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Exposita Notes

A simple proof of Ekeland and Scheinkman's result on the necessity of a transversality condition*

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Summary. Ekeland and Scheinkman (1986) prove the necessity of a standard transversality condition under certain technical conditions. Their result is one of the most powerful on the necessity of a transversality condition currently available in the literature, and their proof involves numerous estimations and relies on Ekeland's variational principle and Fatou's lemma. This note relaxes some of their assumptions and provides a simple proof that uses neither Ekeland's principle nor a convergence result like Fatou's lemma.

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1 Introduction

Under certain technical conditions, Ekeland and Scheinkman (1986) (ES hereafter) prove the necessity of a transversality condition (TVC) for infinite-horizon problems with unbounded return functions. Their result is one of the most powerful on the necessity of a TVC currently available in the literature and, not surprisingly, their proof involves numerous estimations and relies heavily on Ekeland's variational principle (ES, Theorem 3.1) and Fatou's lemma. This note relaxes some of their assumptions and provides a simple proof that uses neither Ekeland's variational principle nor a convergence result like Fatou's lemma.

The next section presents ES's model and TVC result. Section 3 summarizes ES's proof, and Sect. 4 presents our proof. Section 5 concludes the note.

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2 The model and the Ekeland-Scheinkman TVC result

Consider the following maximization problem:

$$\max_{\{c_t, x_{t+1}\}_{t=0}^{\infty}} \quad \sum_{t=0}^{\infty} u_t(c_t) \tag{1}$$

$$\text{s.t. } (\forall t \in \mathbb{Z}_+) \qquad (x_{t+1} + c_t, x_t) \in A_t \subset (\mathbb{R}^N)^2, \quad (x_0 \in \mathbb{R}^N \text{ given}) \tag{2}$$

where $N \in \mathbb{N}$ and $(\forall t \in \mathbb{Z}_+)$ $u_t : \mathbb{R}^N \to \mathbb{R} \cup \{-\infty\}$.[1] Though ES use "$\delta^t u_t$" with $\delta \in (0,1)$ instead of "u_t," their maximization problem is equivalent to ours since they allow u_t to depend on t, virtually eliminating the effect of discounting.

A path $\{c_t, x_{t+1}\}$ is called *feasible* if it satisfies (2).[2] We take an "optimal" path $\{c_t^*, x_{t+1}^*\}$ as given and use the following assumptions. (We indicate where ES make the corresponding assumptions in brackets.)

(S1) [ES, Proposition 5.1] (i) $(\forall t \in \mathbb{Z}_+)$ $u_t(c_t^*) > -\infty$ and (ii) for any feasible path $\{c_t, x_{t+1}\}$,

$$\liminf_{T \uparrow \infty} \sum_{t=0}^{T} (u_t(c_t) - u_t(c_t^*)) \leq 0.^3$$

(S2) [ES, Assumption (b), Proposition 4.1] $\{u_t(c_t^*)\}$ is summable.[4]

(S3) [ES, Proposition 5.1(iii)] $(\forall t \in \mathbb{Z}_+)$ $(\forall \lambda \in [0,1])$ $(\lambda(x_{t+1}^* + c_t^*), \lambda x_t^*) \in A_t$.

(U1) [ES, Assumption (a), Proposition 5.1(i)] $(\forall t \in \mathbb{Z}_+)$ (i) u_t is upper semicontinuous, (ii) the set $F_t \equiv \{c \in \mathbb{R}^N \mid u_t(c) > -\infty\}$ is open, and (iii) u_t is C^1 on F_t.

(U2) [ES, Proposition 5.1(ii)] There exist a constant $\theta \in \mathbb{R}$ and a summable sequence $\{b_t\}$ such that

$$(\forall t \in \mathbb{Z}_+)\ (\forall c \in F_t) \qquad u_t'(c)c \leq \theta u_t(c) + b_t. \tag{3}$$

