A Variance Bounds Test of the Linear Quadratic Inventory Model

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This paper develops and applies a novel test of the Holt et al. linear quadratic inventory model. It is shown that a central property of the model is that a certain weighted sum of variances and covariances of production, sales, and inventories must be nonnegative. The weights are the basic structural parameters of the model. The model may be tested by seeing whether this sum is in fact nonnegative. When the test is applied to some nondurables data aggregated to the two-digit SIC code level, it almost always rejects the model, even though the model does well by traditional criteria.

I. Introduction

The linear quadratic inventory model, originated by Holt et al. (1960), has been the basis of much theoretical and empirical work on manufacturers' inventories of finished goods. The model argues that the basic reason firms hold finished goods inventories is to smooth production in the face of randomly fluctuating sales. In some versions of the model a desire to avoid sales backlogs provides an additional motive for holding inventories.[1] That firms might hold inventories for these reasons seems theoretically compelling (Blinder 1983), and

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[1] Throughout this paper, the word "inventories" used without qualification refers to manufacturers' inventories of finished goods.

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much empirical work has been interpreted as being supportive of the model (e.g., Blanchard 1983).

Some basic facts about finished goods inventories, however, seem to contradict the spirit if not the letter of this model. The model suggests that firms will smooth production by building up inventory stocks when sales are low and drawing down stocks when sales are high (Summers 1981). As is well known, however, manufacturers generally do precisely the opposite. Stocks tend to be decumulated in cyclical downturns and accumulated in cylical upturns (Blinder 1981*a*). In addition, it has been suggested that the fact that production has a larger variance than sales in many industries is inconsistent with the model (Blinder 1982, 1983; Blanchard 1983). The argument presumably is that firms could always make production exactly as variable as sales by holding no inventories. So if firms are holding inventories to smooth production, they do not appear to be doing so very successfully.

It is, however, somewhat difficult to evaluate this seemingly unfavorable evidence and to balance such evidence against the favorable results found in recent econometric studies such as Blanchard (1983). None of the authors cited in the previous paragraph formally establishes any implications of the production smoothing model for variances and covariances of inventories, sales, and production. Still less does any try to quantify the economic or statistical significance of the aspects of inventory behavior apparently inconsistent with the production smoothing model. Therefore, whether these aspects provide no or considerable evidence against the model has not yet been established.

This paper formally establishes an inequality summarizing the implications of the production smoothing model for the variances and covariances of inventories, sales, and production and then uses some aggregate data to test the inequality statistically. It turns out that the model *is* consistent both with accumulation of inventories in cyclical upturns and with production being more variable than sales, at least when a desire to avoid sales backlogs provides a motive for holding inventories (Blanchard 1983). But even the model that allows for such a desire restricts the movements of inventories, sales, and production so that only a certain amount of excess variability of production is consistent with the model. The inequality that this paper derives summarizes these restrictions.

The inequality is derived by comparing how much better off the firm could have expected to have been by ignoring random sales fluctuations and simply letting inventories increase from period to period at their trend rate of growth. This may be calculated as the difference between expected costs under this static policy and the policy that is optimal according to the model. This difference, which should be nonnegative if the model is correct, may be expressed as a simple weighted sum of certain variances and covariances of inventories, sales, and production. The weighted sum includes in particular the excess of production over sales variability. The weights are the basic structural parameters of the model, obtainable in standard fashion from a Euler equation. Even if all the estimates of parameters are right signed and significant, the estimate of this difference in principle may be insignificantly positive or even negative.

If the difference is negative for a given set of data, it seems unlikely that inventories truly are chosen in accordance with the supposedly optimal policy and therefore unlikely that the model is correct. The inequality quantifies the cost savings produced by the optimal inventory policy; that is, it quantifies the extent to which firms cut costs by adjusting inventories in response to random sales fluctuations. If the model is correct, a violation of the inequality indicates nonsensically that firms adjusted inventories to increase costs. Such violation would therefore mean that there is no evidence that production smoothing provided the motive for holding inventories.

And in fact, for almost all of the aggregate nondurables industries studied here, the inequality *is* violated; that is, the allegedly optimal policy for almost all the industries could have been expected to increase costs relative to the static one. The increase is statistically significant about half the time. Moreover, it is economically large, with expected deviations of costs from trend that are up to 50 percent higher than under the static policy. This strongly suggests that in these industries production smoothing does not provide the only motive for holding inventories.

