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DO RATIONAL TRADERS FRENZY?

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*Do Rational Traders Frenzy?**

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Abstract

I develop a simple new model of strategic trade with *endogenous timing*, generalizing Glosten and Milgrom (1985): A competitive market maker faces n risk neutral traders with unit demands or supplies. It is private information whether any given trader is either informed, with a *heterogeneous* informative signal about the asset payoff, or a pure noise trader planning to make a trade at a random time. The market is open for an exponential length of time.

This structure is *recursively soluble* into a sequence of ‘subgames’, and — despite the endogenous timing — I find there is a *unique* separating equilibrium. I prove that necessarily there is *incomplete separation*, since only informed traders whose signals are ‘good news’ ever buy, and only those with ‘bad news’ ever sell! I show that traders can only envision switching sides of the market if the underlying signal distribution has no neutral news signal. Finally, I conjecture that the answer to the title is ‘yes’, that all trades are *self-feeding*, and accelerate the time schedule of any future trades.

My analysis is greatly simplified by focusing on a facetious ‘competitive auction’ model, where the market maker only wishes to sell units.

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1. INTRODUCTION

There has of late been some interest in the question of whether there can be rational asset price movements ‘without news’. That this can in fact arise in equilibrium has in fact already been demonstrated in several papers.¹ In a world in which individuals trade on the basis of private information, the price may move without *the arrival* of news if individuals may optimally time their trades. This brings me to the question of this paper: *In equilibrium*, will traders rationally choose to cluster their buy or sell transactions? One loose intuition in favor such ‘frenzying’ seems to be that the imputed informational content of any given trade acts as a siren call, luring other wavering traders into the fray on the *same* side of the market, and discouraging contrarians. This paper explores a simple model designed in part to test this thesis.

This incentive to herd is perhaps most cleanly seen in Bulow and Klemperer (1994) (hereafter, simply BK). They study a dynamic multiple-unit independent private values Dutch auction: In equilibrium, the highest types purchase first, with each sale potentially sparking a buying ‘frenzy’. Namely, all traders with valuations within some range *immediately purchase* at the current price. This behavior is motivated by an implicit cost of delay, stemming from the possibility of a stock-out. Gul and Lundholm (1992) capture a related phenomenon in a purely informational context.² They describe a simple n -player forecasting game, where everyone wishes to accurately predict $x_1 + x_2 + \dots + x_n$ as soon as possible, where x_k is type k ’s random type. Here, the delay cost is an explicit increasing function of x , so that the highest types forecast first in the unique separating equilibrium. It is shown that endogenous timing results in more *clustered* forecasts, with extreme types able to make more informed decisions.³

Crucially observe that there is *no cost* to frenzying or clustering in either paper. For instance, the auctioneer in BK commits to maintaining the current price after a sale until no further buyers wish to purchase. Yet for many situations in real life, this is hardly realistic. This casts some doubt as to whether it is an apt description of behavior in such settings. I am interested in the most salient example, namely financial markets, where the price mechanism explicitly penalizes frenzying.

¹See Romer (1993), for one, and references therein.

²A more applied take on this is found in Caplin and Leahy (1994).

³Loosely, the frenzying of BK might be seen as an extreme form of clustering, and I shall henceforth lump both phenomena together under the rubric of frenzying.

An Overview of the Model. I have in mind to address the question of endogenous timing and the issue of financial frenzying in a relatively simple and yet natural fashion. Smith (1995) shows that once one ventures outside the world of purely strategic trading, individuals may have no incentive whatsoever to time their trades. I thus reach back to one of the earlier models of insider trading, due to Glosten and Milgrom (1985) (hereafter, simply denoted GM). They consider a market for an asset of uncertain value, in which prices are set by a competitive ‘market maker’ or specialist who faces a sequence of trading orders from potentially informed individuals, the ‘insiders’, each with unit demands or supplies. As their model is primarily concerned with explaining the existence and characteristics of the bid-ask spread in such a market, they treat the arrivals of informed and uninformed traders as arising from an exogenous stochastic process known to the market maker.

My adaptation of GM endogenizes the timing of trades. Specifically, I start with a known number of traders in the market at time zero. An unobservable number are informed insiders with heterogeneous private information, while the others are pure ‘noise’ traders. I then allow the informed traders to strategically time their trades while the noise traders execute their trades according to an exogenously-given Poisson process. It is private information whether any given trader is either informed, having observed a heterogeneous signal about the payoff of the asset, or a pure noise trader planning a random trade at a random time. The market closes after a random exponential length of time, after which the payoff of the asset is revealed. As a key simplifying innovation, I largely focus on a facetious ‘competitive auction’ model, where the auctioneer only wishes to sell units, all at zero expected profit. Later on, I simply expand the strategy space, allowing traders to take both sides of the market.

The Equilibrium. In any separating equilibrium of this game, informed traders possessing the most extreme high signal (and also the lowest signal, in the market maker model) enter first, by analogy to Gul and Lundholm (1992). With this separation continuously occurring, the updated distribution of possible signals possessed by traders is constantly being truncated. Barring termination of the game, the solution of any such *truncation equilibrium* thus consists of a sequence of n subgames that can be recursively solved. That is, each subgame immediately following a trade is simply a rescaled version of the original game — with fewer players, and a truncated

signal distribution, and updated beliefs on the value distribution and the number of informed traders. Note that this is true even of the *one-trader game*! For with only one trader left, separation will still continue to occur, as the lone individual still plays against *nature*, who determines when the game will end. This virtue is the reason for the assumption that trade occurs for a random exponential length of time rather than, for instance, on the unit time interval.⁴

This is not the first instance of a recursive structure in a bid-ask market. Wilson (1986) studies a private values double auction, and looks for a truncation equilibrium. In it, the ask and bid prices set by the uninformed market maker converge between trades, and jump up (resp. down) in response to a purchase (resp. sale). There are a few analogies to be drawn between my equilibrium and his, but by and large, the common values setting here intuitively makes this an inherently different problem.

With the recursive structure, it suffices to study subgames in isolation by backward induction, carefully grafting the solution of each into its predecessor. The assumption of ‘one-trade-and-out’ also allows me to characterize equilibrium strategies as the solutions to a relatively straightforward continuous time optimal stopping problem.

Trade Dynamics. As I have set out expressly to explore trade timing, the model is set in continuous rather than discrete time. For the market maker model, the dynamics in each subgame can be represented by a five-dimensional autonomous system in the ask and bid prices, as well as the highest and lowest signals x and y that have not yet purchased, and the remaining mass u of uninformed noise traders. Across subgames, the specialist’s zero profit condition implies that the signal truncations must be continuous, and so the initial x and y (and more easily u) in any subgame are given. The bid and the ask, however, must jump, and this indeterminacy potentially creates a two-dimensional continuum of equilibria. In fact, I argue that only one set of initial ask and bid prices is consistent with forward-looking equilibrium behavior. That is, *there is a unique separating equilibrium*.

With a unique equilibrium, the assumption of unit trades lends itself to a simple test of the frenzied hypothesis: Conditional on a trade occurring at time t , how do remaining informed traders subsequently adjust the time that they intend to trade? I conjecture that locally at least, rational traders *do* frenzy. At least temporarily,

⁴Also with a finite horizon, a nasty singularity ensues from a last minute ‘mad rush’ to trade.

trades are *self-feeding*. For instance, after a purchase, traders soon planning to jump into the fray will purchase sooner and sell later than planned.

Why Separation Occurs. That separation even occurs here is by no means transparent, and the argument hinges on the assumption that traders' signals are *conditionally* i.i.d. The intuition for why traders with stronger signals move earlier revolves around two distinct lines of out-of-equilibrium reasoning. The first is specifically tied to the dynamic truncation process, while the second stems from the common-values assumption.

First, a truncation equilibrium induces heterogeneous beliefs about future price movements among the informed types (the *price effect*). Since prices are set by an uninformed specialist, their evolution between trades and the magnitudes of their jumps immediately following trades are commonly forecast by all types of insiders; however, with conditionally i.i.d. signals, insiders disagree on the distribution of signals that may be held by other insiders, and thus differentially assess the *likelihoods* of future trades; they also have different estimates of the asset value. In particular, a truncation equilibrium applies more competitive pressure on traders with stronger signals: The highest signal trader is most confident of an intervening purchase, most pessimistic of an intervening sale, and most fearful of the prospect of exogenous market closure.⁵ While short of a proof, I believe that the market-maker model offers a nice cross between an incomplete information game of *pre-emption* and *war of attrition*: Intuitively, traders wish to buy just in advance of competing same side traders and just after their opposite side counterparts. But while all would ideally prefer to wait and trade at the higher bid or lower ask price that may soon prevail, those with stronger signals believe an intervening same-side trade and consequent unfavorable price jump to be more likely; they therefore will be more impatient to trade earlier.⁶

Observe that an analogous 'price effect' also emerges as the driving force behind separation in a pure independent private values (IPV) auction, such as BK. But it is inherently different there, insofar as it is differences in von Neumann-Morgenstern (vNM) *preferences* rather than *beliefs* that matter: Traders with higher absolute val-

⁵That is, except perhaps for some extreme low signal traders, who might have more to gain by shorting the asset.

⁶By corollary, if traders' information signals are merely unconditionally i.i.d., as with Foster and Viswanathan (1993), a truncation equilibrium doesn't exist: Gul and Lundholm (1992), for instance, have an explicit cost function to force traders with high signals to move sooner rather than later.

uations place a lower value on the known marginal gains from further price reductions when weighed against the loss should an intervening trade occur. In the one-trader game, absent further learning, beliefs are fixed, and one can view separation as occurring solely from heterogeneous vNM preferences.

It should be noted that the insight that competition among traders can endogenously generate ‘impatience’ in the absence of explicit discounting originates with the double auction of Wilson (1986). But in that paper, the more subtle *option value of delay* does not emerge: In the common-values setting, for a generic class of signal distributions, inframarginal traders may be uncertain and uncommitted as to which side of the market to take, if any. Having seen the decisions of others, a trader may be persuaded to change the course of action she had previously planned to take in the current subgame — for instance, she may buy rather than sell. But among such traders, one with a relatively stronger signal will place a lower probability on the event that other traders will reveal sufficiently strong opposing signals to overturn her decision, and so her option value of delay is lower than that of an informed trader with a weaker signal. As it turns out, the class of signal distributions without any option value of delay play a key role in this paper.

Is Separation Complete? Since the ‘no-trade’ theorem of Milgrom and Stokey (1982), it is well-known that trade for purely informational motives cannot occur in equilibrium. With noise traders in the picture, such trades may occur provided there exists a bid-ask spread sufficient to address the remaining adverse selection problem. GM admitted to one ‘failing’ of their competitive specialist model, that the required bid-ask spread effectively screens out marginal but nonetheless potentially valuable information: Insiders with only middling information simply do not trade. They write, “*This opens the possibility that another way of arranging trade may exist which is Pareto superior to the competitive system.*”

This paper in part explores whether such an improvement can be effected *within* the competitive system simply by endogenizing the timing of trades. The bid-ask spread is a symptom of an adverse selection problem, and in a world with endogenous timing, if the informed traders enter at a much faster rate than the noise traders, then the spread can vanish. So could *all* informed traders eventually purchase? Do the incentive efficiency characteristics of double auctions that Wilson (1985) finds

extend beyond the IPV setting? No. I show that in the competitive auction model, equilibrium necessarily entails *incomplete separation*, simply because no trader having ‘bad news’ (in the sense of Milgrom (1981)) ever purchases. In the market maker model, additionally no one with ‘good news’ ever sells. While both results are true in GM, with endogenous timing, they are both subtle and tight: For I show that *eventually* all good news traders do buy and, in the market maker model, all bad news traders do sell in the unique equilibrium.

Relationship to the Literature. There are a handful of informational insider trading papers with endogenous timing.⁷ First is the monopolistic model of Kyle (1985), whose analysis focuses on the ‘linear’ equilibrium. Most closely related to this work is Foster and Viswanathan (1993), who have extended the Kyle model to the case of many heterogeneously informed insiders. Their insiders may repeatedly trade in any of finitely many discrete time periods. Yet faithful to Kyle, they have retained linearity,⁸ as have most others.⁹ Our dynamic insider trading paper, by way of contrast, requires no such focus on equilibria with special characteristics.

