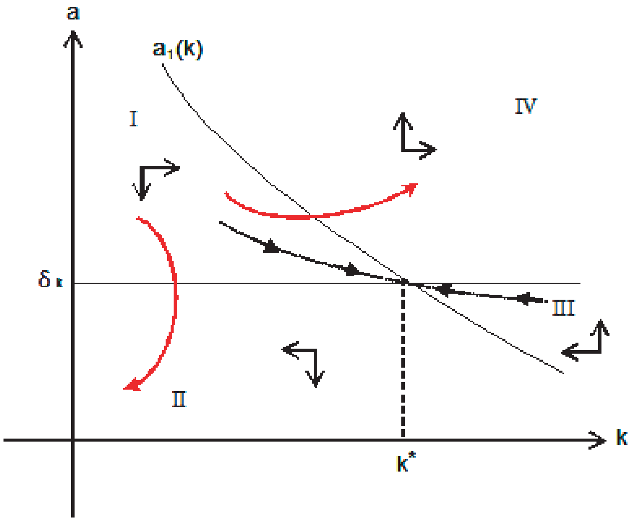

Figure 1: The dynamics of the open economy with adjustment cost

Regarding the interest rates issue, recall that without the adjustment cost ($\phi = 0$)

$$R_t = 1 - \delta_k + r_t = 1 - \delta_k + zf'(k_t).$$

However, this is no more true with the adjustment cost, which even easily can be shown by looking at the steady state case in (2). It is interesting to look at the case when $k_0 < k^*$. In this case, we have that

$$1 - \delta_k + zf'(k_t) > 1 - \delta_k + zf'(k^*) = \frac{1}{\beta} + \phi\left(\frac{\delta_k}{\beta} - \delta_k + \frac{\delta_k^2}{2}\right) > \frac{1}{\beta} = R_t^*$$

In general the interest rates are less than the marginal product of capital (or return from the capital investment). This is because the investment to the capital good is more costly, so if the return from the capital investment is not higher than that from the bond, the agent has no incentive to invest for the capital goods.

1-2. Permanent Increase in z.

From (5) we can easily seen that if $z' > z$, then the locus corresponding to z' is located above the locus corresponding to z. Taking the derivative with

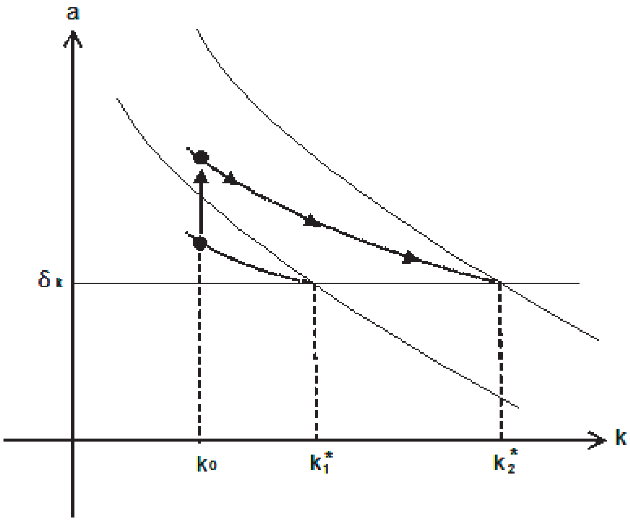

Figure 2: The effect of the permanent increase in productivity z

respect to z on (5), we get

$$(\phi\beta a - \phi(1-\beta+\beta\delta_k) + \beta z k f'')\frac{\partial a_1}{\partial z} = -\beta f' < 0.$$

The coefficient of $\frac{\partial a_1}{\partial z}$ is positive by the same reason in $1-1$. Then, $\frac{\partial a_1}{\partial z} > 0$, which implies there will be an upward jump from the z-manifold to the z'-manifold. We can also check from equation (2) that the steady state k_2^* corresponding z' get larger then k_1^* corresponding z.

From figure 2, we see that for given initial capital k_0, x_0 is increased. This is intuitively clear because this economy now has a better engine for growth that enables it to eventually reach to the more capital accumulation state ($k_2^* > k_1^*$). r_0 and w_0 starts with the higher value and they follows the z'-manifold. So, again r_t is going down and w_t is going up as time goes by since k_t is still strictly increasing.