The conclusion that the model does not adequately explain the data considered here seems particularly compelling since the test performed here requires relatively few economic or statistical assumptions. The test, for example, is consistent with but does not require the assumptions about market structure, causality, and demand made in the recent studies of Eichenbaum (1982) and Blanchard (1983). Also, and again in contrast to Eichenbaum (1982) and Blanchard (1983), it is computationally straightforward, requiring only linear estimation. In fact, in some cases, it could be concluded that the static inventory policy would be expected to cost less than the supposedly optimal policy without even calculating any of the model's parameters. All that was required was the calculation of certain variances and covariances. Since the test easily extends to cover other linear quadratic models, and perhaps some nonlinear models as well, it may be of general interest.

This is especially so since the test appears to be economically more informative than the usual test of cross-equation restrictions, at least in the present case. The significance of a rejection or acceptance of the variance bounds test can be measured not only in statistical but also in economic terms, by the calculation of the increase in expected costs mentioned above. In addition, the test itself suggests a reason for any rejection that occurs: some unexplained factors are making production too volatile. This indicates that the model needs to be modified to account for such excess volatility, and the concluding section to this paper briefly discusses some possible modifications. In contrast, statistical rejections of tests of cross-equation restrictions often appear to be difficult to interpret in economic terms (e.g., Blanchard 1983, p. 387).

To prevent misunderstanding, it should be emphasized at the outset that the innovation in the present paper is not in the model used but in the test performed. Two general formulations of the model are studied, both drawn from the existing literature on the linear quadratic inventory model. The two are motivated only briefly and uncritically. A critical evaluation of the model may be found in Blinder (1983) and West (1983*a*). The two were chosen because they are representative of the many versions of the model that have been formulated. Both are not only quite similar to most versions studied but are even identical to or strictly more general than some (e.g., Holt et al. 1960; Belsley 1969).

But the two of course do not incorporate all aspects of all formulations of the model. It is worth mentioning in particular that both follow the mainstream of work in the model and assume that inventories are held to cut production and possibly backlog costs in the face of randomly fluctuating sales. Some recent formulations of the model such as Blinder (1983) allow inventories also to serve to cut production costs in the face of randomly varying production costs. Extensions of the present paper to cover this and other major extensions to the linear quadratic model are left for future work.

The paper is organized as follows. Section II develops the test, Section III contains empirical results, and Section IV contains conclusions. An Appendix contains econometric details.

II. The Test

This section first describes the model and then derives an inequality that is central to the test.

A. *The Model*

The model under consideration is intended for finished goods inventories in so-called production to stock industries (Abramovitz 1950; Rowley and Trevedi 1975). Its precise formulation varies from author to author, and this paper's empirical work tests two versions. Both may be derived from the following general model. Firms producing a single homogeneous good maximize expected discounted real profits:

$$\max E_0 \sum_{t=0}^{\infty} d_1^t\{(p_t S_t) - d_2^t[a_0(\Delta Q_t)^2 + a_1(Q_t)^2 + a_2(H_t - a_3 S_{t+1})^2]\} \tag{1}$$
$$\text{s.t. } Q_t = S_t + H_t - H_{t-1},$$

where E_0 = mathematical expectations, conditional on information available at time 0; d_1 = fixed real discount rate, $0 < d_1 < 1$; d_2 = fixed rate of technological progress, $0 < d_2 \leq 1$; p_t = real price in period t; S_t = units sold in period t; Q_t = units produced in period t; H_t = units of finished goods inventories at the end of period t; and a_i = strictly positive parameters.

Two general comments on equation (1) will be made before the individual terms of the equation are briefly discussed. First, the firm's choice variables have been left unspecified intentionally. The estimation here is consistent with any of the standard ones: output only (Belsley 1969) or inventories only (Blanchard 1983) in models in which sales are exogenous; output, inventories, and sales in models in which the firm is a perfect competitor (Blanchard and Melino 1981; Eichenbaum 1982);[2] output, price, and inventories in models in which the firm is a monopolist (Blinder 1982). The firm's information set has been left unspecified for the same reason.