The remainder of the paper is organized as follows. In section 2, I describe the fictitious zero profit auction model. This is introduced both as a conceptual device, and because it saves much analytic repetition. Since the model is largely new, I spend some time deriving the equations of the truncation equilibrium and intuitively discussing its properties in section 3. Section 4 is a quick march through the analogous equilibrium laws of motion for the market maker model. In section 5, I explore the theoretical possibility of full separation in either context, and in particular establish the good news and bad news exclusion results. I use this discovery to prove existence and uniqueness of the separating equilibrium in section 6. Section 7 explores the dynamics of trade in equilibrium, with attention to the frenzying question.

⁷See also Vayanos (1993), who studies hedging-motivated trade by insiders.

⁸Foster and Viswanathan underscore the daunting complexity of doing without linearity, noting that they “*avoid the usual forecasting the forecasts ... problem.*” — a.k.a. equilibrium theory. In this framework, this forecasting problem also rears its head: Individuals must know the equilibria for all future subgames.

⁹For references and a discussion, see Rochet and Vila (1994). On this note, I should comment that the ‘justification’ in this latter paper for such a restricted focus and that of normality rely on an assumption that I expressly rule out — that insiders directly observe the amount of noise trading. While an elegant unifying result, this seems an awfully powerful presumption.

2. A ZERO-PROFIT COMMON VALUE AUCTION

Consider an auctioneer who is selling shares of a single asset with uncertain *ex post* value \tilde{V} . Assume that the auction is continuously open on a time interval $[0, \tilde{T})$ of random duration, with N potential *buyers* (or *traders*) present in the market at time zero, and no subsequent arrivals. Each trader may make at most one purchase of unit size, while the market remains open ('one-trade-and-out'). The number of remaining traders and all trades are always publicly observable.

The Uncertainty. The random auction closing time \tilde{T} is distributed over $(0, \infty)$ according to a Poisson process with parameter λ — that is, \tilde{T} exceeds t with chance $e^{-\lambda t}$. When the auction closes, the realized value v of the random variable \tilde{V} is publicly revealed.¹⁰ The time-zero *prior beliefs* about \tilde{V} shared by all participants are described by the distribution $G_0(\cdot)$ on $[V_0, V_1]$. For notational simplicity, I shall assume that G_0 has a continuous density g_0 .

The Auctioneer. The auctioneer continuously announces a temporary *ask price* at which he is currently willing to sell unit shares of the asset.¹¹ That is, at each time- t , given only the information \mathcal{I}_t from the history of purchases in $(0, t)$ and the elapse time t , the ask price a_t earns zero expected profits conditional upon the event β_t of a purchase. For instance, the auctioneer may be an arm of the government, with the designated goal of neither losing nor making money in expectation on any transaction. Later on, I shall allow this auctioneer to take both sides of a trade, in which case it will be the classic zero-expected-profit market maker introduced by Glosten and Milgrom (1985), that may be taken as a proxy to Bertrand competition. This is the ultimate motivation for this preliminary theoretical construct.

The GM assumption automates the auctioneer's decision rule, and thus nicely allows one to focus exclusively on insiders' optimization. In other words, the ask price incorporates any information that is transmitted by the arrival of the order. Just as in GM, I assume that if more than one zero-expected-profit ask price exists, I follow the convention of choosing the lowest one (and later on, the highest bid price too), but unlike GM I simply assume that the auctioneer has no private information.

¹⁰Preston McAfee suggested this infinite-horizon reformulation of an earlier finite horizon model, where the market was open on $[0, 1)$.

¹¹Throughout the exposition, I shall maintain the convention of referring to the traders as females and the auctioneer (and later, the market maker) as male.

The Traders' Private Information. At time zero, each trader with chance q_0 observes a signal $w \in [0, 1]$ of the asset's value. I assume that the event of becoming informed is the trader's private information and is independent across traders.

The signals too are private information, and are *conditionally independent*. This means that each signal w is an independent draw from the continuous distribution $F(\cdot | v)$, given the true value v . I assume there is also a smooth density $f(\cdot | v)$ that satisfies the strict *monotone likelihood ratio property* (MLRP), namely,

$$f(w_H | v_H)/f(w_H | v_L) > f(w_L | v_H)/f(w_L | v_L)$$

whenever $w_H > w_L$ and $v_H > v_L$. Intuitively, signals and values are complementary, so that traders with higher signals are more optimistic about the expectation of \tilde{V} . Observe that trivially, the strict MLRP implies that the signal distribution has full support on $[0, 1]$, so that I can never rule out any signals regardless of v .

Some of the results shall assume that there exists a signal \bar{w} that is *neutral news*: namely, it has the same likelihood for any value v . Other results only need the next weaker notion that necessarily exists with a nonatomic signal space and the MLRP: Signal \bar{w} is *weak neutral news* given a prior g if the posterior expectation of \tilde{V} given w equals its prior mean. Finally, a signal w is (weak) *good news* or (weak) *bad news* as $w \gtrless \bar{w}$. Appendix A.1 summarizes some key implications of the MLRP and some simple integral inequalities that I find very useful.

The Traders' Strategies. Focusing on the informed traders, I assume that the uninformed (*noise*) traders have no discretion over the timing of their trades. Instead, they are prompted to buy by a positive random liquidity shock that arrives according to an independent Poisson process with parameter μ .¹²

In contrast, informed traders purchase for purely speculative motives and fully optimize over the timing of their trades. While the market is open, any informed trader with signal w who has not yet traded chooses an elapse time $\tau(w)$ to buy if no one else has already purchased.¹³ She may well choose never to buy, in which

¹²An arguably more realistic (and more complicated) model with explicit stochastic entry of liquidity traders could also make informational sense if the total number of traders were unobserved. But I feel that the analytic advantage of my approach far outweighs any injustice wrought by my twisting the standard story. Hopefully the reader will soon agree.

¹³One technical question naturally arises with continuous time: What happens if one trader buys

case $\tau(w) = \infty$. All informed traders are risk neutral and do not discount future utility.¹⁴ The type- w trader's expected payoff at each time t is given by $\mathcal{E}_{wt}\tilde{V} - a_t$ if she purchases at an ask price of a_t , and 0 otherwise. The expectation operator \mathcal{E}_{wt} is defined with respect to the public information set \mathcal{I}_t (given the commonly known strategies), and the private signal w . I further let β_t^I and β_t^U be the events analogous to β_t , but where the trader is known to be informed and uninformed, respectively.

The Equilibrium Concept. As I model a dynamic game of incomplete information among the informed traders, the appropriate solution concept is sequential equilibrium. I further wish to prune any equilibria in which decisions are predicated upon payoff irrelevant information. For instance, I only wish to consider equilibria that are anonymous, in the sense that each trader's strategy may depend only upon her signal and the public posterior on the asset value. I shall address these concerns simply by restricting focus to the *Markov sequential equilibria* (MSE) of the model (due to Maskin and Tirole (1992)).

Separating Equilibria. Among the class of MSE, I am interested in those that are *separating*: namely, any traders with different signals who both plan to buy, must plan to do so at different times. Partition the game into N subgames, each ending in a trade. The focus of this paper is an especially salient example of a separating equilibrium called a *truncation equilibrium*: There is a continuous and strictly decreasing *truncation function* $t \mapsto x_t$, differentiable in each subgame, where $0 \leq x_t \leq 1$, such that at time t (i) x_t is the informed type that buys, and thus (ii) all remaining informed traders have signals $w \in [0, x_t]$. I am also interested in the notion of a *completely* (or *fully*) *separating* MSE, in which all informed types eventually buy if the market is open long enough: $\lim_{t \rightarrow \infty} x_t = 0$. Since strategic decisions are no longer made as soon as $x_t = 0$, the subsequent analysis shall restrict focus to $x_t > 0$.

As described in the introduction, I may exploit the recursive structure of a truncation equilibrium and simply analyze a representative n -trader subgame ($1 \leq n \leq N$). So a truncation equilibrium is fully described by the monotonic function x for $t \in (\tau_0, \infty)$, where τ_0 marks the start of the current subgame when the last trade occurred.

'instantaneously after' another? To save on tedious details, I shall simply ignore this possibility as it will be a zero chance event in equilibrium.

¹⁴Everyone behaves identically in the models with interest rate $r > 0$ or, as is the case here, with no discounting and \tilde{T} exponential with parameter $r + \lambda$.

3. EQUILIBRIUM ANALYSIS

3.1 The Accounting Equations Within a Subgame

Any delay of trading activity within a subgame simultaneously reveals information about (i) the informed status of all remaining traders: remaining traders are less (resp. more) likely informed if their equilibrium entry rate exceeds (resp. is less than) the noise traders entry rate; and (ii) the underlying asset value: since the informed traders in particular have not purchased, the asset value is more likely low. In particular, if I condition the probability q that a typical trader is informed upon the underlying asset value v , then the following two posteriors jointly evolve:

- $q_t(v)$, the chance that any given remaining trader is informed, given v and \mathcal{I}_t
- $g_t^n(v)$, the public posterior density of v given \mathcal{I}_t

Note that $q_t(\cdot)$ is an *interim* density in the sense that it does not incorporate the informed trader's private signal. It is relevant for the informed trader and not the auctioneer inasmuch as it assumes uncertainty only over $n - 1$ and not all n traders. This, as it turns out, is what the auctioneer cares about.

Computing $q_t(\cdot)$ is easiest. Given the current truncation $[0, x_t]$,

$$q_t(v) = \frac{F(x_t | v)q_0}{F(x_t | v)q_0 + e^{-\mu t}(1 - q_0)} = \frac{F(x_t | v)q_0}{F(x_t | v)q_0 + u_t(1 - q_0)} \quad (1)$$

where I introduce the notation

- $u_t = e^{-\mu t}$, the (per capita) probability mass of uninformed traders left at time- t

Expression (1) is valid both between and within subgames, and implies that $q_t(v)$ is continuous in t (since x_t is continuous in any truncation equilibrium).

Next, consider an n -trader subgame that begins at time τ_0 , immediately after the last purchase, endowed with the truncation x_{τ_0} , mass u_{τ_0} , density g_{τ_0+} , and beliefs q_{τ_0} .¹⁵ (So $x_0 = u_0 = 1$, $g_{\tau_0+} = g_0$, and $q_{\tau_0} = q_0$.) Before proceeding, it helps to define

¹⁵Functions are assumed *left* continuous in time, since a purchase at time t does not affect the information \mathcal{I}_t upon which all variables depend. At a discontinuity of a function h , $h(\tau_0+)$ denotes the right evaluation of h the 'instant after' the trade at time τ_0 — i.e., the right hand limit of $h(t)$ as $t \downarrow \tau_0$.

- $\Phi_t(\cdot | \cdot)$ and $\phi_t(\cdot | \cdot)$, an informed trader's time- t *conditional signal distribution and density* given the value, namely

$$\Phi_t(w | v) \equiv \frac{F(w | v)}{F(x_t | v)} \quad \text{and} \quad \phi_t(w | v) \equiv \frac{\partial}{\partial w} \Phi_t(w | v) = \frac{f(w | v)}{F(x_t | v)} \quad (2)$$

Intuitively, the MLRP will imply that $\Phi_t(w | v)$ is decreasing in v for fixed $w < x_t$, since higher values are associated with higher signals.

I can now calculate the value density g_t^n during the current subgame as follows:

$$g_t^n(v) = \frac{[q_{\tau_0}(v)\Phi_{\tau_0}(x_t | v) + (1 - q_{\tau_0}(v))(u_t/u_{\tau_0})]^n g_{\tau_0+}(v)}{\int_{V_0}^{V_1} [q_{\tau_0}(z)\Phi_{\tau_0}(x_t | z) + (1 - q_{\tau_0}(z))(u_t/u_{\tau_0})]^n g_{\tau_0+}(z) dz} \quad (3a)$$

$$\propto \left(\frac{q_0 F(x_t | v) + (1 - q_0)u_t}{q_0 F(x_{\tau_0} | v) + (1 - q_0)u_{\tau_0}} \right)^n g_{\tau_0+}(v) \quad (3b)$$

for all $t > \tau_0$. Formula (3a) is simply an application of Bayes rule for the chance that the value is v given that n traders remain at time t . Here, I have (i) exploited the conditional independence of signals held by informed traders, for a fixed value v ; (ii) applied the law of total probability in the numerator and denominator to the unknown number of those traders who are informed, and (iii) used the fact that $N - n$ traders have already bought is already embedded in the prior g_{τ_0+} . The proportionality sign \propto is with respect to v ; therefore, the simplification (3b) ignores the denominator, which is independent of v . For most uses in this paper, the density g_t^n will appear in both numerator and denominator of fractions, and thus it will suffice to substitute this reduced form.