None of the assumptions here is stronger than ES's counterpart. Furthermore we do not need some of ES's assumptions; more specifically, we do not use Proposition 5.1(iv) and Assumptions (a)–(c) in ES except for the upper semicontinuity of u_t and the implication of Assumption (b) assumed in (S2). ES use Assumptions (a)–(b) to show (8) below, which our proof does not need. While Assumptions (a)–(c) also ensure the existence of the infinite sum in (1) for all feasible paths, (S1) allows the possibility that the sum may not always exist and only requires that $\{c_t^*, x_{t+1}^*\}$ be weakly optimal. (S2) is weaker than ES's counterpart, which assumes that $\sum_{t=0}^{\infty} u_t(c_t^*) > -\infty$ and that there exists a summable sequence $\{f_t\}$ such that for any feasible path, $u_t(c_t) \leq f_t$ for all t. Unlike ES's counterpart, (S2) is innocuous by itself since replacing $u_t(c_t)$ in (1) by $[u_t(c_t) - u_t(c_t^*)]$ results in an equivalent problem that trivially satisfies (S2). ES's counterpart to (S3) requires A_t to be star-shaped with respect to $(0,0)$ and

[1] $\mathbb{R} = (-\infty, \infty)$, $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, and $\mathbb{N} = \{1, 2, 3, \ldots\}$.

[2] We use the convention that $\{y_t\} = \{y_t\}_{t=0}^{\infty}$ for any sequence y_t.

[3] "$\uparrow$" means "increases to" and "$\downarrow$" means "decreases to."

[4] A sequence $\{a_t\}$ is summable iff $\sum_{t=0}^{\infty} |a_t| < \infty$

thus immediately implies (S3). (U1) and (U2) are equivalent to ES's counterparts. The key assumptions in this note as well as in ES are (S2) and (U2).

Theorem 2.1. *Let $\{c_t^*, x_{t+1}^*\}$ be a feasible path satisfying (S1)–(S3), (U1), and (U2). Then*

$$\limsup_{T\uparrow\infty} u_T'(c_T^*)x_{T+1}^* \leq 0. \tag{4}$$

3 Ekeland and Scheinkman's argument

Before presenting our proof, we briefly review ES's proof, which uses numerous estimations but seems to be a rather natural argument. We note however that our proof does not use any result or argument in this section. To facilitate our discussion, we first make some preparations.

Let $\{c_t\}$ be such that $\{c_t^* + \alpha(c_t - c_t^*)\}$ is feasible for all $\alpha \in [0, 1]$. For $\alpha \in [0, 1]$, define

$$G(\alpha) = \sum_{t=0}^{\infty} u_t(c_t^* + \alpha(c_t - c_t^*)). \tag{5}$$

Then if G is differentiable at $\alpha = 0$ and if we may differentiate under the summation sign, we must have

$$\sum_{t=0}^{\infty} u_t'(c_t^*)(c_t - c_t^*) \leq 0. \tag{6}$$

If this inequality holds for certain feasible paths, then the TVC (4) can easily be derived (ES, Proposition 5.1). The difficult part is to show (6).

Let P_∞ be the maximization problem (1)–(2) and P_T be the maximization problem (1)–(2) with $\sum_{t=0}^{T}$ replacing $\sum_{t=0}^{\infty}$. Let $\overline{c}^T \equiv \{\overline{c}^T\}$ be a solution of P_T. Then, by an argument similar to that used for (6), optimality requires

$$\sum_{t=0}^{T} u_t'(\overline{c}_t^T)(c_t - \overline{c}_t^T) \leq 0. \tag{7}$$

The underlying idea of ES's argument is to take the limit of (7) as $T \uparrow \infty$ and derive (6). Unfortunately that does not work, because though $\{\overline{c}^T\}_{T=0}^{\infty}$ has a subsequence converging to some solution of P_∞, it need not have any subsequence converging to the given solution $c^* \equiv \{c_t^*\}$. To overcome this difficulty, ES (Proposition 4.1) show that

$$\lim_{T\uparrow\infty} \sum_{t=0}^{T} u_t(\overline{c}_t^T) = \sum_{t=0}^{\infty} u_t(c_t^*), \tag{8}$$

i.e., c^* is an approximate solution of P_T. But there is no guarantee that c^* approximately satisfies (7) with c^* replacing $\overline{c}^T$. ES thus use Ekeland's variational principle (ES, Theorem 3.1) to establish the existence of an approximate solution $\tilde{c}^T \equiv \{\tilde{c}_t^T\}$ of P_T such that $\tilde{c}^T$ approximately satisfies (7) with $\tilde{c}^T$ replacing $\overline{c}^T$ and such that $\tilde{c}^T \to c^*$ pointwise as $T \uparrow \infty$. ES then use (U2) and Fatou's lemma, among other things, to establish (6) by letting $T \uparrow \infty$.