Now consider what happens in consumption. It is not easy to get the result by looking at the FOCs since we need to show how the lagrange multiplier λ changes depending on z. (Can you do this by directly using the FOCs?) Now, by manipulating feasibility constraints, we can derive the following present value of the agent's optimal consumption:

$$\sum_{t=0}^{\infty} \beta^t c^* = \sum_{t=0}^{\infty} \beta_t (zf(k_t^*) - x_t^* - \frac{\phi x_t^{*2}}{k_t^*}) + b_0 \quad \text{or}$$

$$\frac{c^*}{1-\beta} = \sum_{t=0}^{\infty} \beta^t \left(zf(k_t^*) - x_t^* - \frac{\phi x_t^{*2}}{k_t^*} \right) + b_0, \tag{6}$$

where we already know c^* is constant.[2] The right hand side of equation (6) can be interpreted as an adjusted production or endowment of this economy. Notice that fixing n^* the economy is a decreasing returns to scale, so it has positive profits (Why?). Thus, if z gets bigger, then the profits gets bigger. (Note that this is not true for the CRS case.) Hence, if there is a good shock for productivity($z \rightarrow z'$), then c^* gets bigger.($c^* < c'^*$.) In other words, at time 0 the agent decides to spend a larger consumption than what, otherwise, he/she would have chosen, and he/she keeps that consumption level constant forever.

1-3. Temporary shock Increase in z.

As seen in the figure 3, there is a upward jump in investment right before the economy starts (or as soon as the agent learns the shock). But, its size is smaller than that of the permanent shock case in part $1-2$. So, it should start at below the z'-manifold and the optimal path should go south-east until it lands at the z-manifold at time $t = T$. The analysis is quite similar to $1-2$ except that the equilibrium path eventually follows z-manifold and it reaches the steady state k_1^*. The amount of change in investment and consumption at the initial state is smaller than that of the permanent shock.

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[2] By recursion,

$$\sum_{t=0}^{T} \beta^t c^* = \sum_{t=0}^{T} \beta^t (zf(k_t^*) - x_t^* - \frac{\phi x_t^{*2}}{k_t^*}) - \beta^T u'(c_T) b_{T+1} + b_0$$

Then, taking the limit with respect to T, we get the result by TVC2.

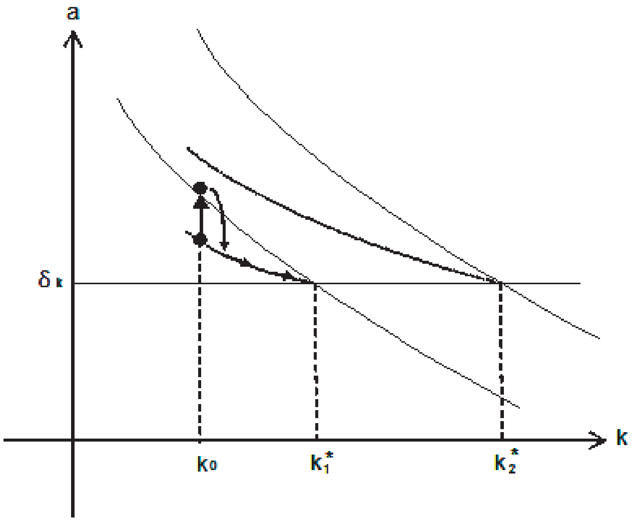

Figure 3: The effect of the temporary increase in productivity z

1-4 and 1-5. Change in a Country's External Asset Position

It's not quite obvious to say what happens about the bond holding for arbitrary k_0. So, let's imagine that this economy is initially at the steady state $k_0 = k_1^*$ with $b_0 = 0$ and there arrives a news about the productivity shock. Then, before the agent knows the shock at the steady state,

$$c^* + x_0^* + \frac{\phi x_0^{*2}}{k_0} = zf(k_0)$$

$$\text{or} \quad 0 = zf(k_0) - \left(c^* + x_0^* + \frac{\phi x_0^{*2}}{k_0}\right).$$

After the shock is realized,

$$b_1 = b_1 - R_{t+1}^* b_0 = z' f(k_0) - \left(c'^* + {x_0'}^* + \frac{\phi {x_0'}^{*2}}{k_0}\right).$$

By looking at the above equation, it doesn't seem that we can determine whether $b_1 > 0$ or $b_1 < 0$. Recall, however, from part $1-3$ and $1-4$ that c'^* and ${x_0'}^*$ for the permanent shock are higher than those for the temporary shock. Thus, the bond holding b_1 for the permanent shock case is greater than the bond holding for the temporary shock case. Notice that along the equilibrium path $b_{t+1} - b_t$ must get close to 0 as time goes by.