Second, for the present, all variables should be assumed to be deviations from trend (where trend should be understood to encompass all deterministic components, seasonal as well as secular). This assumption is made for algebraic simplicity and will be relaxed shortly. We wish to derive some restrictions that are implied for arbitrary trend, and the algebra is less cluttered when trend terms are set to zero.

The first term in parentheses in equation (1) is revenue; the term in brackets is costs. Although the revenue function will play no role in the bulk of this paper, it is worth pointing out some of the implications of its presence at this initial state to emphasize the generality of the tests performed here. The market may be perfect (Eichenbaum 1982) or imperfect (Blinder 1982). Price speculation on the supply side (Eichenbaum 1982) or perhaps even on the demand side may be

[2] The Blanchard and Melino (1981) and Eichenbaum (1982) models do not fit precisely into this framework, mainly because in effect they include the term $a_4 w_{t+j} Q_{t+j}$ in the cost function, where w_{t+j} is the wage and a_4 another positive parameter. As will become apparent, the inequality to be derived here is approximately correct if $a_4[\text{cov}(w, s) - \text{cov}(w, Q)]$ is small compared with the other terms in the inequality.

present. Pricing and production decisions may be made simultaneously (Blinder 1982; Eichenbaum 1982) or separately (Holt et al. 1960). In short, Summers's (1981) criticisms of inventory models that ignore interactions between firms and their customers are not relevant here.

The term in brackets in (1) is costs. These are the focus of the model and, here as elsewhere, are central. Total costs per period are the sum of three terms. The first is the cost of changing production, which is quadratic in the period-to-period change in the number of units produced. This represents, for example, hiring and firing costs. The second is the cost of production, which is quadratic in the number of units produced. This approximates an arbitrary concave cost function that results as usual from a decreasing returns to scale technology.

The third and final term embodies inventory and backlog costs and is quadratic in how far inventories are from a target level. A brief explanation of its rationale is as follows (see West [1983*a*] for a lengthier discussion and critique). Inventory holding costs (e.g., storage and handling charges) are reflected in a_2. The parameter a_3 is the ratio of inventory to expected sales that would be set in the absence of both types of production costs ($a_0 = a_1 = 0$). Not all authors agree that this ratio should be anything but zero, and the two major variations in (1) accommodated in the tests here turn on whether a_3 is allowed to be nonzero. Those who do so (Holt et al. 1960; Eichenbaum 1982; Blanchard 1983) argue that sales sometimes exceed inventories on hand, forcing firms to backlog orders. Firms face costs when such a backlog develops, perhaps because of loss of future sales. Thus, ceteris paribus, when expected sales are higher, inventories should be higher as well. The target level for inventories, $a_3E_tS_{t+1}$, trades off backlog and inventory costs. In this model *with a target level,* inventories can serve two functions.[3] They can buffer production, allowing it to be smoothed in the presence of fluctuating demand, and they can cut backlog costs. Optimal inventories balance production, holding, and backlog costs.

Some other authors, however, insist that in the absence of production costs the target level for inventories would be zero (Belsley 1969; Auerbach and Green 1980; Blinder 1982). They impose $a_3 = 0$. Inventories are then held purely to smooth production. In this model

[3] That is, inventories serve two functions apart from any they may serve on the revenue side. In the general formulation of the model used here, inventories may also serve, say, to allow the price speculation by producers that is emphasized in Eichenbaum (1982). This comment also applies to the model without a target level.

without a target level, optimal inventories balance savings in production costs against the costs of carrying inventories.

The tests performed here will thus accommodate equation (1) both with and without a target level for inventories.

B. *An Inequality*

We now derive an inequality that compactly expressed the production smoothing motive for holding inventories by calculating the effect inventories have on expected costs.[4] (The algebra carries along a_3. The effect in models without a target level is obtained simply by setting $a_3 = 0$ in the manipulations that follow.) According to the model, firms solve (1) subject to transversality and market equilibrium conditions to select optimal H_t^* and/or Q_t^* (and, as noted above, possibly p_t^* and S_t^* as well). In this optimal closed-loop policy, the endogenous control variables are set by a feedback rule, with their optimal period t values a function of their own past values and past and present values of forcing variables.