Later on, I shall require the following differential laws of motion of g_t^n and q_t .

Lemma 1 (Laws of Motion) *Within a subgame, the conditional chance $q_t(\cdot)$ that any given remaining trader is informed, obeys*

$$\dot{q}_t(v) = q_t(v)(1 - q_t(v))(\mu + \phi(x_t | v)\dot{x}_t) \quad (4)$$

while the public posterior density $g_t^n(\cdot)$ over the value v evolves according to

$$\frac{\dot{g}_t^n(v)}{g_t^n(v)} = n \left[\frac{\dot{q}_t(v)}{1 - q_t(v)} - \int_{V_0}^{V_1} \frac{\dot{q}_t(z)}{1 - q_t(z)} g_t^n(z) dz \right] \quad (5)$$

The proof is in the appendix. To see how intuitive is equation (4), observe that $-\phi_t(x|v)\dot{x}_t$ is the flow probability of a purchase by informed traders in equilibrium. It thus simply says that the difference in the rates of change of the proportions of informed and uninformed traders equals the negative difference in their respective trading hazard rates:

$$\frac{d \log q_t(v)}{d \log(1 - q_t(v))} = \frac{\dot{q}_t(v)}{q_t(v)} - \frac{(1 - \dot{q}_t(v))}{1 - q_t(v)} = \mu + \phi_t(x|v)\dot{x}_t$$

3.2 The Auctioneer's Problem

At any moment in time, the auctioneer's sole task is to control the ask price in a purely automated fashion. It is natural to define

- $\bar{v}_t^n \equiv \mathcal{E}_n[\tilde{V}|\mathcal{I}_t] = \int_{V_0}^{V_1} v g_t^n(v) dv$, the (auctioneer's) time- t public expectation of the value given n remaining traders

The auctioneer's pricing rule must reflect his informational disadvantage to the informed traders; therefore, he must equate the ask price not to \bar{v}_t^n , but to his updated expected valuation of the asset conditional on a purchase at time t . Let $p_t(v)$ denote the *purchase hazard rate* for a typical trader at time t , conditional on the true value of \tilde{V} being v . This is the sum of the respective entry rates of informed and uninformed buyers, as follows:

$$p_t(v) = q_t(v)\phi_t(x_t|v)(-\dot{x}_t) + (1 - q_t(v))\mu \quad (6)$$

$$= \frac{q_0 f(x_t|v)(-\dot{x}_t) + (1 - q_0)u_t\mu}{q_0 F(x_t|v) + (1 - q_0)u_t} \quad (7)$$

The auctioneer only sees that a purchase has occurred at a given ask price, but does not observe the informed status of the trade. Hence, following a purchase he updates $g_t(v)$ to the *new posterior value density* $g_{t+}(v) = g_t^{B,n}(v)$, where

$$g_t^{B,n}(v) = \frac{p_t(v)g_t^n(v)}{\int_{V_0}^{V_1} p_t(z)g_t^n(z)dz} \propto p_t(v)g_t^n(v) \quad (8)$$

He carries this new posterior density into the next subgame. The ask price that satisfies the auctioneer's zero-profit constraint is therefore

$$a_t = \mathcal{E}_n[\tilde{V} | \mathcal{I}_t, \beta_t] = \int_{v_0}^{V_1} v g_t^{B,n}(v) dv = \frac{\int_{v_0}^{V_1} v p_t(v) g_t^n(v) dv}{\int_{v_0}^{V_1} p_t(v) g_t^n(v) dv} \quad (9)$$

Observe how I may likewise produce an expression for g_t^n in terms of the original prior g_0 . For if n traders remain, and the first $N - n$ have purchased at times $\tau_1, \tau_2, \dots, \tau_{N-n}$, then by the conditional independence of signals,

$$g_t^n(v) \propto p_{\tau_1}(v) p_{\tau_2}(v) \cdots p_{\tau_{N-n}}(v) [q_0 F(x_t | v) + (1 - q_0) u_t]^n g_0(v) \quad (10)$$

3.3 The Informed Trader's Problem

In order to formulate the stopping time problem faced by a typical informed trader in a truncation equilibrium, I describe in turn her flow of payoffs within the subgame, and then the laws of motion of all payoff-relevant variables.

I remark that any informed trader's time- t *interim posterior density* given \mathcal{I}_t and the fact that she exists is g_t^{n-1} and not g_t^n . For this is the conditional density of v given that $n - 1$ *other* traders remain at time t . And once only $n = 1$ trader remains, no more learning occurs by her (if she is informed), and so $g_t^0(v)$ is time invariant. In particular, $g_t^0(v) \equiv g_0(v)$ if there is only $N = 1$ trader initially, who still remains.

First, to express the type- w informed trader's privately estimated valuation $\mathcal{E}_{wt} \tilde{V}$, introduce the notation

- γ_t^{n-1} , the *joint signal-value density*: $\gamma_t^{n-1}(w, v) \equiv f(w | v) g_t^{n-1}(v)$ is the joint density that a given remaining informed trader has signal w and the asset value is v , given the information \mathcal{I}_t from all previous trades
- $g_t(\cdot | \cdot)$, her time- t *conditional value density* given the private signal and the information \mathcal{I}_t , defined by Bayes' rule as

$$g_t(v | w) = \frac{f(w | v) g_t^{n-1}(v)}{\int_{v_0}^{V_1} f(w | z) g_t^{n-1}(z) dz} \propto f(w | v) g_t^{n-1}(v) = \gamma_t^{n-1}(w, v) \quad (11)$$

Then at time t , an informed trader with signal w expects the asset is worth

$$\mathcal{E}_{wt}\tilde{V} \equiv \mathcal{E}[\tilde{V}|\mathcal{I}_t, w] = \int_{V_0}^{V_1} v g_t(v|w) dv = \frac{\int_{V_0}^{V_1} v \gamma_t^{n-1}(w, v) dv}{\int_{V_0}^{V_1} \gamma_t^{n-1}(w, v) dv} = \frac{\int_{V_0}^{V_1} v f(w|v) g_t^{n-1}(v) dv}{\int_{V_0}^{V_1} f(w|v) g_t^{n-1}(v) dv}$$

An informed trader with signal w can act as if she receives all payoffs as soon as the current subgame ends. This occurs either because she or another trader buys, or the market closes. If she buys first, then she receives the expected profits $\Pi_t^B(w)$ from that purchase evaluated using her concurrent beliefs. If another purchase triggers the next subgame, she receives her expected continuation value $\mathcal{W}(\mathcal{I}_t, \beta_t, w)$. If the market closes, she receives nothing. Her instantaneous payoffs are thus:

$$\begin{aligned} \Pi_t^B(w) &= \mathcal{E}_{wt}\tilde{V} - a_t \\ \Pi_t^D(w) &= \begin{cases} \mathcal{W}(\mathcal{I}_t, \beta_t, w), & \text{if another trader buys at time } t \\ 0, & \text{if the market closes at time } t \end{cases} \end{aligned} \quad (12)$$

Next, I describe an informed trader's personal estimate of the instantaneous probabilities of intervening purchase *by any other given trader*. For one's private signal is an indication of the underlying value, and by implication of the distribution of other informed traders' private signals. The *privately assessed purchase hazard rate* $\hat{p}(w)$ for someone with signal w is thus obtained simply by integrating $p_t(v)$ over the buyer's private beliefs on the value distribution:

$$\hat{p}_t(w) \equiv \int_{V_0}^{V_1} p_t(v) g_t(v|w) dv = \frac{\int_{V_0}^{V_1} p_t(v) \gamma_t^{n-1}(w, v) dv}{\int_{V_0}^{V_1} \gamma_t^{n-1}(w, v) dv} = \frac{\int_{V_0}^{V_1} p_t(v) f(w|v) g_t^{n-1}(v) dv}{\int_{V_0}^{V_1} f(w|v) g_t^{n-1}(v) dv} \quad (13)$$

where I have substituted from (11).

In light of equations (1)–(3a) and (11)–(13), all payoff-relevant information for an informed trader at any time t is captured by initial conditions for the current subgame, as well as (u_t, x_t, a_t) .^{16,17} Crucially observe that *within a subgame all state variables evolve deterministically*, and it is only the concluding time of this subgame

¹⁶For indeed, (1) and (3a) express q_t and g_t^{n-1} given these quantities and $(u_{\tau_0}, x_{\tau_0}, q_{\tau_0}, g_{\tau_0+})$, while ϕ_t is a simple function of x_t via (2). All other variables are functions of x_t, q_t, g_t^{n-1} , and ϕ_t .

¹⁷Observe that the starting time of the current subgame is payoff irrelevant. By way of contrast, in the natural alternative formulation of the model where the market closes almost surely at time 1 (my first choice), the actual calendar time also matters.

that is stochastic: The type- w informed trader thinks it ends with Poisson flow rate $(n - 1)\hat{p}_t(w) + \lambda$ at time t . Confronting the type- w informed trader is the task of optimally stopping this jump process, given the termination payoff function (12). Her optimization can clearly be performed at time τ_0 , as her objective is essentially static: Choose a stopping rule τ so as to maximize her expected payoff, namely

$$\int_{\tau_0}^{\tau} e^{-\int_{\tau_0}^t [(n-1)\hat{p}_s(w) + \lambda] ds} (n-1)\hat{p}_t(w) \mathcal{W}(\mathcal{I}_t, \beta_t, w) dt + e^{-\int_{\tau_0}^{\tau} [(n-1)\hat{p}_s(w) + \lambda] ds} \Pi_{\tau}^B(w) \quad (14)$$

Recall this optimization is performed at time- τ_0 . The first term captures type w 's assessment of the possibility that the subgame will end before time τ , delivering her the continuation value from the ensuing subgame in the case of an intervening trade or zero if the game ends exogenously. The second term describes the remaining possibility that the subgame will persist until her planned purchase time τ .

Incentive Compatibility. Differentiating (14) with respect to the stopping time τ , and dividing through by the discount factor $\exp(-\int_{\tau_0}^{\tau} [(n-1)\hat{p}_r(w) + \lambda] dr)$, I obtain the first-order condition

$$0 = (n-1)\hat{p}_{\tau}(w) [\mathcal{W}(\mathcal{I}_{\tau}, \beta_{\tau}, w) - \Pi_{\tau}^B(w)] + \frac{\partial}{\partial \tau} \Pi_{\tau}^B(w) - \lambda \Pi_{\tau}^B(w) \quad (15)$$

Equilibrium requires that type x_t finds it optimal to buy at time t . Thus, (15) yields the following differential equation:

$$\left[\frac{\partial}{\partial t} \Pi_t^B(w) \right]_{w=x_t} = \lambda \Pi_t^B(x_t) + (n-1)\hat{p}_t(x_t) [\Pi_t^B(x_t) - \mathcal{W}(\mathcal{I}_t, \beta_t, x_t)] \quad (16)$$

As the 'smooth pasting' condition defining the optimal stopping time, this IC equation has a straightforward interpretation. The type who is just willing to buy at time t equates the rate of change of her terminal payoff (on the LHS) with the expected instantaneous change in the value of the game (on the RHS), due to the possibility of the game exogenously ending or someone else buying.

3.4 Analysis of the First Order System

Equation (16) yields an implicit *second-order* differential equation in x : The profit function $\Pi_t^B(w)$ differentiated on the LHS of (16) contains the ask price that itself has \dot{x} embedded in the purchase hazard rate. So I shall instead derive an explicit system of *first-order* autonomous differential equations in the state variables (u, x, a) .

From now on, I suppress time subscripts, and abbreviate: $f(x_t|v) \rightarrow f_x$, $q_t(v) \rightarrow q$, $g_t^n(v) \rightarrow g^n$, $\phi_t(x_t|v) \rightarrow \phi_x$, $\gamma_t^{n-1}(x_t, v) \rightarrow \gamma_x$, $\Pi_t^B(x_t) \rightarrow \Pi_x^B$, and $\mathcal{W}(\mathcal{I}_t, \beta_t, x_t) \rightarrow \mathcal{W}_x^B$.

