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ESTIMATION OF LINEAR RATIONAL EXPECTATIONS MODELS, IN THE PRESENCE OF DETERMINISTIC TERMS

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Hansen and Sargent (1981b) discussed instrumental variables procedures for estimating linear rational expectations models, under the assumption that all variables have zero unconditional means. This note points out two implications if variables instead have nonzero unconditional means: first, there are additional restrictions beyond those noted by Hansen and Sargent (1981b), and second, imposing these restrictions results in more efficient estimates of the parameters that are the focus of Hansen and Sargent (1981b). Explicit formulas are given for the restrictions generated by some commonly assumed deterministic terms.

Hansen and Sargent (1981b) discussed instrumental variables procedures for estimating a broad class of linear rational expectations models. Their analysis assumed that all variables have zero unconditional means. This is of course not true for most macroeconomic data, including the data that seem appropriate for the three examples that Hansen and Sargent use to motivate their analysis – Cagan money demand [e.g., Sargent (1981a)], dynamic labor demand [e.g., Sargent (1981b)], permanent income model of consumption [e.g., Sargent (1978)].

This fact of course does not lessen the relevance of Hansen and Sargent (1981b), since researchers can either remove deterministic terms prior to estimation [e.g., Sargent (1981b)] or include them but leave their coefficients unconstrained [e.g., Sargent (1981a)]. But it does leave open the question of whether there is anything to be learned from the deterministic terms that typically are ignored.[1] The answer is yes.

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[1]Apart from West (1987), Sargent (1978) is the only relevant application that I am aware of that imposes restrictions on deterministic terms. Neither paper, however, notes that imposing such restrictions increases the efficiency of the estimates of coefficients on stochastic terms, nor develops general formulas for cross-equation restrictions, nor even illustrates the point that there are in general such cross-equation restrictions. Ogaki (1988, 1989) has independently made the point that in rational expectations models deterministic terms can supply information about parameters of economic interest. His models, however, do not fall in the class considered in this paper, so the technical details of his restrictions are different than those developed here.

To illustrate this, consider a simple version of the Cagan demand for money, one of Hansen and Sargent's (1981b) examples. Let p_t be the log price level, x_t the log money supply, E the expectations operator (assumed equivalent to linear projections), and Ω_t the information set used by agents in forming expectations. [Here and throughout, notation usually, though not always, matches Hansen and Sargent (1981b).] The Cagan demand for money is

$$x_t - p_t = f - \alpha[(\mathrm{E}p_{t+1} \mid \Omega_t) - p_t] + a_t, \tag{1}$$

where a constant term ($f \neq 0$) is the only deterministic term allowed in this simple example, α is a positive parameter, and a_t is a (possibly difference-stationary) demand shock.

Upon imposing the terminal conditions $\lim_{j\to\infty} \mathrm{E}\{[\alpha(1+\alpha)^{-1}]^j x_{t+j} \mid \Omega_t\} = 0$ and $\lim_{j\to\infty} \mathrm{E}\{[\alpha(1+\alpha)^{-1}]^j a_{t+j} \mid \Omega_t\} = 0$, eq. (1) may be solved forward to get

$$p_t = -(1+\alpha)^{-1}\mathrm{E}\left[\left\{\sum_{j=0}^{\infty}\left[\alpha(1+\alpha)^{-1}\right]^j f - \sum_{j=0}^{\infty}\left[\alpha(1+\alpha)^{-1}\right]^j x_{t+j} + \sum_{j=0}^{\infty}\left[\alpha(1+\alpha)^{-1}\right]^j a_{t+j}\right\}\middle| \Omega_t\right]. \tag{2}$$

The simplest setup to illustrate my basic point assumes that the univariate x_t process follows an AR(1) (possibly a random walk), that $\mathrm{E}a_t x_s = 0$ for all t and s, and that instruments are chosen from the information set $\Phi_t \equiv \{1, x_t, x_{t-1}, x_{t-2}, \ldots\}$. This leads to a three-equation system:

$$x_t - p_t = f - \alpha(p_{t+1} - p_t) + v_{1t}, \tag{3a}$$

$$x_{t+1} = m_0 + \gamma_1 x_t + v_{2t}, \tag{3b}$$

$$p_t = -(1+\alpha)^{-1}\mathrm{E}\left\{\sum_{j=0}^{\infty}\left[\alpha(1+\alpha)^{-1}\right]^j (f - x_{t+j} + a_{t+j}) \mid \Phi_t\right\} + v_{3t}$$

$$\equiv d_0 + \psi_0 x_t + v_{3t}, \tag{3c}$$

where

$$v_{1t} = a_t - \alpha(\mathrm{E}p_{t+1} \mid \Omega_t - p_{t+1}),$$

$$v_{2t} = x_{t+1} - \mathrm{E}x_{t+1} \mid \Phi_t,$$

$$v_{3t} = +(1+\alpha)^{-1}\left\{ \sum_{j=0}^{\infty} \left[\alpha(1+\alpha)^{-1}\right]^j \left[\mathrm{E}x_{t+j} \mid \Omega_t - \mathrm{E}x_{t+j} \mid \Phi_t\right] \right\}$$

$$-(1+\alpha)^{-1}\left\{ \sum_{j=0}^{\infty} \left[\alpha(1+\alpha)^{-1}\right]^j \left[\mathrm{E}a_{t+j} \mid \Omega_t\right] \right\},$$

$$\psi_0 = \psi_0(\alpha, \gamma_1) \equiv +(1+\alpha)^{-1}\left[1 - \alpha(1+\alpha)^{-1}\gamma_1\right]^{-1}.$$