Let us assume that the sequences $\{H_t^*\}$, $\{Q_t^*\}$, and $\{S_t^*\}$ are covariance stationary. Methods for calculating this stationary solution in particular cases may be found in Holt et al. (1960), Eichenbaum (1982), and Blanchard (1983). Let $E_0V_0^*$ be the expectation at time 0 of the value of the objective function that results from this policy:

$$E_0 \sum_{t=0}^{\infty} d_1^t\{(p_t^*S_t^*) - d_2^t[a_0(\Delta Q_t^*)^2 + a_1(Q_t^*)^2 + a_2(H_t^* - a_3S_{t+1}^*)^2]\}. \tag{2}$$

Let $E_0V_0^A$ be the expectation at time 0 of the value of the objective function that would result from the alternative policy of setting $H_t^A = 0$ in every period, $Q_t^A = S_t^A = S_t^*$. Price $p_t^A = p_t^*$ will in general still be consistent with buyers demanding $S_t^A = S_t^*$.[5] The value of the objective function under this alternative policy is then

$$E_0 \sum_{t=0}^{\infty} d_1^t\{(p_t^*S_t^*) - d_2^t[a_0(\Delta S_t^*)^2 + a_1(S_t^*)^2 + a_2(-a_3S_{t+1}^*)^2]\}. \tag{3}$$

[4] I thank both Robert Shiller and Lawrence Summers for (independently) suggesting to me the basic argument of this section.

[5] Except if the firm has some market power and demand depends on actual or expected production or inventories. As far as I know, this assumption has never been made in this class of models.

This alternative decision rule in general is feasible. (The only apparent circumstance under which the policy is not feasible is when production takes place with a lag and inventories absorb sales expectational errors, as in Blinder [1982]. Even here the inequality about to be developed may be considered approximately correct if those errors are small relative to the size of the inventory stock, as seems reasonable.) By assumption, then, since V_0^* is optimal, $E_0V_0^* \geq E_0V_0^A$. Now $E_0V_0^*$ and $E_0V_0^A$ are random with respect to unconditional information and $E_0V_0^* - E_0V_0^A$ is a well-defined random variable with respect to this information set. Since it is nonnegative it has a nonnegative expectation. Thus $E(E_0V_0^* - E_0V_0^A) \geq 0$. By the law of iterated expectations, then,

$$EV_0^* \geq EV_0^A \rightarrow E\sum_{t=0}^{\infty} d_1^t\{(p_t^*S_t^*) - d_2^t[a_0(\Delta Q_t^*)^2 + a_1(Q_t^*)^2 + a_2(H_t^* - a_3S_{t+1}^*)^2]\} \tag{4}$$

$$\geq E\sum_{t=0}^{\infty} d_1^t\{(p_t^*S_t^*) - d_2^t[a_0(\Delta S_t^*)^2 + a_1(S_t^*)^2 + a_2(-a_3S_{t+1}^*)^2]\}.$$

Let $\text{var}(Q^*) = E(Q_t^*)^2$ denote the variance of production and $\text{cov}(Q, Q_{-1}) = E(Q_t^*Q_{t-1}^*)$ its first autocovariance, with analogous notation for other variables. (No time subscripts are necessary by the assumption of covariance stationarity.) Also define $d = d_1d_2$. With this notation (4) becomes

$$\sum_{t=0}^{\infty} d_1^tE(p_t^*S_t^*) - \sum_{t=0}^{\infty} d^t[a_0\,\text{var}(\Delta Q^*) + a_1\,\text{var}(Q^*) + a_2\,\text{var}(H^* - a_3S_{+1}^*)]$$

$$\geq \sum_{t=0}^{\infty} d_1^tE(p_t^*S_t^*) - \sum_{t=0}^{\infty} d^t[a_0\,\text{var}(\Delta S^*) + a_1\,\text{var}(S^*) + a_2\,\text{var}(-a_3S_{+1}^*)]. \tag{5}$$

Using $Q_t = S_t + H_t - H_{t-1}$ where convenient, expanding $\text{var}(H^* - a_3S_{+1}^*) = \text{var}(H^*) - 2a_3\,\text{cov}(H^*, S_{+1}^*) + a_3^2\,\text{var}(S^*)$, moving all terms to the left-hand side of the inequality, and then applying the standard formula for a geometric sum transforms (5) into

$$\begin{aligned} 0 < (1 - d)^{-1}\{&a_0[\text{var}(\Delta S^*) - \text{var}(\Delta Q^*)] \\ &+ a_1[\text{var}(S^*) - \text{var}(Q^*)] \\ &- a_2\,\text{var}(H^*) + 2a_2a_3\,\text{cov}(H^*, S_{+1}^*)\}. \end{aligned} \tag{6}$$

