Econ 712 Problem set #1 - solution prepared by Kyoung Jin Choi

Problem 1

1-1

From the given law of motion, it is easy to get $y_T = y_0 + x_1 + x_2 + \cdots + x_{T-1}$. Then, the problem faced by an entrepreneur is

$$\min \sum_{t=0}^{T-1} \frac{x_t^2}{2} \tag{1}$$

subject to

$$y_T = y_0 + \sum_{t=0}^{T-1} x_t \geq y^* \qquad \text{and} \qquad x_t \geq 0, \forall t = 0, 1, \ldots, T-1 \tag{2}$$

1-2

It is equivalent to solve the following maximization problem.

$$\max - \sum_{t=0}^{T-1} \frac{x_t^2}{2} \tag{3}$$

subject to (2). Now define a Lagrangean

$$\mathcal{L} = -\sum_{t=0}^{T-1} \frac{x_t^2}{2} + \lambda \left(y_0 + \sum_{0}^{T-1} x_t - y^* \right) + \sum_{t=0}^{T-1} \mu_t x_t.$$

The FOCs are given by

$$\begin{aligned}
(i) \quad & -x_t + \lambda + \mu_t = 0, \quad t = 0, 1, \ldots, T-1 \\
(ii) \quad & \lambda \geq 0, \quad y_0 + \sum_{0}^{T-1} x_t - y^* \geq 0, \\
& \mu_t \geq 0, \quad x_t \geq 0, \quad t = 0, 1, \ldots, T-1 \\
(iii) \quad & \lambda (y_0 + \sum_{0}^{T-1} x_t - y^*) = 0, \\
& \mu_t x_t = 0, \quad t = 0, 1, \ldots, T-1
\end{aligned}$$

Since this is the first time, I will show all the cases in detail.

Case 1: $\lambda = 0 \implies x_t = \mu_t, \forall t$ from (i), which implies $x_t = \mu_t = 0$ by (iii). This yields $y_0 - y^* \geq 0$ from (ii), which is a contradiction.

Case 2: $\mu_t > 0 \implies x_t = 0$ from (iii). Then, $\lambda + \mu_t = 0$ by (i), which is contradiction.

Thus, we have $\lambda > 0$ and $\mu_t = 0$ for all $t = 0, 1, \ldots, T-1$. Hence (i) implies $x_t = \lambda$ for all t. By (ii) we have $y_0 + \lambda T - y^* = 0$. Thus, the solution is

$$x_t = \frac{y^* - y_0}{T}, \quad t = 0, \ldots, T-1,$$

which is of course unique.

1-3

Now first we note that we can inductively derive the following constraint from the given law of motion.

$$y_T = (1-\delta)^T y_0 + \sum_{t=0}^{T-1} (1-\delta)^{T-1-t} x_t - y^* \geq 0. \tag{4}$$

Hence now we are left solve

$$\max - \sum_{t=0}^{T-1} \beta^t \frac{x_t^2}{2}$$

subject to (4). Similarly to 2, we have the following FOCs.

$$(i) \quad -\beta^t x_t + \lambda(1-\delta)^{T-1-t} + \mu_t = 0, \quad t = 0, 1, \ldots, T-1$$

$$(ii) \quad \lambda \geq 0, \quad (1-\delta)^T y_0 + \sum_{t=0}^{T-1} (1-\delta)^{T-1-t} x_t - y^* \geq 0,$$

$$\mu_t \geq 0, \quad x_t \geq 0, \quad t = 0, 1, \ldots, T-1$$

$$(iii) \quad \lambda \left[(1-\delta)^T y_0 + \sum_{t=0}^{T-1} (1-\delta)^{T-1-t} x_t - y^* \right] = 0,$$

$$\mu_t x_t = 0, \quad t = 0, 1, \ldots, T-1$$

By using the similar argument in 2, we have $\mu_t = 0$ and $\lambda > 0$. Then, (i) gives

$$x_t = \lambda \beta^{-t}(1-\delta)^{T-1-t}. \tag{5}$$

Putting (5) into (ii) we get

$$\lambda = \frac{(1-\beta(1-\delta)^2)(y^* - (1-\delta)^T y_0)}{1-(\beta(1-\delta)^2)^T} \beta^{T-1}.$$