2. Durable Goods and Productivity Shocks[3]

(HH's problem)

$$\max \sum_{t=0}^{\infty} [u(c_t) + v(z_t)]$$

subject to

$$\begin{aligned} c_t + x_{kt} + q_t x_{zt} + b_{t+1} &\leq w_t + r_t k_t + R_t b_t, \\ k_{t+1} &\leq k_t + x_{kt} \\ z_{t+1} &\leq z_t h(\frac{x_{zt}}{z_t}) \end{aligned}$$

with the additional condition (TVC) that

$$\lim_{T \to \infty} \Pi_{j=0}^{T} R_{j+1} b_{T+1} = 0.$$

(Firm's problem)

$$\max c_t + q_t x_{zt} + x_{kt} - w_t n_t - r_t k_t$$

subject to

$$c_t + q_t x_{zt} + x_{kt} \leq AF(k_t, n_t)$$

Here we set $f(k) = F(k, n)$.

1. Competitive Equilibrium

A Competitive Equilibrium is an allocation $(\{c_t\}, \{z_{t+1}\}, \{x_{zt}\}, \{x_{kt}\}, \{k_{t+1}\}, \{n_t\})_{t=0}^{\infty}$, a price vector $(\{w_t\}, \{q_t\}, \{r_t\}, \{R_{t+1}\})_{t=0}^{\infty}$, a sequence of bond holdings $\{b_{t+1}\}_{t=0}^{\infty}$ such that

(1) Given prices, the allocation solves the household's problem

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[3] This solution is based on the solution to Probem 1 of the 2006 Midterm.

(2) Given prices, the allocation solves the firm's problem

(3) Market clears: For given $b_0 = 0$[4] and $k_0 > 0, z_0 > 0$,

$$
\begin{aligned}
c_t + x_{kt} + q_t x_{zt} + b_{t+1} &\leq w_t + r_t k_t + R_t b_t, \\
k_{t+1} &\leq k_t + x_{kt} \\
z_{t+1} &\leq z_t h(\frac{x_{zt}}{z_t}) \\
n_t &= 1
\end{aligned}
$$

2. Steady State

Before solving the problem we impose INADA conditions for the utility functions:

$$\lim_{x \to 0} u'(x) = \infty, \quad \lim_{x \to \infty} v'(x) = \infty.$$

Assume that there is an interior solution. Fist from the firm's problem we have

$$r_t = Af'(k_t) \quad \text{and} \quad w_t = Af(k_t) - k_t Af'(k_t).$$

Set up a Lagrangean with multipliers λ_t and μ_t for the consumer's problem in a usual way and we have the following FOCs:

$$
\begin{aligned}
[c_t] \quad &: \quad u'(c_t) = \lambda_t \\
[z_{t+1}] \quad &: \quad \beta v'(z_{t+1}) - \mu_t + \beta \mu_{t+1} \left(h(\frac{x_{zt+1}}{z_{t+1}}) - \frac{x_{zt+1}}{z_{t+1}} h'(\frac{x_{zt+1}}{z_{t+1}}) \right) = 0 \\
[k_{t+1}] \quad &: \quad \lambda_t = \beta \lambda_{t+1}(r_{t+1} + 1 - \delta_k) \\
[x_{zt}] \quad &: \quad q\lambda_t = \mu_t h'(\frac{x_{zt}}{z_t}) \\
[b_{t+1}] \quad &: \quad \lambda_t = \beta R_{t+1} \lambda_{t+1}
\end{aligned}
$$

From FOCs it must be satisfied that

$$R_{t+1} = 1 - \delta_k + r_{t+1} = 1 - \delta_k + f'(k_{t+1}).$$

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[4] Without loss of generality it is ok to assume $b_0 = 0$.