It is the two versions of this inequality—with and without a target level—that will be tested:

$$0 < (1 - d)^{-1}\{a_0[\mathrm{var}(\Delta S) - \mathrm{var}(\Delta Q)] + a_1[\mathrm{var}(S) - \mathrm{var}(Q)] - a_2 \mathrm{var}(H)\}, \tag{7.a}$$

$$0 < (1 - d)^{-1}\{a_0[\mathrm{var}(\Delta S) - \mathrm{var}(\Delta Q)] + a_1[\mathrm{var}(S) - \mathrm{var}(Q)] - a_2 \mathrm{var}(H) + 2a_2a_3 \mathrm{cov}(H, S_{+1})\}. \tag{7.b}$$

The * superscripts have been dropped in accordance with the null hypothesis that observed H, S, and Q accord with the optimal solution to (1).

Inequalities (7.a) and (7.b) have been derived assuming that all variables have zero unconditional expectations. They still hold even when such expectations are nonzero and firms account for them when maximizing expected discounted profits. Let the variables in (1) include deterministic components—constant, time trends, seasonal dummies, and so forth—and add linear terms such as $a_{10}(\Delta Q_t)$ to the cost function in equation (1). It is then easily verified (see West 1983*a*) that if the alternative policy is the no-feedback, open-loop one that sets inventories equal to their unconditional expectation each period—$H_t^A = EH_t^*$, $p_t^A = p_t$, $S_t^A = S_t^*$, $Q_t^A = S_t^* + E(H_t^* - H_{t-1}^*)$—the inequalities in (7) still result.[6] (Note that this alternative policy entails varying inventories from period to period if inventories display a time trend or seasonal variation.) For the remainder of the paper, (7.a) and (7.b) will be understood to apply to just such a model with deterministic terms. It should be noted again that for expositional convenience all such terms will be referred to as trend, even though the word "trend" is perhaps somewhat misleading if deterministic seasonal fluctuations are present or if secular growth is not.

In this light, let us interpret (7.a) and (7.b). The right-hand sides of these two equations describe the cost savings that could be (unconditionally) expected to result from setting inventories optimally rather than without feedback. The first two terms express differences of production costs, the third that of inventory costs, and the fourth, in (7.b), that of costs of inventories that deviate from their target level. The expected difference in inventory holding costs, $-a_2 \mathrm{var}(H_t)$, is always negative. Therefore, according to the model, these expected cost increases are more than offset by savings elsewhere (otherwise the optimal policy would not be optimal). Inequality (7.a), applicable

[6] See Bertsekas (1976, pp. 191–92) for a definition of an "open-loop" policy. Strictly speaking, setting $H_t^A = EH_t^*$ is the open-loop policy only if inventories are the only control.

when there is no target level, says that the firm must expect to save either on costs of changing production—var(ΔQ) < var(ΔS)—or on costs of production—var(Q) < var(S)—or both, and the expected savings must be large enough that overall expected costs are lower; that is, (7.a) holds. Similarly, (7.b), applicable when there is a target level, says that the optimal policy must be expected to more than offset increases in expected inventory holding costs with expected savings in production or target level costs.

Thus it would seem to be a minimal economic requirement that (7.a) and (7.b) be satisfied by data that are to be explained by the model. The inequalities merely ask that the optimal policy be expected to cost less than the static one. The static policy is the one that would be optimal in the absence of any random fluctuations in sales. The inequalities therefore summarize how production, sales, and inventories are expected to interact as they are dynamically adjusted in response to sales shocks. And this is precisely what the model purports to explain. It is perhaps reasonable, therefore, to ask that the data not only satisfy (7.a) and (7.b) but do so to an extent that is significant in economic or statistical terms.