Therefore finally we have

$$x_t = \frac{(1-\beta(1-\delta)^2)(y^* - (1-\delta)^T y_0)}{1-(\beta(1-\delta)^2)^T} (\beta(1-\delta))^{T-1-t}.$$

Problem2

2-1

Yes. Note that the object function and all the constraint functions are continuous and concave(linear). The slater condition can be easily checked since $k_0 > 0$. For example, take $c_0 = k_1 = \frac{Ak_0}{3}$, $c_1 = k_2 = \frac{A^2 k_0}{3^2}$, $c_2 = k_3 = \frac{A^3 k_0}{3^3}$, etc, and we have $c_t + k_{t+1} < Ak_t$ for all $t \geq 0$.

2-2

Define a Lagrangean

$$\mathcal{L} = \sum_{t=0}^{T} \beta^t \left[c_t + \lambda_t(Ak_t - c_t - k_{t+1}) + \delta_t c_t + \gamma_t k_{t+1}\right]$$

FOC are given by

$$c_t: \quad 1 - \lambda_t + \delta_t = 0, \quad t = 0, \dots, T \tag{6}$$

$$k_{t+1}: \quad -\lambda_t + \lambda_{t+1}\beta A + \gamma_t = 0, \quad t = 0, \dots, T-1 \tag{7}$$

$$k_{T+1}: \quad -\lambda_T + \gamma_T = 0 \tag{8}$$

and for $t = 0, \ldots, T$

$$\lambda_t \geq 0, \quad \delta_t \geq 0, \quad \gamma_t \geq 0, \quad c_t \geq 0, \quad k_{t+1} \geq 0, \quad Ak_t - c_t - k_{t+1} \geq 0 \quad (9)$$

$$\lambda_t(Ak_t - c_t - k_{t+1}) = 0, \quad \delta_t c_t = 0, \quad \gamma_t k_{t+1} = 0 \quad (10)$$

We first show that that $k_{T+1} = 0$. This is intuitively clear since there is no reason to invest at period T in order to maximize utility. But, we can verify this algebraically. Suppose $k_{T+1} > 0$. Then, $\gamma_T = 0$ from (10), which implies $\lambda_T = 0$ by (8). This contradicts (6). Therefore, $k_{T+1} = 0$.

Now we will divide the problem into 3 cases. Note that the utility function is linear and the feasibility constraint implies that if you invest k_t (or don't consume it) today, then you will increase it by A, i.e., you will get Ak_t tomorrow, thus you have a chance to increase your future consumption. But, the tomorrow's consumption is discounted by β. So, we have to care about the tradeoff. Keeping this in your mind and following the proofs, you will be able to understand how we can characterize such cases.

Case1: $\beta A > 1 \Longrightarrow c_0 = c_1 = \cdots = c_{T-1} = 0, c_T = A^{T+1}k_0$

Case2: $\beta A < 1 \Longrightarrow c_0 = Ak_0, c_1 = c_2 = \cdots = c_T = 0$

Case3: $\beta A = 1 \Longrightarrow$ Any pair of consumption stream satisfying $c_t + k_{t+1} = Ak_t$ with $k_{T+1} = 0$ is a solution, i.e., there are infinitely many solutions.

Proof of Case1: Let $t < T$. Suppose $c_t > 0$. Then, $\delta_t = 0$ by (10) so $\lambda_t = 1$ by (6). On the other hand, $1 - \lambda_{t+1} + \delta_{t+1} = 0$ implies $\lambda_{t+1} \geq 1$. Then, we have $-\lambda_t + \lambda_{t+1}\beta A + \gamma_t > 0$, which is contradictory to (7). Thus, we have $c_t = 0$ for $t < T$. Since we already know $k_{T+1} = 0$, $c_T = k_T = A^{T+1}k_0$ at this case.