4 Proof of Theorem 2.1

Our proof is based on two elementary lemmas. The proof of the first lemma is similar to that of ES's Proposition 5.1. Unlike their proposition, however, the lemma is derived directly from the optimality requirement (S1) rather than an implication of optimality such as (6). The proof of the second lemma effectively replaces the difficult part of ES's proof discussed in the previous section, though the lemma is a simple implication of (U1) and (U2).

Lemma 4.1. *Assume (S1), (S3), and (U1). If there exist a constant* $\underline{\lambda} \in (0,1)$ *and a summable sequence* $\{e_t\}$ *such that*

$$(\forall \lambda \in [\underline{\lambda}, 1))\ (\forall t \in \mathbb{Z}_+) \qquad \frac{u_t(c_t^*) - u_t(\lambda c_t^*)}{1-\lambda} \le e_t, \tag{9}$$

then the TVC (4) holds.

Proof. Assume (S1), (S3), and (U1); suppose there exist $\underline{\lambda} \in (0,1)$ and a summable sequence $\{e_t\}$ satisfying (9). Fix $T \in \mathbb{Z}_+$ and $\lambda \in [\underline{\lambda}, 1)$ for the moment. Define $\{c_t, x_{t+1}\}$ by

$$\begin{array}{llllll} c_t & = c_t^*, & x_{t+1} & = x_{t+1}^* & \text{if } t < T, \\ c_T & = c_T^* + (1-\lambda)x_{T+1}^*, & x_{T+1} & = \lambda x_{T+1}^*, & \\ c_t & = \lambda c_t^*, & x_{t+1} & = \lambda x_{t+1}^* & \text{if } t > T. \end{array} \tag{10}$$

By (S3), $\{c_t, x_{t+1}\}$ is feasible. By (S1) and (10),

$$u_T(c_T^* + (1-\lambda)x_{T+1}^*) - u_T(c_T^*) + \liminf_{s\uparrow\infty} \sum_{t=T+1}^{s} [u_t(\lambda c_t^*) - u_t(c_t^*)] \le 0.$$

Therefore,

$$\frac{u_T(c_T^* + (1-\lambda)x_{T+1}^*) - u_T(c_T^*)}{1-\lambda} \le \limsup_{s\uparrow\infty} \sum_{t=T+1}^{s} \frac{u_t(c_t^*) - u_t(\lambda c_t^*)}{1-\lambda} \le \sum_{t=T+1}^{\infty} e_t, \tag{11}$$

where the last inequality holds by (9). By (S1)(i) and (U1), u_T is differentiable at c_T^*; thus applying $\lim_{\lambda\downarrow 0}$ to (11) yields $u_T'(c_T^*)x_{T+1}^* \le \sum_{t=T+1}^{\infty} e_t$. Further applying $\limsup_{T\uparrow\infty}$ yields the TVC (4). □

Lemma 4.2. *Under (U1) and (U2),*

$$\begin{aligned} &(\forall t \in \mathbb{Z}_+)\ (\forall c \in F_t)\ (\forall \lambda \in (0,1)) \\ &u_t(\lambda c) \ge \lambda^\theta \left[u_t(c) - b_t \int_\lambda^1 z^{-\theta-1} dz \right], \end{aligned} \tag{12}$$

where θ *and* b_t *are given by (U2).*

Proof. Assume (U1) and (U2). Let $t \in \mathbb{Z}_+$ and $c \in F_t$. For $z \in (0,1]$, define $v(z) = u_t(zc)$. Let

$$\tilde{\lambda} = \inf\{y \in [0,1] \,|\, (\forall z \in (y,1])\ v(z) > -\infty\}. \tag{13}$$

Since $v(1) > -\infty$ and F_t is open by (U1), $\tilde{\lambda} < 1$. We claim

$$\tilde{\lambda} > 0 \quad \Longrightarrow \quad v(\tilde{\lambda}) = u_t(\tilde{\lambda}c) = -\infty. \tag{14}$$

To see this, suppose $\tilde{\lambda} > 0$ and $v(\tilde{\lambda}) > -\infty$. Then by the openness of F_t, there exits $y \in [0, \tilde{\lambda})$ such that $(\forall z \in (y,1])\ v(y) > -\infty$, which contradicts (13). Thus (14) must hold.