The next section sees how well some aggregate nondurables data satisfy these inequalities. Given that (7.a) and (7.b) have been derived for a single firm, however, it is appropriate to make a remark on aggregation before examining these empirical results. The inequalities *do* still hold at an aggregate level, provided that all the parameters representing technology (e.g., the a_i's) and the stochastic characteristics of forcing variables (i.e., their autoregressive moving average [ARMA] parameters) are the same for each individual firm. As is explained in detail in West (1983*a*), under these sufficient though perhaps not necessary conditions each firm's behavior is summarized by a set of linear regressions with identical coefficients on the regressors. As usual, therefore, the model aggregates exactly, and aggregate behavior is characterized by the same set of regressions. It is no surprise, then, that aggregate production, sales, and inventories satisfy (7.a) and (7.b) for arbitrary correlations of production, sales, and inventories across firms.

III. Empirical Results

Data and estimation are described briefly before the basic and some additional empirical results are presented.

A. *Data*

The data were real (1972 dollars) and monthly. Both seasonally adjusted and unadjusted data were used. Seasonally adjusted data were available for 1959–80 for aggregate nondurables and for all six two-digit industries that Belsley (1969) identified as operating in production to stock markets: food (SIC 20), tobacco (SIC 21), apparel (SIC 23), chemicals (SIC 28), rubber (SIC 30), and petroleum (SIC 29). Seasonally unadjusted data were available for aggregate nondurables and three two-digit industries (chemicals, petroleum, and rubber). Again, durable goods and the remaining nondurable goods industries were excluded because the model is intended to apply only to industries that produce to stock, and, according to Belsley (1969), none of these other industries produces to stock.

Sales were obtained by using the appropriate wholesale price index to deflate the Bureau of the Census nominal figures for sales (all figures found in the Citibank Economic Database, in Bureau of the Census [1978, 1982] or obtained directly from the Bureau of the Census). The seasonally adjusted inventory figures were obtained by converting the bureau's recently calculated constant-dollar seasonally adjusted finished goods inventory series (Hinrichs and Eckman 1981) from "cost" to "market" so that \$1.00 of inventories represented the same physical units as \$1.00 of sales (see West [1983*b*] for a definition of "cost" and "market" and an explanation of why a conversion was necessary). As in Reagan and Sheehan (1982), the seasonally unadjusted constant-dollar inventory figures were obtained by multiplying the adjusted figures by the corresponding ratio of unadjusted to adjusted figures for book value (nominal) finished goods inventories. (This procedure was adopted since no unadjusted constant-dollar data appear to be available. It makes the plausible assumption that the "seasonal deflator" is the same for book value and constant-dollar inventories.)[7] Production was obtained from the identity $Q_t = S_t + H_t - H_{t-1}$.

B. Estimation

The sample period covered 1959:5–1980:10, with 1980:11 and 1980:12 used for leads and 1959:2–1959:4 used for lags. All regressions included deterministic terms: a constant and a time trend, and, for seasonally unadjusted data, seasonal dummies as well.[8]

[7] An alternative method for calculating unadjusted constant-dollar inventories would be to deflate unadjusted book value inventories by the appropriate wholesale price index. Given the massive switch from FIFO to LIFO accounting in the 1970s and cyclical differences in output price vs. input cost (see Foss et al. 1980), this is likely to lead to estimates substantially inferior to those derived as described in the text.

[8] It should be noted that in Reagan and Sheehan's (1982) time-series study of precisely the unadjusted aggregate data used here, it was found that seasonal dummies alone successfully accounted for the seasonal variation in inventories. There appeared to be no need to allow for indeterministic seasonal components.

Three specific aspects of estimation will be briefly discussed. These are estimation of the a_i, of the second moments of inventories, sales, and production, and, finally, of the standard error of (7). (Throughout this section, references to eq. [7] should be understood to be shorthand for [7.a] and [7.b].) Additional details will be found in the Appendix and in West (1983*a*).

The a_i in the model with a target level were obtained as follows. (The same procedure was applied to the model without a target level, except that $a_3 = 0$ was imposed.) A necessary first-order condition to solve (1) at time $t \geq t_0$ is obtained by differentiating (1) with respect to H_t and setting the result equal to zero:[9]

$$\begin{aligned} E_t 2[d^2 a_0 H_{t+2} &- (2d^2 a_0 + 2da_0 + da_1)H_{t+1} \\ &+ (d^2 a_0 + 4da_0 + a_0 + da_1 + a_1 + a_2)H_t \\ &- (2a_0 + 2da_0 + a_1)H_{t-1} + a_0 H_{t-2} \\ &+ d^2 a_0 S_{t+2} - (d^2 a_0 + 2da_0 + da_1 + a_2 a_3)S_{t+1} \\ &+ (2da_0 + a_0 + a_1)S_t - a_0 S_{t-1} + \text{deterministic terms}] = 0. \end{aligned} \tag{8}$$