Proposition 1 (Truncation Equilibria of the Competitive Auction Model)

Within a subgame, the payoff relevant state variable for traders and the auctioneer in any truncation equilibrium is (u, x, a) , which continuously evolves according to

$$\dot{u} = -\mu u \tag{17}$$

$$\dot{x} = \frac{\mu \int (v - a)(1 - q)g^n}{\int (v - a)q\phi_x g^n} \tag{18}$$

$$\dot{a} = -\lambda \left(\frac{\int v f_x g^{n-1}}{\int f_x g^{n-1}} - a \right) + (n - 1) \frac{\int (\mathcal{W}_x^B - (v - a)) (q\phi_x(-\dot{x}) + (1 - q)\mu) f_x g^{n-1}}{\int f_x g^{n-1}} \tag{19}$$

where q , f_x , ϕ_x , g^{n-1} , and g^n are explicit functions via (1) and (3a) of (x, a, u) and the initial conditions $(n, u_{\tau_0}, g_{\tau_0}, q_{\tau_0}, x_{\tau_0})$.

The derivation of the equations is in the appendix, but they have an intuitive reformulation. The informed traders control their rate of entry $-\dot{x}$ so as to enable the auctioneer to break even on all trades,¹⁸ while the auctioneer selects the rate of decline \dot{a} of the ask price allowing the traders to behave in an incentive compatible fashion. Thus, the law of motion for x comes from the ask price equation (9), while the law of motion for a comes from the informed traders' IC equation (16):

$$0 = \underbrace{(-\dot{x}) \int (v - a)q\phi_x g^n}_{\text{informed traders' flow (+) profits}} + \underbrace{\mu \int (v - a)(1 - q)g^n}_{\text{uninformed traders' flow (-) profits}}$$

¹⁸More generally, with a profit-maximizing auctioneer, traders would choose $-\dot{x}$ so that the auctioneer could satisfy his optimization.

$$\dot{a} = \underbrace{-\lambda \Pi_x^B}_{\lambda \times (\text{expected loss from exogenous termination})} + \underbrace{\frac{\int (\mathcal{W}_x^B - (v - a)) (n - 1) p \gamma_x}{\int \gamma_x}}_{\text{expected (purchase hazard rate)} \times (\text{loss from another purchase})}$$

REMARK. By continuity of (u, x) , I have initial conditions $u_{\tau_0+} = u_{\tau_0}$ and $x_{\tau_0+} = x_{\tau_0}$ in every subgame. But the initial ask price is not pinned down (intuitively, $a_{\tau_0+} > a_{\tau_0}$ after a purchase). This indeterminacy *potentially* creates a multiplicity of equilibria. Whether in fact one or more equilibria do in fact exist has not yet been shown.

3.5 Are There Other Separating Equilibria?

With the continuation payoffs nailed down by the recursive structure, any given subgame can be thought of as a *static* signalling game by the remaining informed traders; their message space is simply $(\tau_0, \infty]$, where $t = \{\infty\}$ is the message never to purchase. In this light, I now briefly revisit the choice of the truncation equilibrium. A standard feature of many distributions satisfying the MLRP is a single-peakedness of the conditional density,¹⁹ or

* *Most Likely Signal Property*: For each for $v \in (0, 1)$, the map $w \mapsto f(w|v)$ has a unique interior maximum $w(v) \in (0, 1)$.

Lemma 2 *Any separating MSE of the one trader game is a truncation equilibrium, and if the most likely signal property holds, then this is the unique separating MSE of the n -trader game too.*²⁰

The (appendicized) proof is in three steps. First, I use the single-crossing property to argue that higher types purchase first. In a word, while all traders prefer the presumably lower price that comes with time, the higher one's signal the greater the expected payoff that one places at risk by waiting. But with more than one trader, this argument is incomplete: A subgame may end from another purchase, and a trader with a lower signal may well have a *higher estimate* of the purchase hazard rate by

¹⁹For instance, the normal family $f(w|v) = ce^{-(w-v)^2}$, but not the Poisson family $f(w|v) = ve^{-vw}$. (Of course, these popular examples are ruled out by the bounded signal support assumption.)

²⁰It is very plausible that no 'pooling equilibria' exist, in the sense of two traders of different types planning to buy at the same time. In other words, *any MSE is a truncation equilibrium*.

those other traders if, perversely, other lower signal traders are expected to enter first. If the equilibrium in the continuation subgame is particularly dreaded, they may be more eager to purchase earlier than the higher signal traders. As it turns out, I show that this logic cannot be sustained.

The final two steps rule out either atoms of informed traders entering (x_t drops discontinuously), or no informed traders entering ($\dot{x}_t = 0$). Essentially, the entry rates of informed and uninformed traders always have to be ‘mutually absolutely continuous’, neither infinitely larger than the other.

3.6 The One-Trader Endgame

The differential equations for the n -trader subgame contain continuation values from the respective $(n - 1)$ -trader subgames that can be triggered at each point by a purchase. This significantly complicates the analysis. I therefore specialize the equations to the one-trader endgame, upon whose solution the equilibria of all subgames are constructed via backwards induction.

With $n = 1$, equation (19) reduces to

$$\dot{a}_t = -\lambda \Pi_t^B(x_t) \equiv -\lambda \frac{\int_{V_0}^{V_1} (v - a_t) f(x_t|v) g_t^0(v) dv}{\int_{V_0}^{V_1} f(x_t|v) g_t^0(v) dv} \quad (20)$$

Simply put, the lone informed trader buys when her marginal profits from waiting for a lower price are rising at the rate of time discounting induced by the flow probability λ of exogenous market closing. When only one trader remains in the market, neither the competitive pressures nor the learning opportunities associated with the possibility of intervening trade in the many-player subgame are present, and thus the informed trader’s only strategic consideration is to hide her signal from the auctioneer. Notice in particular that the fully pooling no-purchase equilibrium with $a_t \equiv 1$ is not a solution of (20).

But as is standard with the single-crossing property (from section 3.5), all strictly inframarginal traders must make strictly positive profits $\Pi_t^B(x_t) > 0$. So (20) implies

Corollary *In any truncation equilibrium of the one trader subgame, the ask price must strictly monotonically decline.*

I cannot be as definite about subgames with more than one trader pending a better understanding of the continuation values. In particular, to establish a monotonic decline in the ask price, it would be sufficient to argue that the traders are de facto engaged in a *pre-emption game*, with an expected loss from entry by a trader with a slightly higher signal — i.e. the second term in (19) is negative. Deducing this remains an open problem.

4. THE MARKET MAKER MODEL

4.1 The Revised Model

I now complete the story of a competitive specialist or market-maker in the spirit of GM, rather than the preceding facetious competitive auctioneer. The market maker continuously announces *bid* and *ask* prices at which he is currently willing to buy and sell unit shares of the asset, respectively. The market maker chooses these temporary (time- t) prices for each type of transaction so as to earn zero expected profits, conditional upon the occurrence of the transaction at time t .

The risk neutral informed traders' information structure is as before, but all strategy spaces are enlarged. Each of the N traders may make at most one publicly observable transaction of unit size, either a purchase or a sale of a single share, while the market remains open. Each noise trader is prompted to *buy or sale with equal probabilities* according to an independent Poisson process with parameter μ .

The separating *truncation equilibrium* now consists of continuous and strictly decreasing (resp. increasing), and hence piecewise differentiable, functions $t \mapsto x_t$ (resp. $t \mapsto y_t$), where $0 \leq y_t < x_t \leq 1$, such that at time t , (i) x_t and y_t are the informed types that respectively buy and sell, and thus (ii) all remaining informed traders have signals $w \in [y_t, x_t]$. Such an equilibrium is *completely* (or *fully*) *separating* provided $\lim_{t \rightarrow \infty} x_t = \lim_{t \rightarrow \infty} y_t$. If $x_\infty = y_\infty$ but $x_t > y_t$ for all finite t , then it may always be true that some informed types have not traded while the market is open. I shall call such an equilibrium *eventually separating*.

4.2 The Analysis of the Market Maker Model

The Accounting Equations. Given the current truncation $[y_t, x_t]$,

- in contrast to (1) and (4), the probability q of a typical trader being informed is now

$$q_t(v) = \frac{(F(x_t | v) - F(y_t | v))q_0}{(F(x_t | v) - F(y_t | v))q_0 + u_t(1 - q_0)} \quad (21)$$

and its derivative is $\dot{q}_t(v) = q_t(v)(1 - q_t(v))(\mu + \phi(x_t | v)\dot{x}_t - \phi(y_t | v)\dot{y}_t)$

- the public posterior beliefs $g_t^n(v)$ are still given by (3a), but the conditional signal distribution and density are now conditioned on the truncation $[y_t, x_t]$, as in

$$\Phi_t(w | v) \equiv \frac{F(w | v) - F(y_t | v)}{F(x_t | v) - F(y_t | v)} \quad \text{and} \quad \phi_t(w | v) = \frac{f(w | v)}{F(x_t | v) - F(y_t | v)}$$

The Market Maker's Problem. Equations for $\gamma_t(w, v)$, $g_t(v | w)$, and $\mathcal{E}_{wt}\tilde{V}$ remain unchanged. Since noise traders are assumed equally likely to buy or sell, the *purchase* and *sale hazard rates* of traders (informed and uninformed alike) are now

$$\begin{aligned} p_t(v) &= q_t(v)\phi_t(x_t | v)(-\dot{x}_t) + (1 - q_t(v))(\mu/2) \\ s_t(v) &= q_t(v)\phi_t(y_t | v)\dot{y}_t + (1 - q_t(v))(\mu/2) \end{aligned}$$

Given a trade that has occurred at a given ask or bid, the specialist updates the value density $g_t(v)$ following a purchase to $g_{t+}(v) = g_t^{B,n}(v)$ as in (8), and to $g_{t+}(v) = g_t^{S,n}(v)$ following a sale, where

$$g_t^{S,n}(v) = \frac{s_t(v)g_t^n(v)}{\int_{V_0}^{V_1} s_t(z)g_t^n(z)dz} \propto s_t(v)g_t^n(v)$$

With this equation, there is a natural (but omitted) analogue to the formula (10) for the posterior density over valuations v . Also, the required ask and bid prices that meet the market maker's zero-profit constraint are given by (9) and

$$b_t = \int_{V_0}^{V_1} v g_t^{S,n}(v) dv = \frac{\int_{V_0}^{V_1} v s_t(v) g_t^n(v) dv}{\int_{V_0}^{V_1} s_t(v) g_t^n(v) dv}$$

The Informed Trader's Problem. The type- w trader's instantaneous payoffs at time $t < T$ from buying (B), selling (S), and delay (D) are as follows:

$$\begin{aligned}\Pi_t^B(w) &= \mathcal{E}_{wt}\tilde{V} - a_t \\ \Pi_t^S(w) &= b_t - \mathcal{E}_{wt}\tilde{V} \\ \Pi_t^D(w) &= \begin{cases} \mathcal{W}_t^B(\mathcal{I}_t, \beta_t, w), & \text{if another trader buys at time } t \\ \mathcal{W}_t^S(\mathcal{I}_t, \sigma_t, w), & \text{if another trader sells at time } t \\ 0, & \text{if the market closes at time } t \end{cases}\end{aligned}$$

The informed trader's private assessment of the instantaneous probabilities of intervening trade by any other given trader are $\hat{p}_t(w)$ as before, and now also

$$\hat{s}_t(w) = \int_{V_0}^{V_1} s_t(v)g_t(v|w)dv = \frac{\int_{V_0}^{V_1} s_t(v)\gamma_t(w, v)dv}{\int_{V_0}^{V_1} \gamma_t(w, v)dv} = \frac{\int_{V_0}^{V_1} s_t(v)f(w|v)g_t^{n-1}(v)dv}{\int_{V_0}^{V_1} f(w|v)g_t^{n-1}(v)dv}$$

Note that the informed trader's objective at time τ_0 is now entails choosing a 'directed stopping rule', or a time and an *action* ($\alpha = B$ buy or $\alpha = S$ sell) so as to maximize her expected eventual payoff from this subgame, namely:

$$\begin{aligned}& \int_{\tau_0}^{\tau} e^{-\int_{\tau_0}^t [(n-1)(\hat{p}_r(w) + \hat{s}_r(w)) + \lambda]dr} (n-1) [\hat{p}_t(w)\mathcal{W}_t^B(w) + \hat{s}_t(w)\mathcal{W}_t^S(w)] dt \\ & + e^{-\int_{\tau_0}^{\tau} [(n-1)(\hat{p}_r(w) + \hat{s}_r(w)) + \lambda]dr} \Pi_{\tau}^{\alpha}(w)\end{aligned}$$

Thus incentive compatibility requires that types x_t and y_t find it optimal to buy and sell, respectively, at time t .

