Department of Economics **Rody Manuelli**
University of Wisconsin

Econ 712: Macroeconomic Theory
Midterm Exam (10/30/99)

- Please answer all questions. If you get stuck in one section move to the next one. Do not waste a lot of time on questions that you find hard to solve.
- Partial credit will be awarded if it is clear that you were approaching the problem in an essentially correct manner.
- This is a closed book exam. Students may bring two pages of notes.
- This is a two and a half hour exam. Please hand it in at 11:30 AM.
- Each question is worth 50 points. The point total for each section is indicated at the beginning of the section.
- Look at the point total for each section when deciding how to allocate your time!!
- Please use **one blue book** for each question and write only on the “right” page. (The odd numbered page in a newspaper.)
- Please remember to put your name in each blue book.
- Good luck !

Problem # 1 **Imperfect Capital Mobility and Growth** (50 points)

Consider an economy populated by a large number of identical households. The utility function of each household is,

$$\sum_{t=0}^{\infty} \beta^t u(c_t),$$

where $0<\beta<1$, u is differentiable, increasing and strictly concave, and c_t is consumption per family member. All households are endowed with one unit of labor that is supplied inelastically.

Assume that there is a large number of firms that produce output using capital and labor. Each firm has a production function given by F(K,N) which is increasing, differentiable, concave and homogeneous of degree one. Here K is total (not per capita) capital, and N total population (or workforce), which we normalize to one. Let f(k) = F(k,1). In this notation, k is capital per worker.

Assume that this economy has "limited" access to international capital markets. More precisely, there is a distortion (you may think of it as a tax) that increases the cost of borrowing from the rest of the world from R* (the world interest rate) to (1+a)R*, for some a > 0. This means that if an individual wants to borrow in the international market he/she has to pay (1+a)R*, while if he/she lends (or deposits) he/she receives R*. Assume that $R^* = \beta^{-1}$.

To formulate the planner's problem, it is convenient to use different notation for the amount borrowed from abroad at time t (denoted as b_{t+1}), and the amount lent, or deposited, in the rest of the world at time t, denoted d_{t+1}. Each of these quantities must be non-negative. The planner's problem for this economy is,

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to,

$$c_t + x_t + (1+a)R^*b_t + d_{t+1} \leq f(k_t) + b_{t+1} + R^*d_t,$$
$$k_{t+1} \leq (1-\delta)k_t + x_t,$$
$$(c_t, k_{t+1}, x_t) \geq (0,0,0), \text{ and } (b_{t+1}, d_{t+1}) \geq (0,0)$$
$$k_0 > 0,\ d_0 = b_0 = 0.$$

Note that the non-negativity constraints on foreign assets and liabilities are critical: if we allowed d_{t+1} to be negative we are, effectively, letting the country borrow at R*, and this is contrary to the spirit of this model. Assume that $[1-\delta + f'(k_0)] \ggg (1+a)R^*$ or, alternatively, that the initial capital stock is very small.

i) (5 points) Describe the Lagrangean associated with the planner's problem.

ii) (15 points) Show that, at t = 0, there will be an influx of loans from abroad, but that, from then on, the country will no longer borrow or lend abroad. (There will be some loan repayment, of course)

iii) (10 points) Show that the country never lends to the rest of the world. What can you say about the interest rate in this country vis a vis the rest of the world?

iv) (10 points) Describe the time path of consumption, and compare it to the path of consumption of another identical country except that it has a = 0.

v) (10 points) How will output per capita in this economy differ from output per capita in an a = 0 economy in the long run? What about consumption per capita?

Problem # 1 **Imperfect Capital Mobility and Growth** - Solution

i) The Lagrangean for the planner's problem is:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \{u(c_t) + \lambda_t[f(k_t) + (1-\delta)k_t + b_{t+1} + R^*d_t - c_t - k_{t+1} - d_{t+1} - (1+a)R^*b_t] + \gamma_{bt}b_{t+1} + \gamma_{dt}d_{t+1}\},$$

where I have ignored the nonnegativity constraints on c_t, x_t. The first order conditons for this problem are:

(1) c_t: $u'(c_t) = \lambda_t$,

(2) k_{t+1}: $\lambda_t = \beta\lambda_{t+1}[(1-\delta) + f'(k_{t+1})]$,

(3) b_{t+1}: $\lambda_t = \beta\lambda_{t+1}(1+a)R^* - \gamma_{bt}$,

(4) d_{t+1}: $\lambda_t = \beta\lambda_{t+1}R^* + \gamma_{dt}$,

and the feasibility constraints. First, note that (3) and (4) show that it is not possible for both b_{t+1} and d_{t+1} to be positive. If this were the case, $\gamma_{bt} = \gamma_{dt} = 0$, which, in turn, implies (1+a) = 1, a contradiction. Next we want to argue that for t = 1, 2, ...,T-1 (for some T) $b_t > 0$. Under this guess, it follows that,