After defining lower-case $q_t = dQ_t - Q_{t-1}$ and dividing this first-order condition by two, the Euler equation (9) results:

$$\begin{aligned} E_t\{a_0 dq_{t+2} &- [a_1 + a_0(1 + d)]q_{t+1} + a_0 q_t \\ &+ a_2 H_t - a_2 a_3 S_{t+1} + \text{deterministic terms}\} = 0. \end{aligned} \tag{9}$$

A normalization is required to estimate the a_i. The normalization chosen is arbitrary since changing the a_i by a scale factor does not change inequality (7). The normalization used was $a_1 + (1 + d)a_0 = 1$, so (9) becomes

$$\begin{aligned} q_{t+1} = a_0(dq_{t+2} + q_t) &+ a_2 H_t - a_2 a_3 S_{t+1} \\ &+ u_{1t} + \text{deterministic terms}, \end{aligned} \tag{10}$$

where the disturbance u_{1t} has a moving average component.[10] With a monthly value of d imposed (10) can be estimated by instrumental variables. The results here report $d = .995$ (corresponding annual rate is about 6 percent); results with $d = .990$ and $d = .999$ were

[9] This assumes that $dp_{t+j}S_{t+j}/dH_t = 0$. This is consistent with any linear quadratic inventory model that I am aware of, including not only those in which sales are exogenous (e.g., Belsley 1969) but also those in which they are jointly endogenous with inventories (Blinder 1982; Eichenbaum 1982).

[10] If production and sales decisions are made simultaneously, u_{1t} is MA(1). But if production is decided before sales are known, as in Blinder (1982), u_{1t} is MA(2). It seemed desirable to adopt a procedure that was consistent under those circumstances, so the estimation procedure allowed for an MA(2) disturbance.

virtually identical. The six instruments used apart from the deterministic terms in (10) were three lags each of inventories and sales. The estimation required two steps, as described in Hansen and Singleton (1982). The first step calculated the variance-covariance matrix of the u_{1t}, and the second obtained the optimal instrumental variables estimator. See the Appendix and West (1983*a*) for further details. Since the equation is overidentified—the model without a target level has four fewer right-hand-side variables than instruments, and the one with a target level has three—Hansen's (1982) test of overidentifying restrictions was calculated.

Variances and covariances were calculated from bivariate (inventories, sales) autoregression of order three:[11]

$$\begin{aligned} H_t &= \text{deterministic terms} + \phi_{11}H_{t-1} + \phi_{12}H_{t-2} \\ &\quad + \phi_{13}H_{t-3} + \phi_{14}S_{t-1} + \phi_{15}S_{t-2} + \phi_{16}S_{t-3} + u_{2t}, \\ S_t &= \text{deterministic terms} + \phi_{21}H_{t-1} + \phi_{22}H_{t-2} \\ &\quad + \phi_{23}H_{t-3} + \phi_{24}S_{t-1} + \phi_{25}S_{t-2} + \phi_{26}S_{t-3} + u_{3t}. \end{aligned} \tag{11}$$

The Yule-Walker equation using the estimated ϕ_{ij} was then used in the standard way (Anderson 1971, p. 182) to obtain the needed second moments of sales and inventories. The second moments of production were derived from the identity $Q_t = S_t + H_t - H_{t-1}$; for example, $\text{var}(Q) = \text{var}(S) + 2\,\text{cov}(S, H) - 2\,\text{cov}(S, H_{-1}) + 2\,\text{var}(H) - 2\,\text{cov}(H, H_{-1})$.

