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Econ 712: Macroeconomic Theory
Midterm Exam (10/21/00)

- Please answer all questions. If you get stuck in one section move to the next one. Do not waste a lot of time on questions that you find hard to solve.
- Partial credit will be awarded if it is clear that you were approaching the problem in an essentially correct manner.
- This is a closed book exam. Students may bring two pages of notes.
- This is a two and a half hour exam. Please hand it in at 11:30 AM.
- Each question is worth 50 points. The point total for each section is indicated at the beginning of the section. Look at these "prices" when deciding how to allocate your time!!
- Please use **one blue book** for each question and write only on the "right" page. (The odd numbered page in a newspaper.)
- Please remember to put your name in each blue book.
- Good luck !

Problem # 1 **Interest Rates and Output** (50 points)

Consider an economy populated by a large number of identical households that maximize the utility function,

$$\sum_{t=0}^{\infty} \beta^t c_t^{1-\sigma}/(1-\sigma) ,$$

where $0<\beta<1$, $\sigma > 0$, and c_t is consumption per household. Each household is endowed with one unit of labor which is supplied inelastically. Output in this economy is produced using capital, k, and labor, n, with a concave, differentiable, and homogeneous of degree one function F(k,n). Assume that firms and households behave competitively.

This economy has access to an international bond market with gross interest rates given by $1 + r_t^*$. Assume that

$$1 + r_t^* = \begin{cases} (1+\gamma)/\beta & \text{for } t = 0, 1, \ldots, T-1 \text{ and } \gamma > 0 \\ 1/\beta & \text{for } t = T, T+1, \ldots \end{cases}$$

i) (15) Describe the steady state of this economy.

ii) (15) Describe the dynamic path of consumption, domestic interest rates and wage rates. Go as far as you can describing the time path of trade deficits in this economy. On the basis of data from this economy what would you conclude is the connection between changes in the trade deficit, world interest rates and changes in domestic consumption?

iii) (15) What is the relationship, if any, between:
   a) world interest rates and domestic output?
   b) world interest rates and wages?
   c) world interest rates and consumption growth?
   d) consumption growth and domestic output?

iv) (5) What is the effect of T --the time at which the world interest rate will decrease-- on the level of initial consumption? Does T influence the <u>qualitative</u> behavior of the trade deficit?

Problem # 1 **Interest Rates and Output** (Notes on the Solution)

i) The problem faced by an "open economy" planner is,

$$\max \quad \sum_{t=0}^{\infty} \beta^t c_t^{1-\sigma}/(1-\sigma)$$

subject to,

$$c_t + k_{t+1} + b_{t+1} \leq f(k_t) + (1-\delta)k_t + (1+r_t^*)b_t.$$

The standard first order conditions are (where λ_t is the Lagrange multipliers assocayed with the constraint):

c_t: $c_t^{-\sigma} = \lambda_t$
k_{t+1}: $\lambda_t = \beta\lambda_{t+1}\,[1-\delta + f'(k_{t+1})]$
b_{t+1}: $\lambda_t = \beta\lambda_{t+1}\,(1+ r^*_{t+1})$,

and the TVC,

$\lim_{T\to\infty} \beta^T \lambda_T k_{T+1} = 0$, and $\lim_{T\to\infty} \beta^T \lambda_T b_{T+1} = 0$. It follows that,

$$f'(k_{t+1}) - \delta = r^*_{t+1},$$

and that,

$$\lambda_t = \lambda_0\,(1+\gamma)^{-t} \quad \text{for } t = 1,2, ..., T-1,\ \lambda_t = \lambda_0\,(1+\gamma)^{-(T-1)}, \text{ for } t > T-1.$$

It follows that, in the steady state,

$$f'(k^*) - \delta = \rho, \quad \text{where } \beta = (1+\rho)^{-1},$$
$$\text{and } c^* = f(k^*) - \delta k^* + \rho b^*,$$

where $\rho b^* > f(k^*) - \delta k^*$ in order to guarantee positive consumption.