[see Hansen and Sargent (1981a)] and [see below]

$$d_0 = d_0(f, m_0, \alpha, \gamma_1)$$

$$\equiv -f - \left\{\left[\alpha(1+\alpha)^{-1}\right] / \left[1 - \alpha(1+\alpha)^{-1}\gamma_1\right]\right\} m_0. \tag{4}$$

Note that d_0 depends not only on m_0 and f but on γ_1 and α as well: the relationship between unrestricted estimates of d_0, m_0, and f gives information about the values of γ_1 and α. Thus, imposing the restriction across the constant terms increases the efficiency of the estimates of γ_1, α, and ψ_0 relative to an estimation procedure that ignores these restrictions.[2]

The general point illustrated by this example is that when one calculates a present value [as in eq. (3c)] in the environment posited by Hansen and Sargent (1981b), the resulting deterministic terms are overidentified. It should be noted that unless some deterministic terms are known a priori [e.g., it is known that there are no deterministic terms in the Euler equations (the analogue to (3a)), as in the permanent income model in Sargent (1978)], this overidentification is contingent on estimating Euler equations simultaneously with the other equations of the system [see Hansen and Sargent (1981b, pp. 293–295)]. See West (1987) for an example of the usefulness of testing such overidentifying restrictions.

To facilitate test and imposition of these restrictions in future work, this note concludes with explicit formulas for some commonly assumed determin-

[2]A formal argument may be developed along the lines of the proof of Theorem 2 in Newey and West (1987); details are available on request. Technical point: Newey and West (1987) assumes stationarity, so while its argument allows a constant and seasonal dummies [Ogaki (1989)], time trends are ruled out I have not worked out the extension to establish the asymptotic properties of the relevant estimators when time trends are present.

istic terms: a constant, polynomial in time, and seasonal dummies. Let $S_{1t},\ldots,S_{st}$ be seasonal dummies (e.g., $s=4$ for quarterly data), $S_{it}=1$ in season i, $S_{it}=0$ otherwise. Let m_t be a vector deterministic term, with

$$m_t = m_0 + m_1 t + \cdots + m_n t^n + m_2^s S_{2t} + \cdots + m_s^s S_{st}. \tag{5}$$

Let L be the lag operator. Let x_t be a vector stochastic process dimensioned conformably with m_t that follows an ARMA(p,q) with innovation u_t,

$$\gamma(L)x_t = \theta(L)u_t + m_t, \tag{6}$$

$$\gamma(L) \equiv I - \gamma_1 L - \cdots - \gamma_p L^p, \qquad \theta(L) \equiv \theta_0 + \theta_1 L + \cdots + \theta_q L^q,$$

roots of $|\gamma(z)|$ and $|\theta(z)|$ on or outside unit circle.

Note that unit autoregressive roots are allowed. This is to facilitate derivation of the formulas. For estimation, one might impose such roots by differencing or imposing cointegrating relationships. [For example, in (3a) to (3c), if a_t were difference-stationary and $\gamma_1 = 1$, then prior to estimation one would difference (3a), impose $\gamma_1 = 1$ in (3b), and difference (3c).] In seasonally adjusted data one would set $m_2^s = \cdots = m_s^s = 0$. For much data many researchers would set $m_1 = \cdots = m_n = 0$ as well [Nelson and Plosser (1981)]. The m_t and m_t^s of course have zeroes in appropriate places if, say, one element of x_t has a deterministic seasonal and another does not.

Let Φ_t be the information set $\{1, t, \ldots, t^n, S_{2t}, \ldots, S_{st}, x_t, x_{t-1}, \ldots\}$. We wish to calculate

$$y_t \equiv \sum_{j=0}^{\infty} \mathrm{E}\left(b^j x_{t+j} \mid \Phi_t\right), \tag{7}$$

where $0 < b < 1$ is a discount rate. (In the example above, $b = \alpha(1+\alpha)^{-1}$, and (7) is calculated for (a) $m_t = -f$, $\gamma(L) = 1$, $\theta(L) = 0$ [first term inside braces in (2)] and (b) $m_t = m_0$, $\gamma(L) = 1 - \gamma_1 L$, $\theta(L) = 0$ [second term inside braces in (2)].) If one is interested in only one element of x_t, say the first [as in Hansen and Sargent (1980)], then the solution given below is multiplied by the vector $(1\ 0 \ldots 0)$.