Proof of Case2: Let $t = T$. Suppose $c_t > 0$. Then, $\delta_t = 0$, so $\lambda_t = 1$. Since $1 - \lambda_{t-1} + \delta_{t-1} = 0$, we have $\lambda_{t-1} \geq 1$. But, $-\lambda_{t-1} + \lambda_t\beta A + \gamma_t = 0$ and $\beta A < 1$ imply that $\gamma_t > 0$. This contradicts $k_{T+1} = 0$. Hence we have $c_T = 0$. From this, we can derive $k_T = 0$ by the same argument when we derive $k_{T+1} = 0$. Now applying the above procedure inductively to $t = T-1, T-2, \ldots$, we get $c_t = 0$ for all $t = 1, 2, \ldots, T$.

Proof of Case3: Any combination of consumption stream do not violate the FOCs as long as $c_t + k_{t+1} = Ak_t$ with $k_{T+1} = 0$. Expanding

$$\begin{aligned}
c_0 + k_1 &= Ak_0 & & \\
c_1 + k_2 &= Ak_1 & & \times \beta \\
\cdots \quad & \cdots & & \cdots \\
\cdots \quad & \cdots & & \cdots \\
c_{T-1} + k_T &= Ak_{T-1} & & \times \beta^{T-1} \\
c_T + k_{T+1} &= Ak_T & & \times \beta^T
\end{aligned}$$

and summing them up on each side using $\beta A = 1$, we can get

$$c_0 + \beta c_1 + \beta^2 c_2 + \cdots + \beta^T c_T = Ak_0.$$

Here Ak_0 is the current value of discounted consumption stream $(c_0, c_1, \ldots, c_T)$.

Problem 3

3-1

Note that the objective function $f(x, y) = x^2 + y^2$ is not concave. Thus, Kuhn-Tucker Theorem cannot be applied.[1]

First see that the maximum attains where the constraint binds, i.e., if $f(x^*, y^*)$ is the maximum, then $x^* + y^* = K$. Suppose not, i.e, there exists (x_0, y_0) where $f(x_0, y_0)$ is the max and $x_0 + y_0 < K$. Then, we can choose $\epsilon > 0$ such that $x_0 + y_0 + \epsilon < K$ and $f(x_0, y_0 + \epsilon) > f(x_0, y_0)$, which is a contradiction.

Hence we can rewrite the problem as following.

$$\max_{0 \leq x \leq K} x^2 + (K - x)^2$$

Then, it is easy to check that the FOC and SOC gives that $(x, y) = (0, K)$ and $(K, 0)$ are the maximizers.

[1] There is a version of Kuhn-Tucker with the objective functions that are generally not convex nor concave.

3-2-(a)

Let us define the full set of Lagrangean

$$\mathcal{L}_F(x, y, \lambda, \mu, \gamma) := \sin x + \sqrt{y} + \lambda(1 - mx - y) + \mu x + \gamma y.$$

In many cases it is not easy to get a saddle point without aids of FOCs. This is the case because $\mathcal{L}_F$ is not differentiable especially where $y = 0$. So, we approaches this problem from which the differentiation procedure works, i.e., we are looking at the interior first.

We will show that there is no interior solution. Suppose there is an interior solution (x^*, y^*), i.e., $x^* > 0$ and $y^* > 0$. Then, we have the following FOCs.

$$\cos(x) - \lambda m = 0 \tag{11}$$

$$\frac{1}{\sqrt{y}} - \lambda = 0 \tag{12}$$

$$1 - mx - y = 0 \tag{13}$$

From (11) and (12), we have

$$x^* = \cos^{-1}(\lambda m), \quad y^* = \frac{1}{m}\left(1 - \frac{1}{4\lambda^2}\right). \tag{14}$$