ii) Since $c_t^{-\sigma} = \lambda_t$, and $\lambda_t = \lambda_0\,(1+\gamma)^{-t}$, it follows that consumption increases at the constant rate $(1+\gamma)^{1/\sigma}$ from t = 0 to t = T-1, and it is constant afterwards. Output is given by $f(k_t)$ and it $k_t = k_l$ for t = 0, 1,..T-1, and $k_t = k^*$ for t > T-1. Finally, wages are an increasing function of capital, $w_t = f(k_t) - k_t f'(k_t)$. Thus, wages are “low” during [0, T-1] and “high” afterwards. The trade deficit is given by $f(k_l) - \delta k_l - c_t$ for t = 0, 1, T-2. It is $f(k_l) - \delta k_l - (k^* - k_l) - c_{T-1}$, at T-1, where $c_{T-1} = c^*$; and $f(k^*) - \delta k^* - c^*$ afterwards. Thus the period of low output and high growth rate of consumption coincided with the period of increasing trade deficits.

iii) High world interest rates imply low output, low wages, and high growth rate of consumption (se above). This, in turn, implies that there is a negative correlation between the growth rate of consumption and the level of output.

iv) The present value version of the budget constraint implies that,

$$c_0\,\{[1-((1+\gamma)^{1+1/\sigma}\beta)^T/[1-\beta(1+\gamma)^{1+1/\sigma}] + [\beta(1+\gamma)^{1/\sigma}]^T/[(1+\gamma)(1-\beta)]\} =$$
$$(f'(k_l)-\delta k_l)[1-((1+\gamma)\beta)^T/[1-\beta(1+\gamma)] + (f'(k^*)-\delta k^*)\beta^T/((1-\beta).$$

Some (painful) manipulation shows that as T increases, initial consumption decreases. The qualitative features of the time path of the trade surplus are not affected by T.

Problem # 2 **Land Prices, Slave Prices, Emancipation and Growth** (50 points)

Consider an economy in which output is produced using capital, k_t, land, a_t, and labor, n_t according to the production function,

$y_t \leq Bk_t^{\alpha}a_t^{\theta}n_t^{1-\alpha-\theta}$, $0 < \alpha, \theta < 1$, and $\alpha + \theta < 1$.

The workforce in this economy, n_t, is composed of free workers, z_t, and slaves, s_t. Thus, at any point, $n_t = s_t + z_t$. Assume that landowners rank consumption streams according to the function,

$$\sum_{t=0}^{\infty} \beta^t u(c_t),$$

where $0<\beta<1$, u is differentiable, increasing and strictly concave, and c_t is consumption per landowner family.

Assume that each landowner can buy and sell both land and slaves. Thus, landowners are also slave-owners. If s_t slaves are used in production at time t, net output is

$$Bk_t^{\alpha}a_t^{\theta}n_t^{1-\alpha-\theta} - ms_t,$$

where m is the minimum consumption needed to keep each slave alive. Assume that both new slaves and new land purchased at time t are not productive until time t+1. Let the supply of land per landowner be $\bar{a}$ (you may assume that, in equilibrium, all landowners end up with the same amount of land). Let the total number of free workers per landowner be $\bar{z}$, and let the wages of free workers be denoted by w_t.

To simplify, only landowners face a truly dynamic problem: at any time t, their budget constraint is:

$$c_t + k_{t+1} - (1-\delta)k_t + p^*(s_{t+1} - s_t) + r_t\,(a_{t+1} - a_t) \leq Bk_t^{\alpha}a_t^{\theta}n_t^{1-\alpha-\theta} - ms_t - w_t z_t, \qquad n_t \leq s_t + z_t$$

where r_t is the price of land, and p^* is the price of slaves (determined, in the absence of restrictions, in international markets). Assume that slaves and free workers consume in every period their "income," m and w_t, respectively.

i) (20) Go as far as you can describing the steady state (assume that it exists) of the economy.

ii) (15) Compare this economy with another economy that has an endowment of $\bar{a}'$ units of land, with $\bar{a}' > \bar{a}$. Go as far as you can describing the impact of the higher endowment of land upon:
   a) the steady state stock of slaves per landowner, s^*.
   b) the steady state capital stock per landowner, k^*.
   c) the steady state wage rate.
   d) the steady state level of output per landowner.