$$\begin{aligned}\left[\frac{\partial}{\partial t} \Pi_t^B(w) \right]_{w=x_t} - \lambda \Pi_t^B(x_t) &= (n-1) \{ \hat{p}_t(x_t) [\Pi_t^B(x_t) - \mathcal{W}_t^B(x_t)] \\ & \quad + \hat{s}_t(x_t) [\Pi_t^B(x_t) - \mathcal{W}_t^S(x_t)] \} \\ \left[\frac{\partial}{\partial t} \Pi_t^S(w) \right]_{w=y_t} - \lambda \Pi_t^S(y_t) &= (n-1) \{ \hat{p}_t(y_t) [\Pi_t^S(y_t) - \mathcal{W}_t^B(y_t)] \\ & \quad + \hat{s}_t(y_t) [\Pi_t^S(y_t) - \mathcal{W}_t^S(y_t)] \}\end{aligned}$$

The Autonomous First Order System. By analogy to the solution of the auction model, I now produce an explicit system of *first-order* autonomous differential equations in the state variables (u, x, y, a, b) . Rather than write down an analogue to Proposition 1, I shall just include below the intuitive statement of the laws of motion. Clearly, $\dot{u} = -\lambda u$. In addition, I have

$$0 = \underbrace{(-\dot{x}) \int (v-a)q\gamma_x^{n-1}}_{\text{informed traders' flow profits of trading}} + \underbrace{(\mu/2) \int (v-a)(1-q)g^n}_{\text{uninformed traders' flow profits of trading}} \quad (22)$$

$$0 = \underbrace{\dot{y} \int (b-v)q\gamma_y^{n-1}}_{\text{informed traders' flow profits of trading}} + \underbrace{(\mu/2) \int (b-v)(1-q)g^n}_{\text{uninformed traders' flow profits of trading}} \quad (23)$$

$$\dot{a} = \underbrace{-\lambda \Pi_x^B}_{\lambda E} + \underbrace{\frac{\int (\mathcal{W}_x^B - (v-a)) (n-1) p \gamma_x^{n-1}}{\int \gamma_x^{n-1}}}_{\text{expected (purchase hazard rate) } \times \text{ (}\Delta \text{ profit from another purchase)}} + \underbrace{\frac{\int (\mathcal{W}_x^S - (v-a)) (n-1) s \gamma_x^{n-1}}{\int \gamma_x^{n-1}}}_{\text{expected (sale hazard rate) } \times \text{ (}\Delta \text{ profit from another sale)}} \quad (24)$$

$$\dot{b} = \underbrace{\lambda \Pi_y^S}_{\lambda E} + \underbrace{\frac{\int ((b-v) - \mathcal{W}_y^B) (n-1) p \gamma_y^{n-1}}{\int \gamma_y^{n-1}}}_{\text{expected (purchase hazard rate) } \times \text{ (}\Delta \text{ profit from another purchase)}} + \underbrace{\frac{\int ((b-v) - \mathcal{W}_y^S) (n-1) s \gamma_y^{n-1}}{\int \gamma_y^{n-1}}}_{\text{expected (sale hazard rate) } \times \text{ (}\Delta \text{ profit from another sale)}} \quad (25)$$

Here, E is the profit loss or gain from nature's exogenous termination of the market.

REMARKS:

1. Just as the competitive auction model had a single degree of freedom in the choice of a_{τ_0} , in the market maker model there are two degrees of freedom in (a_{τ_0}, b_{τ_0}) .

2. As before, strict individual rationality implies that the ask price is rising and the bid price falling in the one trader game, since there are no continuation values. To be more definite about the n -trader game ($n > 1$), I need the sum of the final two terms to be negative in (24) and positive in (25). This result, however, remains a conjecture as I cannot yet sign these terms.

3. Intuitively (to me), individuals strictly prefer that an opposite side trade just precede them, but just wish to pre-empt same-side trades (as with the competitive auction model), i.e. the second term is negative and the third positive in (24), with the reverse true of (25). I am forced to leave open this intriguing question.

5. IS FULL SEPARATION POSSIBLE?

In this section, I consider the extent of separation. This is a very fundamental issue: *If the market is open long enough, is all information eventually impounded in the price?* As discussed earlier, GM tantalizingly hint at the impossibility of full-separation in the common value setting, and it was certainly true of their model. This would provide a sharp contrast to the theory developed in Wilson (1985), who showed that in standard double auction environments, full separation is incentive compatible.

5.1 The Competitive Auction Model

This paradigm constrains the action set of the traders, who can only purchase. It so happens that this provides a very natural limit on the extent of entry.

Proposition 2 (Incomplete Separation in the Competitive Auction Model)

In a truncation equilibrium of the n -trader subgame, only those whose signals are good news given the prior g_t^{n-1} ever purchase. Moreover, if a neutral signal \bar{w} exists, then $x_t > \bar{w}$ for all t . In either case, complete separation is thus impossible.

Proof: In any n -buyer truncation equilibrium, (18) yields $\dot{x} < 0$ exactly when

$$\int (v - a)(1 - q)g^n \leq 0 \iff \int (v - a)q\phi_x g^n \geq 0$$

or, equivalently, when either of the following inequality pairs obtains:

$$\mathcal{E}_n[\bar{V}|\mathcal{I}_t, \beta_t^U] \equiv \frac{\int v(1 - q_t)g_t^n}{\int (1 - q_t)g_t^n} \leq a_t \leq \frac{\int vq_t\phi_x g_t^n}{\int q_t\phi_x g_t^n} \equiv \mathcal{E}_n[\bar{V}|\mathcal{I}_t, \beta_t^I] \quad (26)$$

The appendix shows how this implies

$$\bar{v}_t^{n-1} \equiv \frac{\int v g_t^{n-1}}{\int g_t^{n-1}} < a_t < \frac{\int v f_x g_t^{n-1}}{\int f_x g_t^{n-1}}$$

Thus, bad and neutral news given the prior g_t^{n-1} is screened out.

If a neutral signal \bar{w} doesn't exist, then incomplete separation is still in doubt: The density g_t^{n-1} is not constant in time, and so \bar{v}_t^{n-1} might well tend to 0 over time. But I will with probability one eventually reach the one trader game — unless the two

smallest signals are tied, a zero chance event. As soon as $n = 1$, the only informed traders who purchase have signals above weak neutral news given the prior $g_{\tau_0+}^0$ for that subgame. Thus complete separation cannot occur (almost surely). \square

5.2 The Market Maker Model

Proposition 2 has the following immediate extension, whose proof I omit.

Proposition 3 (On Eventual Separation in the Market Maker Model)

In a truncation equilibrium of the n -trader subgame, only those with good (resp. bad) news signals given the prior g_t^{n-1} ever purchase (resp. sell). Moreover, if a neutral signal \bar{w} exists, then $x_t > \bar{w} > y_t$ for all t . So with an arbitrary number of traders, complete separation cannot occur in finite time.

In other words, there is no option value of changing sides if the signal distribution has a neutral signal.

Since $\langle x_t \rangle$ and $\langle y_t \rangle$ are each monotonic, they have well-defined limits as $t \rightarrow \infty$. Observe that eventual separation, or $x_\infty = y_\infty$, does not imply that the chance of informed trade vanishes, since by (1) an *adverse selection problem* (namely $q_\infty > 0$) will arise if the informed traders have entered more slowly than the noise traders. For eventual separation requires that $x_t - y_t$ vanish at a faster than the exponential rate that $e^{-\mu t}$ does. On the other hand, if I do not achieve eventual separation, or $x_\infty > y_\infty$, then necessarily $q_\infty = 1$ by (1), and all trade is choked off. I shall try to resolve which of these possibilities obtains in the next section by characterizing the equilibrium.

Corollary (Switching Sides of the Market)

If a neutral news signal exists, then no traders are ever unsure of which side of the market they will trade on, if any, in future subgames. Conversely, if a neutral news signal does not exist, then for some intervening series of transactions, a trader might well change plans (buy rather than sell, or vice versa).

6. EXISTENCE AND UNIQUENESS

I first must understand how well behaved are the laws of motion governing the state variables. It turns out that the model is sufficiently well behaved that the slopes of the equilibrium paths nicely fan out from any given initial point — at least in the one trader game.

Lemma 3 (Derivative Monotonicity) *In a truncation equilibrium, $\partial \dot{a}/\partial a > 0$, with $\partial \dot{a}/\partial x < 0$ in the one trader game. In the market maker model, additionally: $\partial \dot{b}/\partial b > 0$, with $\partial \dot{x}/\partial y > 0$ and $\partial \dot{y}/\partial x > 0$ if $n > 1$, and $\partial \dot{b}/\partial y < 0$ if $n = 1$. If also a neutral signal exists, then $\partial \dot{x}/\partial x > 0$, $\partial \dot{x}/\partial a < 0$, $\partial \dot{y}/\partial y > 0$, and $\partial \dot{y}/\partial b < 0$.*

REMARK. The signs of partials of x and y with respect to x or y crucially rely on the fact that only informed traders whose signals are good (resp. bad) news will purchase (resp. sell) in equilibrium.

Absent the encumbering continuation values, results for both models are more definite with one than with $n > 1$ traders. I now exploit the above result to provide a proof of existence and uniqueness of a truncation equilibrium for the $n = 1$ trader game. Essentially, this entails nailing down exactly what the initial ask and bid prices are for each subgame. For instance, with extremely precise signals, the initial ask price must be set very high, and the bid price very low.

I argue below that the maximal amount of separation occurs in equilibrium.

Proposition 4 (Existence and Uniqueness) *There exists a truncation equilibrium for either model, which is unique for the one trader subgame. Also, eventually any informed trader with good news purchases in the competitive auction setting, and in the market maker model, any informed trader with bad news eventually sells.*

Proof: I provide a constructive and visual proof, first for the competitive auction model. The idea is to revert to the nonautonomous dynamical system in (x, a) alone, by using time t rather than $u = e^{-\mu t}$, as in figure 1. The uniqueness essentially will arise because the desired equilibrium is saddle point stable — although I make no use of this given the nonautonomous structure.

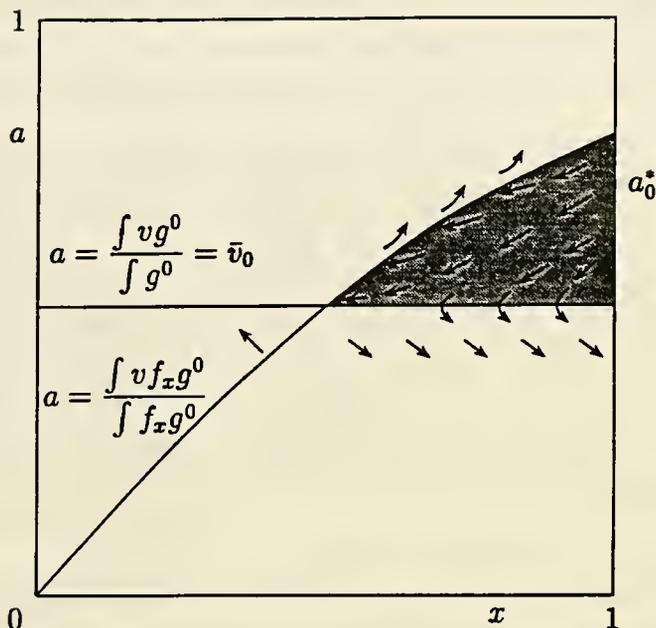


Figure 1: Phase Plane Diagram of One Trader Game. For the competitive auction model with $n = 1$, I project the solutions of (17), (18), and (19) onto x - a space. Notice that a unique equilibrium arises (namely, a stable manifold starting at a_0^* tending to the origin), and it entails the theoretically maximal amount of separation, with convergence to the neutral signal.

In x - a space, a truncation equilibrium is a path with $x_0 = 1$ and $\dot{x} < 0$ for all $t > 0$. Now, $\dot{x} > -\infty$ ²¹ and $\dot{x} \leq 0$ imply

$$\bar{v}^0 \equiv \frac{\int v g^0}{\int g^0} \leq a \leq \frac{\int v f_x g^0}{\int f_x g^0} \quad (27)$$

The only possible destination for the dynamical system is the *intersection point* \mathcal{P} of the two boundaries. For as the dynamical system approaches the upper boundary, \dot{x} starts to explode while \dot{a} vanishes, and so da/dx vanishes; therefore, the paths tend towards that boundary horizontally and fast.²² Conversely, as the system approaches the lower boundary, \dot{x} vanishes but \dot{a} does not, so that da/dx vanishes; therefore, the paths must transverse the boundary vertically. In either case, the system will hit either boundary in finite time, which cannot possibly be an equilibrium (for then x will change sign). Thus, *only if* the system (x, a) approaches \mathcal{P} can $|\dot{x}| + |\dot{a}| \rightarrow 0$ as

²¹This equivalently comes from individual rationality.