By the definition of v and (3),

$$(\forall z \in (\tilde{\lambda}, 1]) \qquad z v'(z) = u_t'(zc)zc \le \theta u_t(zc) + b_t = \theta v(z) + b_t.$$

Multiplying the leftmost and rightmost sides by $z^{-\theta-1}$ yields

$$(\forall z \in (\tilde{\lambda}, 1]) \qquad z^{-\theta} v'(z) - \theta z^{-\theta-1} v(z) \le z^{-\theta-1} b_t.$$

Integrating with respect to z from λ to 1 yields

$$(\forall \lambda \in (\tilde{\lambda}, 1)) \qquad z^{-\theta} v(z)|_\lambda^1 = v(1) - \lambda^{-\theta} v(\lambda) \le b_t \int_\lambda^1 z^{-\theta-1} dz.$$

Multiplying through by λ^θ and rearranging, we get

$$(\forall \lambda \in (\tilde{\lambda}, 1)) \qquad v(\lambda) \ge \lambda^\theta \left[v(1) - b_t \int_\lambda^1 z^{-\theta-1} dz \right]. \tag{15}$$

Since this inequality is equivalent to that in (12), we only need to show $\tilde{\lambda} = 0$ to complete the proof of the lemma. Suppose by way of contradiction that $\tilde{\lambda} > 0$. By the upper semicontinuity of u_t and (15), $v(\tilde{\lambda}) \ge \limsup_{\lambda \downarrow \tilde{\lambda}} v(\lambda) \ge \tilde{\lambda}^\theta [v(1) - b_t \int_{\tilde{\lambda}}^1 z^{-\theta-1} dz] > -\infty$, which contradicts (14). This completes the proof of Lemma 4.2. □

To prove Theorem 2.1, assume (S1)–(S3), (U1), and (U2). Let $\underline{\lambda} \in (0,1)$, $\lambda \in [\underline{\lambda}, 1)$, $\mu = \max_{z \in [\underline{\lambda},1]} \theta z^{\theta-1}$, $\nu = \max_{z \in [\underline{\lambda},1]} \lambda^\theta z^{-\theta-1}$, and $t \in \mathbb{Z}_+$. By (S1) and (U1), $c_t^* \in F_t$. Thus by Lemma 4.2

$$\begin{aligned} u_t(c_t^*) - u_t(\lambda c_t^*) &\le (1-\lambda^\theta) u_t(c_t^*) + \lambda^\theta b_t \int_\lambda^1 z^{-\theta-1} dz \\ &= u_t(c_t^*) \int_\lambda^1 \theta z^{\theta-1} dz + b_t \int_\lambda^1 \lambda^\theta z^{-\theta-1} dz \\ &\le |u_t(c_t^*)| \mu (1-\lambda) + |b_t| \nu (1-\lambda). \end{aligned} \tag{16}$$

Let $e_t = |u_t(c_t^*)|\mu + |b_t|\nu$. Then the sequence $\{e_t\}$ is summable by (S2) and (U2) and satisfies (9) by (16). Thus the TVC (4) holds by Lemma 4.1. This completes the proof of Theorem 2.1.

5 Concluding comments

In this note, we simplified the proof of ES's TVC result while relaxing some of their assumptions. One of the assumptions we used that may not be particularly easy to verify in applications would be (S2), i.e., the finiteness of the objective function at the optimal path. This finiteness condition is crucial to ES's approach since their proof uses the convergence result (8), which relies on the finiteness condition. In view of Lemma 4.1, however, one can see that the finiteness condition is not crucial in establishing the TVC (4). In fact, any summable sequence $\{e_t\}$ satisfying (9) would allow one to prove the TVC without the finiteness condition. In the context of asset pricing models, Kamihigashi (1998a) offers conditions on marginal utility that guarantee the TVC without the finiteness condition.

Lemma 4.1, perhaps the most useful result in this note, can easily be adapted to the reduced-form model or even stochastic models. The lemma can be adapted to the reduced-form model essentially by replacing "$u_t(c_t^*) - u_t(\lambda c_t^*)$" in (9) with "$v_t(x_t^*, x_{t+1}^*) - v_t(\lambda x_t^*, \lambda x_{t+1}^*)$", where v_t is the period utility function of the reduced-form model. It can be adapted to stochastic models by replacing "u_t" with "Eu_t" throughout the statement and proof of the lemma, where E is the expectation operator. A general version of the lemma is used in Kamihigashi (1998b) to obtain results on the TVC for deterministic, continuous-time models.

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