(5) $\lambda_{t+1} = \lambda_t/\beta(1+a)R^* = \lambda_t/(1+a)$, t = 0,1, T-1,

(6) $\beta[(1-\delta) + f'(\bar{k})] = 1+a$, t = 0,1, T-1,

(7) $b_{t+1} = (1+a)R^*b_t + c_t - [f(\bar{k}) - \delta \bar{k}]$ t = 1, T-1, (the first period is different because k_0 is given)

It follows that $\bar{k} < k^*$, the closed economy steady state since $\beta[(1-\delta) + f'(\bar{k})] = 1+a$, while $\beta[(1-\delta) + f'(k^*)] = 1$. It is immediate to show that $[f(\bar{k}) - \delta \bar{k}] < 0$, and that c_t is increasing over time (see (5)). Then, we guess that at t = T, $b_T = 0$. If this is the case, then at t = T this economy "looks" like a closed economy with initial capital $\bar{k}$. As such, it will choose consumption on the stable manifold. Let this level of consumption be $\bar{c}$. Note that the sequence of consumption must satisfy the appropriate version of (5) -- $u'(c_{t+1}) = u'(c_t)/(1+a)$--, with a terminal condition $c_T = \bar{c}$. This, given T, pins down $\{c_t\}$ for t = 0, 1, ..., T-1. Given this, (7) gives b_t.

How do we know that there is a T such that foreign liabilities are 0? Suppose not. Then (5) holds for all t, and it implies that consumption goes to infinity, while output is bounded. This is a contradiction.

Finally we have to check that our "guess" that $d_{t+1} = 0$ is correct. It clearly holds for t =0, 1, ..., T. After time T, this is a closed economy, with a domestic interest rate given by $R_t = (1-\delta) + f'(k_t) > R^*$ since $k_t < k^*$ --the steady state capital per worker-- for all t.

ii) Note that at t = 1 the capital stock is $\bar{k} > k_0$. This is financed by borrowing from the rest of the world at t = 0.

iii) We already showed --in i)-- that deposits in the rest of the world are zero.

iv) From equation (5) it follows that consumption is increasing from t = 0 to t = T. From then on it increases as well, given that --from T on-- this economy behaves as a closed economy (it neither borrows nor lends in the international market) with initial capital $\bar{k} < k^*$ (the closed economy steady state).

v) This economy, which behaves as a closed economy after t = T, converges to the steady state. Output per capita is just $f(k^*)$, while consumption per capita is $c^* = f(k^*) - \delta k^*$. On the other hand, a similar economy with a = 0 converges in one period to the steady state. Thus, in the long run, both economies will have the same level of output per worker.

It is not possible to compare the long run levels of consumption. The reason is simple: for the a = 0 economy its long (and short) run level of consumption is given by, the solution to the following two equations,

$$c + k^* - (1-\delta)k_0 = f(k_0) + b,$$

$$c + (R^*-1)b = f(k^*) - \delta k^*.$$

Thus, it matters whether b is positive or negative. If the a = 0 country has a small initial capital stock, then $b > 0$, and $c < c^*$. If the opposite is the case, then $c > c^*$.

Problem # 2 **Wars and Interest Rates** (50 points)

Consider a simple Lucas' tree economy. The representative household has preferences defined over stochastic processes of consumption given by,

$$\sum_{t=0}^{\infty} \beta^t u(c_t),$$

where $0<\beta<1$, and $u(x) = x^{(1-\theta)}/(1-\theta)$, for some $\theta > 0$. (If $\theta = 1$, the utility is logarithmic.) Each household owns one tree. Thus, the number of households and trees coincide. The amount of output that grows in a tree satisfies,

$$d_{t+1}/d_t = \gamma \exp[\varepsilon_{t+1} - \sigma^2/2],$$

where $\{\varepsilon_{t+1}\}$ is a sequence of i.i.d. random variables, $N(0,\sigma^2)$. (This means that each is distributed Normal with mean 0 and variance σ^2.) At time 0, the country finds itself fighting a war. The cost of the war --in terms of output-- is $x_t = gd_t$, $0 < g < 1$. Thus, during the war years, consumption is $(1-g)d_t$, while in peacetime it is d_t. Assume that, at time 0, it is known that the war will end at t = T (year T is the first year of "peace"), and that the country will never fight another war. Suppose that the government has issued an infinitely lived bond that pays, in peacetime, r units of consumption in every period and in every state of nature. However, during wartime, the government suspends the payments. That is, during the war years the bond pays nothing.