In the steady state we must have $x_z^* = \delta z^*$ since $h(\frac{x_z^*}{z^*}) = 1$ where $h(\cdot)$ is monotone with $h(\delta_z) = 1$. It follows that

$$\begin{aligned}
u'(c^*) &= \lambda^* && (7)\\
\beta v'(z^*) &= \mu^* \left(1 - \beta h(\delta_z) + \beta \delta_z h'(\delta_z)\right) && (8)\\
q\lambda^* &= \mu^* h'(\delta_z) && (9)\\
1 &= \beta R^* &&\\
1 &= \beta(Af'(k^*) + 1 - \delta_k) && (10)\\
c^* &= Af(k^*) - \delta_k k^* - \delta_z z^* && (11)
\end{aligned}$$

First notice that k^* is pinned down by (10). (11) is derived by using $b^* = 0$ since $b_{t+1} = 0$ for all $t \geq 0$. (See claim 60 in page 58 of Lecture notes. The proof is the same even for this model.) Then putting the first three equations (7), (8) and (9) together we have

$$v'(z^*) = \frac{q(1 - \beta h(\delta_z) + \beta \delta_z h'(\delta_z))}{\beta h'(\delta_z)} u'(c^*) \tag{12}$$

Then, Putting (10) into (12), we have

$$v'(z^*) = Xu'(Y - \delta_z z^*) \tag{13}$$

with some constants $X = \frac{q(1-\beta(h(\delta_z)-\delta_z h'(\delta_z)))}{\beta h'(\delta_z)}$ and $Y = Af(k^*) - \delta_k k^*$. Here note that by strict concavity of h with $h(0) = 0$,

$$\frac{h(\delta_z) - h(0)}{\delta_z - 0} > h'(\delta_z) \quad \text{or} \quad 1 = h(\delta_z) > \delta_z h'(\delta_z),$$

which guarantees $X > 0$. It has been shown (in the lecture) that $Y > 0$. Now we are left to show that (13) has an unique solution. Let us define a function $H_Y : (0, Y/\delta) \to R$ such that

$$H_Y(z) = v'(z) - Xu'(Y - \delta z)$$

Then, $H_Y'(z) = v''(z) + \delta X u''(Y - \delta z) < 0$ and

$$\lim_{z \to 0} H_Y(z) = \infty \quad \text{and} \quad \lim_{z \to Y/\delta} H_Y(z) = -\infty.$$

by the INADA conditions. Thus, the intermediate value theorem, implies that there exists a unique solution $z^* \in (0, Y/\delta)$ such that $H_Y(z^*) = 0$.

Now regarding to the effect of changing TFP, we first easily see that $\frac{dk^*}{dA} > 0$ from (10). Then,

$$\frac{dY}{dA} = \frac{1}{dA}(Af(k^*) - \delta k^*) = f'(k^*) + (Af'(k^*) - \delta)\frac{dk^*}{dA} > 0.$$

This implies that solution $z^*(Y)$ corresponding to $H_Y(z) = 0$ is increasing because $H_Y(z)$ is shifting upward as A (thus Y) is increasing. (Sketch the graph of $H_Y(z)$ to check this!) To get the sign of $\frac{dc^*}{dA}$, we need to invoke (7), (8), and (9). From (8) we have $\frac{d\mu^*}{dA} < 0$ and so $\frac{d\mu^*}{dA} < 0$ by (9). Finally, (7) implies $\frac{dc^*}{dA} > 0$. This also establishes that c^* is increasing in A.

2-3. The Impact of an Increasing TFP (A) on $\frac{c^*}{z^*}$

To say the conclusion first, the theory does not pin down the impact of increasing TFP on the ratio of durable to nondurable goods. It depends on the curvature of the utility functions.

Recalling (12) and rewriting this by using the given functional forms, we get

$$z^* = X^{-\frac{1}{\theta}} c^{*\frac{\eta}{\theta}} \implies \frac{c^*}{z^*} = X^{-\frac{1}{\theta}} c^{*1-\frac{\eta}{\theta}}$$

where X is independent of A. It follows that

$$\frac{d(\frac{c^*}{z^*})}{dA} = \left(1 - \frac{\eta}{\theta}\right) X^{-\frac{1}{\theta}} \, c^{*-\frac{\eta}{\theta}} \, \frac{dc^*}{dA},$$

which establishes that

$$\begin{aligned} & \frac{c^*}{z^*} \uparrow \quad \text{if} \quad \eta < \theta \\ \text{and} \quad & \frac{c^*}{z^*} \downarrow \quad \text{if} \quad \eta > \theta \end{aligned}$$

as A is increasing whereas $\frac{c^*}{z^*}$ is not changed if $\eta = \theta$.

1-4. 1-period Convergence in an Open Economy

It is easy to see that $\lambda_t = \lambda^*$(constant) for all $t \geq 0$. $u'(c_t) = \lambda^*$ implies the consumption converges to the steady state from $t = 0$. Also, $\beta(Af'(k_{t+1} + 1 - \delta_k) = 1$ gives that the capital stock converges to the steady state from period 1. Now we are left to show that z_t converges to the steady state in one period. But this is not always the case. If certain conditions are satisfied, then it will be true. We will derive those conditions.