To solve (7), note that (7) implies that

$$y_t = x_t + b\mathrm{E}(y_{t+1} | \Phi_t). \tag{8}$$

Guess a solution for y_t of the form

$$\begin{aligned} y_t = {} & d_0 + d_1 t + \cdots + d_n t^n + d_2^s S_{2t} + \cdots + d_s^s S_{st} \\ & + \psi_0 x_t + \psi_1 x_{t-1} + \cdots + \psi_{p-1} x_{t-p+1} \\ & + \pi_0 u_t + \cdots + \pi_{q-1} u_{t-q+1}, \end{aligned} \tag{9}$$

where the number of lags of x_t and u_t is motivated by Hansen and Sargent (1981a). Putting (9) into the left-hand side of (8), (9) led one period into the right-hand side, and using (5) and (6) to solve for $\mathrm{E}(x_{t+1} \mid \Phi_t)$, allows one to solve for $d_0, \ldots, d_n, d_2^s, \ldots, d_s^s$ [as well as for the ψ_t and π_t, which of course turn out to be as in Hansen and Sargent (1981a)]. The algebra is straightforward though very tedious, so only the final solution will be presented.

To do so, some additional notation is required. Define $\gamma(b) \equiv I - \gamma_1 b - \cdots - \gamma_p b^p$. For $1 \le i \le h$ let $\binom{h}{i}$ be the binomial coefficient $h!/[(h-i)!i!]$. For $0 \le i \le k \le n$ define d_{ik} recursively backwards from $i = k$ to $i = 0$ by

$$d_{kk} = b(1-b)^{-1}, \qquad d_{k-1,k} = b(1-b)^{-1}\left[\binom{k}{1}(d_{kk}+1)\right],$$

$$d_{ik} = b(1-b)^{-1}\left[\binom{i+1}{1}d_{i+1,k} + \binom{i+2}{2}d_{i+2,k} + \cdots \right.$$

$$\left. + \binom{k-1}{k-i-1}d_{k-1,k} + \binom{k}{k-i}(d_{kk}+1)\right], \tag{10}$$

$$i = k-2, \ldots, 0.$$

For $s < i \le 2s$, let $m_i^s = m_{i-s}^s$, and define $m_1^s = 0$ (e.g., for quarterly data, $m_6^s = m_2^s$, $m_7^s = m_3^s$, $m_8^s = m_4^s$, and $m_5^s = m_1^s = 0$). Finally, let

$$m_{is} = \left[b(1-b^s)^{-1}\right]\left[m_{i+1}^s + bm_{i+2}^s + \cdots + b^{s-1}m_{i+s}^s\right], \tag{11}$$

$$i = 1, \ldots, s.$$

Then the coefficient vectors on deterministic terms in eq. (9) are

$$d_0 = d_{00}\gamma(b)^{-1}m_0 + d_{01}\gamma(b)^{-1}m_1 + \cdots + d_{0n}\gamma(b)^{-1}m_n$$

$$+ \gamma(b)^{-1}m_{1s},$$

$$d_i = d_{ii}\gamma(b)^{-1}m_i + d_{i,i+1}\gamma(b)^{-1}m_{i+1} + \cdots + d_{in}\gamma(b)^{-1}m_n,$$

$$i = 1, \ldots, n,$$

$$d_i^s = \gamma(b)^{-1}(m_{is} - m_{1s}), \qquad i = 2, \ldots, s. \tag{12}$$

Note that the presence of a trend term t^k in x_t puts terms in $t^k, t^{k-1}, \ldots, 1$ in y_t.

If one estimates with s seasonal dummies rather than with a constant and $(s-1)$ dummies, $d_1, \ldots, d_n$ are unchanged, $d_0 = 0$, and

$$d_i^s = d_{01}\gamma(b)^{-1}m_1 + \cdots + d_{0n}\gamma(b)^{-1}m_n + \gamma(b)^{-1}m_{is}, \qquad i = 1, \ldots, s, \tag{13}$$

where the m_i^s used in computing m_{is} of course are numerically different from m_i^s when only $(s-1)$ dummies are included.

The formulas for the trend terms are complicated enough that it might be useful to close by writing these out explicitly for a third-order polynomial, since $n \leq 3$ seems to apply in practice. Then

$$\begin{aligned}
&d_{00} = b(1-b)^{-1}, \quad d_{01} = b(1-b)^{-2}, \\
&d_{02} = b(1+b)(1-b)^{-3}, \quad d_{03} = b(1+4b+b^2)(1-b)^4, \\
&d_{11} = b(1-b)^{-1}, \quad d_{12} = 2b(1-b)^{-2}, \\
&d_{13} = 3b(1+b)(1-b)^{-3}, \quad d_{22} = b(1-b)^{-1}, \\
&d_{23} = 3b(1-b)^{-2}, \quad d_{33} = b(1-b)^{-1}.
\end{aligned} \tag{14}$$

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