Once we can determine λ, we can get the saddle point. From (11) it must be satisfied that $0 < \lambda < \frac{1}{m}$. Put (14) into (13) and we have

$$1 - m\cos^{-1}(\lambda m) - \frac{1}{4\lambda^2} = 0, \quad 0 < \lambda < \frac{1}{m} \tag{15}$$

Unfortunately, however, there is no solution to (15), which is shown as following. Let

$$F(\lambda) := 1 - m\cos^{-1}(\lambda m) - \frac{1}{4\lambda^2}$$

It is easy to see that F is continuously differentiable in $(0, 1/m)$ and strictly increasing. Then, we have

$$F(0) = -\infty, \quad F\left(\frac{1}{m}\right) = 1 - \frac{m^2}{4} < 0$$

since $m > 2$. Note that what we have shown is that there is no interior solution. If it had an interior solution, then it should have satisfied FOCs (11), (12), and (13) by the second part of Kuhn-Tucker. (It is easy to see that the Slater condition holds.)

So, we are left to check the boundary where $x = 0$ or $y = 0$. We know that there should be an optimum, so the maximum attain at the boundary since there is no interior solution.[2] When $x = 0$, the candidate maximum is 1 at $y = 1$. When $y = 0$, the candidate maximum is $1/m$ at $x = 1/m$. Therefore, the solution is $(x^0, y^0) = (0, 1)$, which is unique. In other words, $(0, 1, \lambda^0, \mu^0, \gamma^0)$ is a saddle point of $\mathcal{L}_F$ for some λ^0, μ^0, and γ^0 by the second part of the Kuhn-Tucker theorem.

3-2-(a)

Now, if $m < 1$, the situation changes a lot. There are two issues here. (i) If m is sufficiently small, then the given objective function is no more concave. Thus, we cannot apply the Kuhn-Tucker theorem any more. Then, the computation will be much more complicated. (ii) Take a reasonable open interval such that $m \in (a, 1)$ for some $a > 0$ so that we can keep the objective function concave in the given constraint.

Here we only look at Case (ii). Note that $m < 1$ implies that

$$F(0) = -\infty, \quad F\left(\frac{1}{m}\right) = 1 - \frac{m^2}{4} > 0.$$

Hence, there is an unique solution (say $\lambda^* > 0$) to (15) by the intermediate value theorem, which implies that $(x^*, y^*, \lambda^*, \mu^*, \gamma^*)$ is a saddle point of $\mathcal{L}_F$, where x^* and y^* are given by (14) and $\mu^* = 0$ and $\gamma^* = 0$, which will turn out to be unique.

Now we are left to check that $(x, y) = (0, 1)$ is no more a saddle point if $m < 1$. Suppose it is a saddle point. Then, $(0, 1, \hat{\lambda}, \hat{\mu}, \hat{\gamma})$ maximizes $\mathcal{L}_F((x, y, \hat{\lambda}, \hat{\mu}, \hat{\gamma})$ for some $\hat{\lambda}, \hat{\mu}$ and $\hat{\gamma}$. Let us consider a small perturbation $(\frac{\epsilon}{m}, 1 - \epsilon)$ for some $\epsilon > 0$. Then, we can easily see that

$$\mathcal{L}_F((x, y, \hat{\lambda}, \hat{\mu}, \hat{\gamma}) < \mathcal{L}_F(\frac{\epsilon}{m}, 1 - \epsilon, \hat{\lambda}, \hat{\mu}, \hat{\gamma})$$

[2] Note that the given objective function is continuous and the support is compact.

since the left hand side is 1 and the right hand side is

$$\mathcal{L}_F(\frac{\epsilon}{m}, 1-\epsilon, \hat{\lambda}, \hat{\mu}, \hat{\gamma}) = \frac{\epsilon}{m} + \sqrt{1-\epsilon} > \epsilon + (1-\epsilon) = 1,$$

which is a contradiction.

In conclusion, if $m < 1$ and m is not so small such that the objective function is still concave in the domain of the constraint, then there is an unique maximizer given by (14) and (15).