iii) (15) Let the economy be at the steady state that you described in i). Assume that at t = 0 there is an <u>unexpected</u> change in the legal status of slaves: all former slaves are declared free workers. However, there are restrictions to migration and all former slaves remain in the country (but, of course, they are now free workers). Thus, the supply of free labor is $\bar{z}$ + the steady state number of slaves from i) Go as far as you can describing the dynamic path of output after emancipation (i.e. after t = 0). Determine, if possible, the impact of emancipation upon land prices and wage rates. Do you have enough information to determine if free workers and or landowners would have favored emancipation?

iv) (10) <u>Extra Credit</u>: Suppose that, at time 0, the economy starts at $k_0 < k^*$, where k^* is the level of capital corresponding to the steady state analyzed in ii). At that time, it is announced that the maximum number of slaves per landowner will be capped at $\hat{s}$ where $\hat{s} < s^*$. Go as far as you can determining the effect of this trade restriction on the steady state values of output, landowner's wealth (land plus capital plus slaves), and wages of free workers.

Problem # 2 **Land Prices, Slave Prices, Emancipation and Growth** (Sketch of the solution)

i) The problem faced by landowners is to

$$\max \quad \sum_{t=0}^{\infty} \beta^t u(c_t),$$

subject to,

$$c_t + k_{t+1} - (1-\delta)k_t + p^*(s_{t+1} - s_t) + r_t (a_{t+1} - a_t) \leq Bk_t^{\alpha} a_t^{\theta} n_t^{1-\alpha-\theta} - ms_t - w_t z_t, \qquad n_t \leq s_t + z_t.$$

To simplify notation let $F(t) = Bk_t^{\alpha} a_t^{\theta} n_t^{1-\alpha-\theta}$, and use subindices to indicate partial derivatives. The first order conditions are (where λ_t is the Lagrange multiplier associated with the constraint)

c_t: $\quad c_t^{-\sigma} = \lambda_t,$
k_{t+1}: $\quad \lambda_t = \beta\lambda_{t+1} [1-\delta + F_k(t+1)],$
s_{t+1}: $\quad p^*\lambda_t = \beta\lambda_{t+1} [p^* + F_n(t+1) - m],$
a_{t+1}: $\quad r_t\lambda t = \beta\lambda_{t+1} [r_{t+1} + F_a(t+1)],$
z_t: $\quad w_t = F_n(t)$

Thus, in a steady state,

$$F_k^* = \delta + \rho$$
$$F_a^* = \rho r^*$$
$$F_n^* = \rho p^* + m.$$

Using the specific functional forms, it is possible to show that,

$$k^* = \bar{a}\ (B\alpha/(\rho+\delta))^{1/\theta}[(1-\alpha-\theta)(\rho+\delta)/\alpha(p^*\rho+m)]^{(1-\alpha-\theta)/\theta}$$
$$r^* = \theta(\rho+\delta)/(\alpha\rho)\ [B\alpha/(\rho+\delta)]^{1/\theta}[(1-\alpha-\theta)(\rho+\delta)/\alpha(p^*\rho+m)]^{(1-\alpha-\theta)/\theta},$$
$$s^* = \bar{a}\ (B\alpha/(\rho+\delta))^{1/\theta}[(1-\alpha-\theta)(\rho+\delta)/\alpha(p^*\rho+m)]^{(1-\alpha)/\theta} - \bar{z}\,.$$

Finally in the steady state consumption of landowners is just $F^* - F_n^*\bar{z} - \delta k^* - ms^*$.

ii) From the description above it follows that capital and total labor, $s^*+\bar{z}$, are linear functions of the stock of land. Thus even though capital increases proportionately with acreage, the number of slaves increases more than proportionately. Thus, the capital-slaves ratio is decreasing as a function of acreage. The steady state wage rate is completely determined by the “capital cost” of slaves which is given by $p^*\rho+m$, that is, the sum of the interest cost on the capital (ρp^*), and the maintenance cost (m). Finally, since all inputs are linear functions of land, so is output.

iii) At the time of emancipation the economy remains at the steady state with no changes in the prices of land or labor. The only difference is that consumption of landowners now drops, immediately, to $F^* - F_n^*(s^*+\bar{z}) - \delta k^*$. This corresponds to a drop of $\rho p^* s^*$, the interest cost (or capital income) associated with slave ownership. It follows that free workers are indifferent between the slave and the free economy, while landowners strictly prefer the slave economy.