²²Note that \dot{x} is discontinuous across the boundary, and so the dynamical system immediately reverses course upon touching it, as seen in the figure.

it must — for since $\partial x/\partial a < 0$ by Proposition 3, a must also come to a rest. The appendix in fact verifies that any such approach must take forever.

By continuity of the dynamical system between these two boundaries, such an approach path (part of the so-called *stable manifold*) must exist. For when a_0 is just below $(\int v f_x g^0)/(\int f_x g^0)$, the system crosses the top boundary, while for a_0 just above $(\int v g^0)/(\int g^0)$, it crosses the lower one. On the other hand, strict monotonicity of the derivatives of the dynamical system implies uniqueness, for the paths monotonically ‘bend down’ as a_0 decreases: Namely, if I compare two paths π and $\hat{\pi}$ starting at ask prices a_0 and $\hat{a}_0 > a_0$ at any later time $t > 0$, x_t is strictly higher and a_t lower for π than for $\hat{\pi}$ — so long as (x_t, a_t) lies strictly between the boundaries (and thus the inequalities apply) for both paths at time t ; therefore, there cannot exist two distinct approach paths to the intersection point.

The market maker problem with $n = 1$ is no more difficult, since the law of motion for (x, y, a, b) decouples into two independent systems in (x, a) and (y, b) , with analogous analyses. The existence proof for $n > 1$ is as yet missing, but will exploit the same fact as above that if the ask or bid prices start too close to the respective boundaries of $\dot{x} \in (-\infty, 0)$ and $\dot{y} \in (0, \infty)$, then very quickly x or y becomes nonmonotonic. Thus, it should be possible to prove that for some choice, both x and y are monotonic.²³ □

REMARK. The uniqueness for $n > 1$ remains an open question, awaiting my sharper understanding of the continuation values. For instance, in the competitive auction model, I only lack demonstration that $\partial \dot{a}/\partial x < 0$ when $n > 1$, since the appendix proofs of all other monotonicity results obtain for general n .

But I can say that if there exists a truncation equilibrium for $n > 1$ (which no doubt there does), then I almost surely eventually reach the one trader game — at which point, the above characterization obtains. Thus, *in any truncation equilibrium of the competitive auction or market maker model with $n > 1$, there necessarily obtains complete separation, with convergence to the neutral signal.*

²³Note that the key to this argument will be the fact that if x and y take forever to reach this point, then they do not hit it at different times.

7. PRICE AND TRADE DYNAMICS

7.1 Inter-Subgame Price Jumps

The intuitive result, true in GM's exogenous entry model, that prices must jump up in response to any purchase is much harder to deduce with endogenous timing. For even with a positive chance that some remaining trader is informed (with $x_t > 0$), it is difficult to conclude that any purchase is good news: A trade may also come from pure liquidity reasons, which garbles its informational content. For by strict FSD of $\langle F(\cdot|v) \rangle$ in v , the chance of informed trade $q_t(v)$ is decreasing in v by (1): The higher is v , the more likely it is that there are fewer informed buyers, given that no one has yet bought in the current subgame.

Fortunately, I can now be more definite.

Lemma 4 (Transactions are Informative) *Assume there exists a neutral signal. Let $x_t > 0$ in the competitive auction model, or $x_t > y_t$ in the market maker model. Then if a neutral signal exists, all $p_t(v)$ is increasing in v and $s_t(v)$ decreasing in v , and so all purchases are good news and all sales bad news about the asset value.*

Proof: I just consider the auction model. Proposition 2 assures us that only traders whose signals are good news will purchase in equilibrium. Thus, Fact 1-4 implies that $f_v(x_t|v) > 0$ for all t , and so the purchase hazard rate

$$p_t(v) = \frac{q_0 f(x_t|v)(-\dot{x}) + (1 - q_0)\mu}{q_0 F(x_t|v) + (1 - q_0)u}$$

is increasing in v , as the numerator is increasing and denominator decreasing. □

Corollary (Bid and Ask Prices) *Thus, $a_t > \bar{v}_t^n > b_t$. Also, both prices jump up following a purchase, and down following a sale.*

Proof: Be definition (9) of an ask price, Lemma 4 and Fact 2.1 then yield $a_t > \bar{v}_t^n$ for all t and n . Next, after a purchase at time τ_0 , the public expectation of the asset value is the immediately preceding ask price, or $\bar{v}_{\tau_0+}^{n-1} = a_{\tau_0}$. But I have just shown that $a_{\tau_0+} > \bar{v}_{\tau_0+}^{n-1}$, and so $a_{\tau_0+} > a_{\tau_0}$, as claimed. □

7.2 Transaction Prices

I wish to extend the martingale property of GM, and show that realized transaction prices are expected to remain constant. To this end, define

- π_k , the k 'th realized *transaction price*, $k = 1, 2, \dots, N$ (if the market doesn't close before everyone has a chance to purchase)

Thus, π_k is a deterministic function of the publicly known first k trade times $\mathcal{I}^k \equiv (\tau_1, \dots, \tau_k)$, namely $\pi_k = \mathcal{E}[\tilde{V}|\mathcal{I}^k]$, but π_{k+1} is clearly a random variable conditional only on the information \mathcal{I}_k known as of the previous purchase. In this notation, the martingale property would assert that $\mathcal{E}[\pi_{k+1}|\mathcal{I}^k] = \pi_k$; however, in light of Propositions 2 and 3, the mere fact that there exists a next transaction, or that $\tau_{k+1} < \infty$, is a valuable piece of information in the competitive auction model: This means that not all remaining signals are bad news. So let's define $\pi_k \equiv \lim_{t \rightarrow \infty} \bar{v}_t$ when $\tau_k = \infty$. Then this allows me to state

Lemma 5 *Transaction prices $\{\pi_{k+1}\}$ are a martingale w.r.t. $\{\mathcal{I}^k\}$: $\mathcal{E}[\pi_{k+1}|\mathcal{I}^k] = \pi_k$.*²⁴

The argument is a standard application of the Law of Iterated Expectations.²⁵

$$\mathcal{E}[\pi_{k+1}|\mathcal{I}^k] = \mathcal{E}[\mathcal{E}[\tilde{V}|\mathcal{I}^{k+1}]|\mathcal{I}^k] = \mathcal{E}[\mathcal{E}[\tilde{V}|\mathcal{I}^k, \tau_{k+1}]|\mathcal{I}^k] = \mathcal{E}[\tilde{V}|\mathcal{I}^k] = \pi_k$$

In section 7, I shall conclude that following any purchase, the ask price must immediately jump up, and then monotonically fall during the next subgame. Thus, with the endogenous timing, the martingale property will have a slightly more pointed implication than its precursor in GM: The initial jump in the ask price at the outset of a subgame is balanced by an expected subsequent gradual decline during the next subgame.

Note that Lemma 5 is *not* an arbitrage argument, since any informed trader entails a risk of the market closing by deferring trade, but no implicit discounting appears in the martingale expression! Rather, it is simply a deduction based on the auctioneer's information structure, and he is not directly affected by the market closing.

²⁴More generally, this is a martingale relative to the auctioneer's information.

²⁵In fact, the martingale result is stronger than that. Namely, I can (and will in the next version) show that $\mathcal{E}[\pi_{k+1}|\mathcal{I}_t] = \bar{v}_t$, the current expected value of the asset.

7.3 Do Rational Traders Frenzy?

I now come to the motivational question for this paper, which I am hopeful can be tackled given the theory I have developed to this point. Consider the competitive auction model, and imagine that a purchase occurs at time t . How do informed traders with signals just below x_{τ_0} (and thus just about to enter) respond? On the one hand, favorable information about the asset value has just been released, which one might presume provides an incentive to jump into the fray sooner. But this urge is tempered by the immediate jump in the ask price. Which effect dominates?

First note that frenzying is an *equilibrium* phenomenon, and it is not possible to deduce it on the basis of one side of the market alone. For instance, let a purchase occur at time τ_0 . Recall from (18) that

$$\dot{x} = \frac{\mu \int (v - a)(1 - q)g^n}{\int (v - a)q\phi_x g^n}$$

Lemma 3 tells us that $\partial \dot{x} / \partial a < 0$. So when the ask price jumps up, this forces down \dot{x} at time τ_0 . The posterior on the valuation is also updated from $g_{\tau_0}^n(v) \mapsto g_{\tau_0}^{B,n}(v) \propto p_{\tau_0}(v)g_{\tau_0}^n(v)$ in the \dot{x} expression. By Lemma 4, $p(v)$ is increasing in v , and so by Fact 2.1, this discretely *increases* \dot{x}_{τ_0+} . Thus, it is not obvious whether in fact $\dot{x}_{\tau_0+} < \dot{x}_{\tau_0}$, or equivalently $|\dot{x}_{\tau_0+}| > |\dot{x}_{\tau_0}|$.

Let's briefly outline a possible avenue of thinking about this problem. I believe I can show that after a purchase occurs, the shift $g_{\tau_0}^n(v) \mapsto g_{\tau_0}^{B,n}(v)$ causes \dot{a} to fall. That is, there *is* frenzying on the part of the market maker. For indeed, the basic integral inequality can be used to show that both of the externality terms strictly diminish, irrespective of whether a war of attrition or a preemption game arises; less obviously so, I believe (but cannot yet prove) that the profits to the marginal trader also discretely increase from the shift. Indeed, following a price jump, an informed trader with a nearby signal places a higher weight on the informed status of the purchase being than does the market maker, and thus ought to be more encouraged to enter by the purchase than she is discouraged by the higher ask price. By the same token, someone about to sell places a lower weight on the informed status of the transaction than the market maker, and thus the bid price rise should lower profits for those about to sell. But I need a proof of this inequality.

Let's assume that I in fact have $\dot{a} < 0$ in equilibrium. As it turns out, I have not yet been able to demonstrate this intuitive property of the equilibrium. Now $\dot{x} = \dot{a}/(da/dx)$, and I have just argued that the numerator of the RHS grows in absolute size. Unfortunately, with the higher ask price that will obtain in the continuation subgame, the required slope da/dx also grows. This is the equilibrium aspect of the analysis, and relies on simple geometry, and may not admit a tight proof. At any rate, this means that it is not at all obvious that \dot{x} falls, and thus frenzying is not clear.

REMARK. Regardless of whether frenzying occurs or not, by (30), the new posterior density g_t^{n-1} immediately but continuously starts to diverge from the prior density $g_{\tau_0}^{B,n}$ for the next subgame at time $t > \tau_0$ by the factor

$$g_t^{n-1}(v)/g_{\tau_0}^{B,n}(v) = \left(\frac{q_0 F(x_t | v) + (1 - q_0)u_t}{q_0 F(x_{\tau_0} | v) + (1 - q_0)u_{\tau_0}} \right)^{n-1} \quad (28)$$

The appendix proves in fact that in the competitive auction model, this term is diminishing in v when $t > \tau_0$, and so by Fact 2.1, the effect of this is to gradually increase \dot{x}_t (that is, lower $|\dot{x}|$), thus putting a damper on any frenzy that occurs, or accelerating any anti-frenzying.²⁶

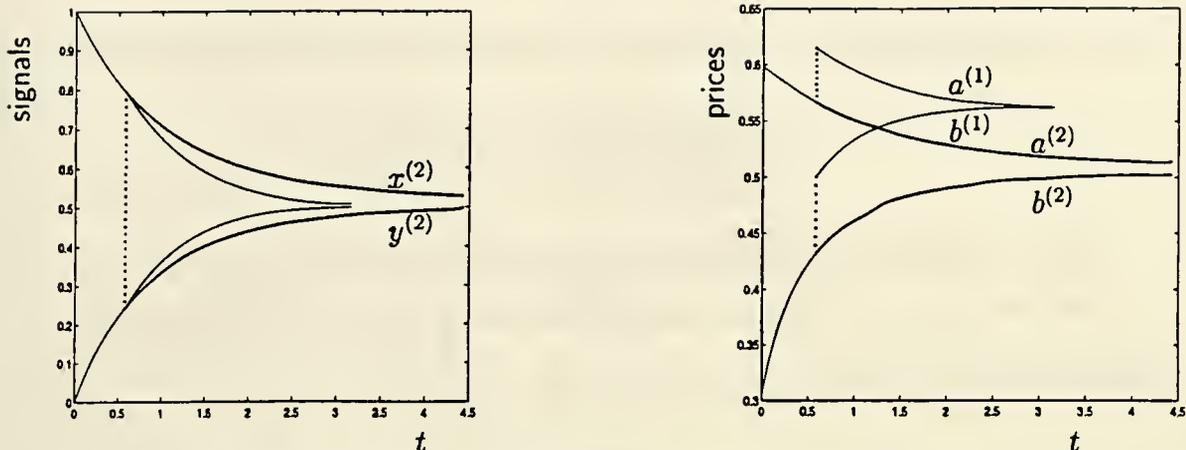
8. A WORKED EXAMPLE

In this section I plan to illustrate the general solution developed up to this point, turning to a simple example with two states of the world. In other words, I temporarily abandon the assumption of an atomless distribution for \tilde{V} , upon which none of the results depended. I shall let g_0 be a two-point distribution that places equal probabilities on the values 0 and 1, and use the two signal densities $f(x | 0) = 1$ and $f(x | 1) = 2x$, both defined on $[0, 1]$. These signal distributions belong to the family of generalized uniform distributions $F(x) = x^{\frac{1}{1-\alpha}}$, $\alpha \in [0, 1]$.