Let $M \equiv \beta(\gamma)^{-\theta}\exp[\theta(1+\theta)\sigma^2/2]$. Assume that $M < 1$.

i) (15 points) Let the price of the government bond after the war ends be v^P_t. Show how this price depends on the coupon rate. Let the rate of return in peacetime -- the interest rate-- be defined as $(v^P_{t+1} + r)/v^P_t$. Describe how economic fundamentals affect the rate of return. Does it depend on r?

ii) (20 points) Let the price of the bond when the war has n periods to go be v^W_{T-n}. Describe this price for n = 1, 2 and 3. What happens to bond prices during the war? Show your work.

iii) (15 points) Let the interest rate during the war years be v^W_{T-n-1}/v^W_{T-n}, if $n\geq 2$, and, in the last period of the war, it is defined by $(v^P_T + r)/v^W_{T-1}$. (Remember that during war years the government does not pay the coupon.) What can you say about wartime interest rates at the beginning of the war? Are they lower than in peacetime? Do they depend on fundamentals? Are interest rates high or low at the end of the war?

Extra Credit: Consider the previous model with just one change: The end of the war is random; in every period the war ends with probability π. However, once the war has ended it never restarts. Go as far as you can describing the behavior of interest rates during war years and in peacetime, including the ex-post rate of return during the transition from wartime to peacetime.

Problem # 2 **Wars and Interest Rates** - Solution

i) Consider first the situation after the war ends. The standard asset pricing equation for these infinitely-lived bonds is just,

(1) $u'(c_t)v^P_t = \beta E_t[u'(c_{t+1})(v^P_{t+1} + r)].$

Using the specific functional forms in this example, (1) reduces to

(2) $v^P_t = \beta E_t[\ \gamma^{-\theta} \exp(-\theta)[\varepsilon_{t+1} - \sigma^2/2](v^P_{t+1} + r)]$.

Note, however, that the right hand side is an i.i.d. random variable, since ε_{t+1} is i.i.d. Then, a natural guess is that the right hand side will be constant, since we are taking the expectation. This, implies, in turn, that v^P_t is constant and given by,

(3) $v^P = M(v^P + r)$,

where M is as defined in the problem. Thus it follows that,

(4) $v^P = rM/(1-M) > 0$,

while the interest rate, R^P is,

(5) $R^P = (v^P + r)/v^P = M^{-1}$.

In this economy the interest rate does not depend on the coupon rate. This, of course, is just a special case of the Modigliani-Miller theorem. Moreover, the interest rate is independent of current output and consumption. It does depend on fundamentals as captured by M. In particular, it is higher the higher the growth rate (Who said that high interest rates impinge growth?), and it decreases with the variability of the growth rate of consumption: Economies with very uncertain growth rates of consumption have low interest rates. Finally, the higher the level of risk aversion, the lower the interest rate.

ii) Consider next what happens in the last period of the war. The relevant version of (2) is,

(6) $v^W_{T-1} = \beta E[\ \gamma^{-\theta}(1-g)^{\theta} \exp(-\theta)[\varepsilon_{t+1} - \sigma^2/2](v^P + r)] = \beta E[\ \gamma^{-\theta}(1-g)^{\theta} \exp(-\theta)[\varepsilon_{t+1} - \sigma^2/2]\ r/(1-M)]$,

where the "key" difference is that c_{T-1} is just (1-g) of the endowment. The solution to (6) is,

(7) $v^W_{T-1} = (1-g)^{\theta}\ rM/(1-M) < v^P$.

During the "interior" war years, we get that the appropriate version of (2) is,

(8) $v^W_{T-n-1} = \beta E[\ \gamma^{-\theta} \exp(-\theta)[\varepsilon_{t+1} - \sigma^2/2]v^W_{T-n}]$, $\quad n = 1,2,..., T-1$.

This equation captures the fact that the war costs fraction 1-g of consumption. This lowers the level of consumption --and increases the marginal utility-- but does not change the marginal rate of substitution between consumption at t and t+1, since the war has the same effect on both. In addition to this effect --or lack of it-- (8) also captures the assumption that, during the war, the government does not pay its coupon rate, i.e. r = 0. The solution to (8) is just,

(9) $v^W_{T-n-1} = Mv^W_{T-n}$.

Thus, during the war years the price of government bonds increase in anticipation of the resumption of coupon payments.

During the war years there are two interest rates. The interior rate is $R^W_{T-n} = v^W_{T-n}/v^W_{T-n-1} = M^{-1} = R^P$, for n = 1, 2, ..., T-1. On the other hand, $R^W_{T-1} = (v^P + r)/v^W_{T-1} = (M(1-g)^{\theta})^{-1} > R^P$. Thus, in the last period of the war --but not before-- the interest rate is high. Note that, only in this last period, the "size" of the war, as measured by g, affects interest rates. The direction is clear: the larger the war, the higher the interest rate.