First, FOCs $[z_{t+1}]$ and $[x_{zt}]$ yield

$$\left[h'(\frac{x_{zt}}{z_t})\right]^{-1} = \beta\left[h'(\frac{x_{zt+1}}{z_{t+1}})\right]^{-1}\left(h(\frac{x_{zt+1}}{z_{t+1}}) - \frac{x_{zt+1}}{z_{t+1}}h'(\frac{x_{zt+1}}{z_{t+1}}) + \frac{v'(z_{t+1})}{\lambda^* q}\right)$$

If z_t converges from period 1, then there must exist z^* and x_{z0} such that

$$z^* = z_0 h(\frac{x_{z0}}{z_0})$$

$$\left[h'(\frac{x_{z0}}{z_0})\right]^{-1} = \beta\left[h'(\delta_z)\right]^{-1}\left(h(\delta_z) - \delta_z h'(\delta_z) + \frac{v'(z^*)}{\lambda^* q}\right)$$

In other words, we must have a solution x_{z0} to

$$\left[h'(\frac{x_{z0}}{z_0})\right]^{-1} = \beta\left[h'(\delta_z)\right]^{-1}\left(h(\delta_z) - \delta_z h'(\delta_z) + \frac{v'(z_0 h(\frac{x_{z0}}{z_0}))}{\lambda^* q}\right) \tag{14}$$

which, however, is not a crazy assumption for standard functions v and h.

Now we are left to determine the consumption profile fitting x_{z0} and z^* given above. Write down the consumer's budget constraint one by one:

$$\begin{aligned}
t = 0, &\quad c^* + k^* + qx_{z0} + b_1 = Af(k_0) + (1-\delta_k)k_0 \\
t = 1, &\quad c^* + \delta_k k^* + q\delta_z z^* + b_2 = Af(k^*) + R^* b_1 \\
t = 2, &\quad c^* + \delta_k k^* + q\delta_z z^* + b_3 = Af(k^*) + R^* b_2 \\
\cdots &\quad \cdots \qquad \cdots
\end{aligned}$$

($b_0 = 0$) Fist notice that $b_{t+1} = b_1$ can be proved by using the TVC for the bond. (Check the argument in the solution to part 2-2 of Homework #2.)

Now we need to fit time 0 and time 1 budget constraint. By multiplying the first equation by $(R^* - 1)$ and adding it to the second equation, we have

$$\begin{aligned} R^* c^* + & (R^* - 1 + \delta_k)k^* + q(R^* - 1)x_{z0} + q\delta_z z_0 h(\frac{x_{z0}}{z_0}) \\ & = (R^* - 1)Af(k_0) + (R^* - 1)(1 - \delta_k)k_0 + Af(k^*) \end{aligned} \tag{15}$$

In conclusion, if there is a solution in (14) and (15) is satisfied, then this economy converges to the steady state in one period.

2-5 and 2-6. Iso-elastic Utility Functions

As seen in part $2 - 4$, the economy reaches the steady state in one period. Since the initial capital starts with the steady state level, (15) is collapsed to

$$c^* = (Af(k^*) - \delta_k k^*) - q\left((1 - \beta)x_{z0} + \beta\delta_z z^*\right)$$

with $z^* = z_0 h(\frac{x_{z0}}{z_0})$ or $x_{z0} = z_0 g(\frac{z^*}{z_0})$ where g is the inverse of h. Note that g is strictly increasing. Hence, we have

$$c^* = (Af(k^*) - \delta_k k^*) - q\left((1 - \beta)z_0 g(\frac{z^*}{z_0}) + \beta\delta_z z^*\right) \tag{16}$$

Then, the steady state after period 1 is already pinned down inpart $2 - 2$. We are left to derive the equation to determine $z_{t+1} = z^*$, which is given by

$$\left[h'(g(\frac{z^*}{z_0}))\right]^{-1} = \beta\left[h'(\delta_z)\right]^{-1}\left(h(\delta_z) - \delta_z h'(\delta_z) + \frac{v'(z^*)}{u'(c^*)q}\right).$$

Note that $\frac{df(k^*)}{dA} > 0$ and $\frac{dz^*}{dA} > 0$ are the same as part $2 - 2$, and the latter can be checked by taking a derivative with respect to A on the above equation.