Observe that in the simulation of this example below, rational traders do frenzy after a purchase.

(to be continued)

²⁶While the proof does not work for the market maker model, I presume the result still holds.



1

Figure 2: **Frenzying after a Purchase.** Here is graphed the dynamic behavior of the prices (a, b) and the entering signals (x, y) against time in the model with two traders. A purchase occurs at time .58, and in response both ask and bid prices jump up, as expected. Also, x and y accelerate: the entry rate of both buyers and sellers rises, and convergence of $x - y$ is much faster.

A. APPENDIX

A.1 Useful Mathematical Facts

Fact 1 (MLRP) *Let $\langle f(w|v) \rangle$ be a smooth family of conditional densities. Then*

- (1) *$\langle f(w|v) \rangle$ has strict MLRP $\Leftrightarrow f_v(w|v)/f(w|v)$ increasing in w , or equivalently f is log-supermodular: $f(w|v)f_{wv}(w|v) > f_w(w|v)f_v(w|v)$*
- (2) *$\langle f(w|v) \rangle$ has strict MLRP $\Rightarrow f(w|v)/F(w|v)$ increasing in v , or equivalently the distribution function F is log-supermodular: $F(w|v)F_{wv}(w|v) > F_w(w|v)F_v(w|v)$*
- (3) *$\langle f(w|v) \rangle$ has strict MLRP $\Rightarrow F(\cdot|v)$ is ordered in v by strict first order stochastic dominance (FSD), or $F(w|v)$ decreasing in v if $F(w|v) < 1$*
- (4) *$\langle f(w|v) \rangle$ has strict MLRP, and a neutral news signal exists $\Rightarrow f_v(x|v) \geq 0$ as the signal x is good news or bad news*

Part (1) is due to Milgrom (1981), and part (3) is trivial. Part (2), less widely known, follows by means of an integral inequality. Finally, to see part (4), let x be good news. Then $f(x|v)/f(x|v') > 1$ iff $v > v'$, and so $f_v(x|v)$ follows.

Another result is used sufficiently often that it is summarized for easy reference.

Fact 2 (Key Integral Inequalities) (1) Let $\psi'(v) > 0$ and $N'(v) > 0$. Then

$$\frac{\int_{V_0}^{V_1} \psi(v)N(v)dv}{\int_{V_0}^{V_1} \psi(v)D(v)dv} > \frac{\int_{V_0}^{V_1} N(v)dv}{\int_{V_0}^{V_1} D(v)dv}$$

if the numerator and denominator in both cases are positive and $D'(v) \leq 0$, or if both numerators are positive and both denominators negative and $D'(v) \geq 0$.

(2) Let $N(v)/D(v)$ be increasing in v , and $N(v), D(v) > 0$. Then

$$N(V_1)/D(V_1) > \frac{\int_{V_0}^{V_1} N(v)dv}{\int_{V_0}^{V_1} D(v)dv} > N(V_0)/D(V_0)$$

A.2 Proof of Lemma 1: Accounting Laws of Motion

I now describe the evolution of q_t and g_t as time elapses with no trade. It is simplest to first consider q . If no trade has occurred between t and $t + \Delta$, then I may use Bayes' Rule to update q_t pointwise, as follows:

$$\begin{aligned} q_{t+\Delta}(v) &= \frac{\frac{F(x_{t+\Delta}|v)}{F(x_t|v)} q_t(v)}{\frac{F(x_{t+\Delta}|v)}{F(x_t|v)} q_t(v) + \exp(-\mu\Delta)(1 - q_t(v))} \\ &= \left[1 + e^{-\mu\Delta} \left(\frac{1 - q_t(v)}{q_t(v)} \right) \left(\frac{F(x_t|v)}{F(x_{t+\Delta}|v)} \right) \right]^{-1} \\ \frac{1 - q_{t+\Delta}(v)}{1 - q_t(v)} &= e^{-\mu\Delta} \left(\frac{q_{t+\Delta}(v)}{q_t(v)} \right) \left(\frac{F(x_t|v)}{F(x_{t+\Delta}|v)} \right) \end{aligned}$$

After taking logs, dividing by Δ , and letting $\Delta \rightarrow 0$, I obtain (4) via l'Hôpital's rule.

The law of motion for $g_t(v)$ within the n -trader subgame uses this result and the following consequence of Bayes' rule:

$$g_{t+\Delta}(v) = \frac{\left((1 - q_t(v))e^{-\mu\Delta} + q_t(v) \frac{F(x_{t+\Delta}|v)}{F(x_t|v)} \right)^n g_t(v)}{\int_{V_0}^{V_1} \left((1 - q_t(z))e^{-\mu\Delta} + q_t(z) \frac{F(x_{t+\Delta}|z)}{F(x_t|z)} \right)^n g_t(z) dz}$$

Taking logs and limits as before, I obtain

$$\begin{aligned}
\frac{\dot{g}_t(v)}{g_t(v)} &= \lim_{\Delta \rightarrow 0} \frac{n \log \left((1 - q_t(v))e^{-\mu\Delta} + q_t(v) \frac{F(x_t + \Delta | v)}{F(x_t | v)} \right)}{\Delta} \\
&\quad - \lim_{\Delta \rightarrow 0} \frac{\log \int_{V_0}^{V_1} \left((1 - q_t(z))e^{-\mu\Delta} + q_t(z) \frac{F(x_t + \Delta | z)}{F(x_t | z)} \right)^n g_t(z) dz}{\Delta} \\
&= n [-\mu + q_t(v) (\mu + \phi(x | v) \dot{x})] - n \left[-\mu + \int_{V_0}^{V_1} q_t(z) (\mu + \phi(x | z) \dot{x}) g_t(z) dz \right] \\
&= n \left[\frac{\dot{g}_t(v)}{1 - q_t(v)} - \int_{V_0}^{V_1} \frac{\dot{g}_t(z)}{1 - q_t(z)} g_t(z) dz \right] \quad \square
\end{aligned}$$

A.3 Proof of Proposition 1: Truncation Equilibria of the Auction Model

From the ask price equation (9),

$$a_t = \frac{\int v p_t(v) g_t^n(v) dv}{\int p_t(v) g_t^n(v) dv} = \frac{\int v [q_t(v) \phi(x_t | v) (-\dot{x}_t) + (1 - q_t(v)) \mu] g_t^n(v) dv}{\int [q_t(v) \phi(x_t | v) (-\dot{x}_t) + (1 - q_t(v)) \mu] g_t^n(v) dv}$$

where I have substituted from (7) for the purchase hazard rate p . This yields equation (18) for \dot{x} upon simplification.

Next, the law of motion for a comes from the informed traders' IC equation (16), which I simplify in parts. Firstly, I need to know

$$\left[\frac{\partial}{\partial t} \Pi^B(\mathcal{I}_t, w) \right]_{w=x_t} = \left[\frac{\partial}{\partial t} \int z \gamma_t(w, z) dz \right]_{w=x_t} = \frac{\int \gamma_x \int z \dot{\gamma}_x - \int \dot{\gamma}_x \int z \gamma_x}{(\int \gamma_x)^2}$$

Now, (5) can be rewritten using (4) as

$$\dot{g}_t^{n-1}(v) = (n-1) g_t^{n-1}(v) \left[q_t(v) (\mu + \phi(x_t | v) \dot{x}_t) - \int_{V_0}^{V_1} q_t(z) (\mu + \phi(x_t | z) \dot{x}_t) dz \right]$$

As the right hand integral is constant in v , it cancels in $\int \gamma_x \int z \dot{\gamma}_x - \int \dot{\gamma}_x \int z \gamma_x$, which therefore expands to

$$\begin{aligned}
&\int \gamma_x \int z f_x \dot{g}^{n-1} - \int z \gamma_x \int f_x \dot{g}^{n-1} \\
&= \int \gamma_x \int z f_x (n-1) g^{n-1} q (\mu + \phi_x \dot{x}) - \int z \gamma_x \int f_x (n-1) g^{n-1} q (\mu + \phi_x \dot{x})
\end{aligned}$$

$$= \int \gamma_x \int z \gamma_x (n-1) q(\mu + \phi_x \dot{x}) - \int z \gamma_x \int \gamma_x (n-1) q(\mu + \phi_x \dot{x})$$

Thus, equation (16) becomes

$$\begin{aligned} \dot{a} &= -\lambda \Pi_x^B + (n-1) \frac{\int [q \phi_x(-\dot{x}) + (1-q)\mu] \gamma_x}{\int \gamma_x} \left[\mathcal{W}_x^B - \frac{\int z \gamma_x}{\int \gamma_x} + a \right] \\ &\quad + \frac{\int \gamma_x \int z \gamma_x (n-1) q(\mu + \phi_x \dot{x}) - \int z \gamma_x \int \gamma_x (n-1) q(\mu + \phi_x \dot{x})}{(\int \gamma_x)^2} \\ &= -\lambda \Pi_x^B + (n-1) \frac{\int (\mathcal{W}_x^B - (z-a)) (q \phi_x(-\dot{x}) + (1-q)\mu) \gamma_x}{\int \gamma_x} \end{aligned}$$

which yields the desired equation (19). \square

A.4 Completion of Proof of Lemma 2

STEP 1: HIGHER TYPES PURCHASE FIRST. Start with the one-trader subgame. The idea is in the spirit of many related results in auction design, only here it is common value. Think of the trader of type w as sending a message, namely the probability $x = e^{-\lambda \tau}$ of acquiring the good, where τ is the time to purchase, and having to pay a transfer $T = a_\tau e^{-\lambda \tau}$, and receiving an expected reward

$$u(x, T, w) = x \mathcal{E}_w[\tilde{V}] - T$$

For a fixed x and T , higher types place a greater likelihood on the asset value being high than do lower types. Thus, the Spence-Mirrlees single-crossing property applies: $\partial(u_x/u_T)/\partial w = -\partial \mathcal{E}_w[\tilde{V}]/\partial w < 0$. Standard results (see, for instance, Theorem 7.2 in Fudenberg and Tirole (1991)) imply that the only implementable (and thus potentially equilibrium) separating outcomes are weakly monotonic in x , or $\dot{x} \leq 0$.

Next, with $n > 1$ traders, the informed trader with the lowest signal cannot be the first to enter, for then $\mathcal{E}[\tilde{V} | \beta_t^I, \mathcal{I}_t] < \mathcal{E}[\tilde{V} | \beta_t, \mathcal{I}_t] \equiv a_t$. This violates individual rationality of that informed trader, as the purchase earns her negative profits in expectation. Next suppose, by way of contradiction, that some trader $\underline{w} \in (0, 1)$ enters first, again by the logic that she is more confident of an early entry by other informed traders with signals close to \underline{w} (and the putative bad continuation payoff). But then for small enough $\varepsilon > 0$, a trader with signal $\underline{w} + \varepsilon$ has a first order increase

in expected profits from entry, and only a second order fall in the private beliefs of entry $\hat{p}_t(\underline{w} + \varepsilon)$. This follows from (13) and the most likely signal property. The earlier single-crossing argument once more will tell us that if \underline{w} has a weak incentive to enter, then $\underline{w} + \varepsilon$ has a strict incentive to do so.

STEP 2: AN ATOM OF INFORMED TRADERS NEVER PURCHASES. With a sudden influx of informed traders, the inelastic noise trade is swamped at that instant in time, and so the standard ‘no-trade’ result ought to preclude this in equilibrium. Without being overly formal: The ask price will coincide with the expected value of the asset conditional on which types are buying. If more than one type is buying, the profits of such a transaction must be negative for the lowest type. Thus, only one (atomless) type can enter at any ‘instant’.

STEP 3: $\dot{x}_t < 0$ ALWAYS HOLDS. I want to rule out ‘flats’ in x . For if $\dot{x}_t = 0$, then the noise trade swamps the informed trade, and the ask price coincides with the expected value of the asset. But then marginally higher types $x_t + \varepsilon$ must have purchased at an ask price that is discontinuously higher (i.e. when $\dot{x}_{x_t+\varepsilon} < 0$). For small enough $\varepsilon > 0$, this could not have been incentive compatible. \square

A.5 Proof of Key Inequality for Proposition 2: Incomplete Separation

First notice that the $>, >$ inequalities in (26) cannot obtain: The ask price must lie below the expected value given an informed purchase (and thus exceed the expected value given an uninformed purchase). This follows more formally from

$$a_t < \frac{\int v f_x g_t^{n-1}}{\int f_x g_t^{n-1}} = \frac{\int v q_t \phi_x [q_0 F_{x_{\tau_0}} + (1 - q_0) u_{\tau_0}] g_t^n}{\int q_t \phi_x [q_0 F_{x_{\tau_0}} + (1 - q_0) u_{\tau_0}] g_t^n} \leq \frac{\int v q_t \phi_x g_t^n}{\int q_t \phi_x g_t^n} \quad (29)$$

since (in order)

- by the informed traders’ strict individual rationality, or $a_t < \Pi_t^B(x_t)$
- equation (3b) and then (1) and (2) permit the implication

$$\begin{aligned} g_t^n(v) &\propto \left(\frac{q_0 F(x_t | v) + (1 - q_0) u_t}{q_0 F(x_{\tau_0} | v) + (1 - q_0) u_{\tau_0}} \right) g_t^{n-1}(v) \\ \implies f(x_t | v) g_t^{n-1}(v) &\propto q_t(v) \phi_t(x_t | v) g_t^n(v) [q_0 F(x_{\tau_0} | v) + (1 - q_0) u_{\tau_0}] \end{aligned} \quad (30)$$

- by Fact 1.3, $F(x_{\tau_0} | v)$ is decreasing in v when $x_{\tau_0} < 1$, and constant in v if $x_{\tau_0} = 1$ — and thus equality holds, for instance, if $n = 1$

Thus, the $<, <$ inequalities must obtain. Consequently,

$$\bar{v}_t^{n-1} \equiv \frac{\int v g_t^{n-1}}{\int g_t^{n-1}} \leq \frac{\int v [q_0 F_x + (1 - q_0)u]^{-1} g_t^n}{\int [q_0 F_x + (1 - q_0)u]^{-1} g_t^n} = \frac{\int v(1 - q_t) g_t^n}{\int (1 - q_t) g_t^n} < a_t < \frac{\int v f_x g_t^{n-1}}{\int f_x g_t^{n-1}} \quad (31)$$

The only nontrivial step above is the first inequality: It ensues from (30) and Fact 1.3. This is the desired inequality. \square

A.6 Proof of Lemma 3: Derivative Monotonicity

I prove the inequalities one at a time, and analogize proofs whenever possible — although the techniques used for the market maker model are more tricky than for the auction model.

CLAIM 1: $\partial \dot{x} / \partial x > 0$ AND $\partial \dot{y} / \partial y > 0$. Consider the first inequality in the auction model. Equation (3b) allows me to rewrite (18) as

$$\dot{x} \propto \frac{\int (v - a) [q_0 F_x + (1 - q_0)u]^{n-1} h}{\int (v - a) f_x [q_0 F_x + (1 - q_0)u]^{n-1} h} \quad (32)$$

where the density $h(v) = [q_0 F(x_{\tau_0+} | v) + (1 - q_0)u]^{1-n} g_{\tau_0+}(v)$, and cancellation of terms here is similar to the proof of Proposition 2. Note that the proportionality refers to the variable x , and thus the sign of the partial derivative in x of the two sides must be the same. Now, if I show that whenever $v_H > v_L$,

$$\frac{\partial}{\partial x} \frac{f(x|v_H)}{f(x|v_L)} > 0 \quad \text{and} \quad \frac{\partial}{\partial x} \left(\frac{q_0 F(x|v_H) + (1 - q_0)u}{q_0 F(x|v_L) + (1 - q_0)u} \right) > 0$$

then it follows that the negative numerator and positive denominator of (32) both are increasing in x , and thus $\partial \dot{x} / \partial x > 0$, by Fact 2.1. But the first inequality is a statement of the MLRP, while the second is also true:

$$\begin{aligned} & [q_0 F(x|v_L) + (1 - q_0)u] q_0 f(x|v_H) - [q_0 F(x|v_H) + (1 - q_0)u] q_0 f(x|v_L) > 0 \\ \iff & q_0 [F(x|v_L) f(x|v_H) - F(x|v_H) f(x|v_L)] + (1 - q_0)u [f(x|v_H) - f(x|v_L)] > 0 \quad (33) \end{aligned}$$

$$\iff F(x|v_L) > F(x|v_H) \text{ and } f(x|v_H) > f(x|v_L)$$

where $f(x|v_H)/f(x|v_L) > 1$ obtains because x is good news.

Now suppose I am in the market maker model. The first term in (33) is now

$$\begin{aligned} & q_0 f(x|v_H)[F(x|v_L) - F(y|v_L)] - q_0 f(x|v_L)[F(x|v_H) - F(y|v_H)] \\ &= q_0 f(x|v_H) ([F(x|v_L) - F(\bar{w}|v_L)] + [F(\bar{w}|v_L) - F(y|v_L)]) \\ & \quad - q_0 f(x|v_L) ([F(x|v_H) - F(\bar{w}|v_H)] + [F(\bar{w}|v_H) - F(y|v_H)]) \end{aligned}$$

and the latter difference is positive, because Fact 2.2 and $f(x|v_H) > f(x|v_L)$ yields

$$\frac{F(\bar{w}|v_L) - F(y|v_L)}{F(\bar{w}|v_H) - F(y|v_H)} > 1 > \frac{F(x|v_L) - F(\bar{w}|v_L)}{F(x|v_H) - F(\bar{w}|v_H)} > \frac{f(x|v_L)}{f(x|v_H)}$$

CLAIM 3: $\partial \dot{x}/\partial y > 0$ AND $\partial \dot{y}/\partial x > 0$ IF $n > 1$. The proofs are similar to the latter part of the proof of claim 2.

CLAIM 4: $\partial \dot{x}/\partial a < 0$ AND $\partial \dot{y}/\partial b < 0$. To see $\partial \dot{x}/\partial a < 0$, differentiate (18):

$$\begin{aligned} & \frac{\partial}{\partial a} \left(\frac{\mu \int (v-a)(1-q)g^n}{\int (v-a)q\phi_x g^n} \right) < 0 \\ \iff & - \int (v-a)q\phi_x g^n \int (1-q)g^n + \int (v-a)(1-q)g^n \int q\phi_x g^n < 0 \\ \iff & \frac{\int (v-a)(1-q)g^n}{\int (1-q)g^n} < \frac{\int (v-a)q\phi_x g^n}{\int q\phi_x g^n} \end{aligned}$$

which is true since $<, <$ obtains in (26).

CLAIM 5: $\partial \dot{a}/\partial a > 0$ AND $\partial \dot{b}/\partial b > 0$. The first follows immediately from inspection of (19), and the second is analogous.

CLAIM 6: $\partial \dot{a}/\partial x < 0$ AND $\partial \dot{b}/\partial y < 0$ IF $n = 1$. Again, the two inequalities are similar, so let's consider just the first. In this case, the second term of the \dot{a} expression vanishes, and what remains in (20) is obviously decreasing in x , by the strict MLRP.

REMARK. The continuation values in (19) make it impossible (for us) to sign the partial derivatives of a and b w.r.t. x and y when $n > 1$. This means that I just fall short of a proof of existence and uniqueness for $n > 1$. \square

A.7 Completion of Proof of Proposition 4

INFINITE TIME DURATION. I wish to show that the approach to the intersection point takes the required infinite amount of time. To see this, consider the fact that on the upper boundary, $\dot{x} = -\infty$ and $\dot{a} = 0$, while on the horizontal line, $\dot{x} = 0$ while \dot{a} is bounded. This suggests looking at the product

$$\dot{x}\dot{a} = \frac{\mu \int (v-a)(1-q)g^1}{\int f_x g^0} \propto e^{-\mu t} \int (v-a)g^0$$

where \propto refers to a and t . But $da/dx = \dot{a}/\dot{x}$ is boundedly finite near the intersection point, as the slope of the upper boundary is finite. Thus, \dot{a} vanishes at most at an extremely slow exponential rate near the intersection point, and so necessarily $a > \bar{v}^0$ for all finite time.

EQUILIBRIUM OF THE MARKET MAKER MODEL. I shall just address the harder market maker model for $n > 1$. The proof is by induction, and it suffices to consider the very first truncation equilibrium where $x_0 = 1$, $y_0 = 0$. I need $\dot{x} < 0$ and $\dot{y} > 0$ for all $t > 0$, and thus I must converge to

This yields the extra constraint

$$\frac{\int v f_y g^0}{\int f_y g^0} \leq b \leq \frac{\int v g^0}{\int g^0} \quad (34)$$

with the derivative \dot{y} exploding as (y, b) approaches the lower boundary and vanishing as (y, b) approaches the upper one. Once more, the dual requirement that x and y settle down rules out all but the intersection point \mathcal{P} of these boundaries as a possible destination point — with convergence by (x, a) and (y, b) from opposite sides. To see that such an equilibrium must exist, consider the locus of exit points of paths from the region \mathcal{R} in (x, y, a, b) -space defined by (27) and (34). The exit point is clearly a continuous function of (a_0, b_0) .

Also, just as in the competitive auction proof, for any $b_0 \in (\int v f_0 g^0 / \int f_0 g^0, \bar{v}^0)$, the exit from \mathcal{R} occurs when (x, a) hits the upper boundary for a_0 just below it, and when (x, a) hits the lower horizontal line $a = \bar{v}^0$ when a_0 is just above it. Likewise, for any $a_0 \in (\bar{v}^0, \int v f_1 g^0 / \int f_1 g^0)$, the exit from \mathcal{R} occurs when (y, b) hits the lower boundary when b_0 is just above it, and hits the upper horizontal line $b = \bar{v}^0$ when

b_0 is just below it. Since the exit locus is a continuous image of a connected set (in (a_0, b_0) -space), it must be connected. Thus, for some (a_0, b_0) , this first exit from \mathcal{X} occurs when (x, a) hits \mathcal{P} . But by the earlier logic from the competitive auction proof, this takes infinite time, and because the (y, b) path can only slow down to zero near \mathcal{P} , (y, b) must also tend to \mathcal{P} along that same path also in infinite time.

A.8 Dousing the Frenzy: Follow-up to Remark in Section 7.3

I wish to show that (28) is diminishing in v , and so it suffices to establish that

$$\begin{aligned} & [q_0 F(x_{\tau_0} | v) + (1 - q_0) u_{\tau_0}] F_v(x_t | v) - [q_0 F(x_t | v) + (1 - q_0) u_t] F_v(x_{\tau_0} | v) < 0 \\ \Leftrightarrow & (1 - q_0) u_t [F_v(x_{\tau_0} | v) - F_v(x_t | v)] + (1 - q_0) (u_{\tau_0} - u_t) F_v(x_{\tau_0} | v) \\ & + q_0 \left(\frac{F_v(x_t | v)}{F(x_t | v)} - \frac{F_v(x_{\tau_0} | v)}{F(x_{\tau_0} | v)} \right) F(x_{\tau_0} | v) F(x_t | v) < 0 \end{aligned}$$

Since $u_t < u_{\tau_0}$, the second term is negative since $F_v < 0$ always (Fact 1.3). Next, because $x_t < x_{\tau_0}$, Fact 1.2 tells us that the third term is negative. Finally, Fact 1.4 implies that $f_v(x|v) > 0$ whenever x is a good news signal, which is true in equilibrium. This implies that the first term is negative because

$$F_v(x_{\tau_0} | v) - F_v(x_t | v) = \int_{x_t}^{x_{\tau_0}} f_v(w|v) dv \quad \square$$

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