Finally, the standard error of the statistic (7) was derived as follows. Let $\boldsymbol{\theta}$ be the parameter vector needed to calculate (7). Vector $\boldsymbol{\theta}$ consists of the coefficients on the right-hand-side (RHS) variables in the three-equation system consisting of (10) and (11) and the three elements of the covariance matrix of the error terms in (11). Thus $\boldsymbol{\theta}$ is (1×24) for seasonally adjusted data ($24 = 15$ RHS variables explicitly listed in [10] and [11] + 6 constant and trend terms + 3 elements of the variance-covariance matrix of the residuals in [11]). Similarly, $\boldsymbol{\theta}$ is (1×57) for seasonally unadjusted data. The estimated $\boldsymbol{\theta}$ is asymptotically normal with a covariance matrix $\mathbf{V}$ defined in the Appendix. The statistic (7) is a function of $\boldsymbol{\theta}$, say $g(\boldsymbol{\theta})$, and thus is asymptotically normal with covariance matrix $(dg/d\boldsymbol{\theta})\mathbf{V}(dg/d\boldsymbol{\theta})'$. The standard error of (7) is the square root of $(dg/d\boldsymbol{\theta})\mathbf{V}(dg/d\boldsymbol{\theta})'$. The derivatives $dg/d\boldsymbol{\theta}$ were calculated numerically.

[11] This is not to say that the model (1) implies that inventories and sales follow such an autoregression. In general, however, it does imply that they follow a bivariate ARMA process of some order (Hansen and Sargent 1981). The order of the ARMA process cannot be tied down without making auxiliary assumptions that I have been at pains to avoid making. The AR process assumed in the text should be considered an approximation to this ARMA process.

It is to be noted that this procedure takes into account not only the uncertainty in the estimates of the a_i but also in the estimates of the first and second moments. The procedure also accounts for the covariance between the estimates of the a_i and of moments. Again, for details see the Appendix and West (1983*a*).

C. *Results*

I will shortly present estimates of the size and the standard errors of the right-hand sides of (7.a) and (7.b) for the data described above. This will require estimates not only of the appropriate variances and covariances of inventories, sales, and production but of the a_i parameters as well. First, however, let us consider whether these data are qualitatively consistent with the inequalities by examining the appropriate second moments. Tables 1 and 2 present these for seasonally adjusted and unadjusted data, respectively.

It follows immediately from the trivial calculations underlying the entries in tables 1 and 2 that for both seasonally adjusted and unadjusted data, the model without a target level violates (7.a) for almost all industries. (The only possible exception is chemicals.) Columns 5–7 indicate that for all but the chemical industry $\text{var}(\Delta S) - \text{var}(\Delta Q) < 0$, $\text{var}(S) - \text{var}(Q) < 0$, and, of course, $\text{var}(H) > 0$. Since the a_i are known a priori to be positive, it follows that for all but chemicals $0 > a_0[\text{var}(\Delta S) - \text{var}(\Delta Q)] + a_1[\text{var}(S) - \text{var}(Q)] - a_2\,\text{var}(H)$. In other words, according to the model itself, the static, no-feedback policy of letting inventories grow at their trend rate would have been expected to be preferable to the optimal policy that the model claims actually was followed: lower costs of changing production, lower costs of production, and lower inventory costs. From these simple calculations we can conclude that, with the possible exception of the chemical industry, the data studied here are inconsistent with the model without a target level. This suggests that backlog costs, whose existence is used to rationalize a nonzero target level, are of crucial importance to this model.

It also follows from tables 1 and 2 that even the model with a target level is inconsistent with the seasonally unadjusted behavior of the petroleum industry, since inventories here covary negatively with next period's sales. Relative to the static policy, the optimal policy that supposedly was followed would have been expected to increase all the costs just noted and the cost of being away from a target level as well. Thus this data set is incompatible with the model, with or without a target level. For the remaining industries, (7.a) and (7.b) cannot be signed without the a_i. Let us therefore turn to precise calculation of the inequalities.

Tables 3 and 4 present the a_i for the models with and without a

TABLE 1

BASIC VARIANCES AND COVARIANCES (Raw Data Seasonally Adjusted)

	var(ΔS) (1)	var(ΔQ) (2)	var(S) (3)	var(Q) (4)	(1) − (2) (5)	(3) − (4) (6)	var(H) (7)	cov(H, S_{+1}) (8)
Aggregate nondurables	124,428	170,407	1,288,650	1,311,890	−45,979	−23,239	1,326,370	715,300
Food (SIC 20)	39,789	51,744	46,801	50,628	−11,955	−38,732	97,779	21,828
Tobacco (SIC 21)	657	2,145	647	1,401	−1,488	−754	1,542	233
Apparel (SIC 23)	5,032	10,728	15,369	18,866	−5,698	−3,497	17,320	4,689
Chemicals (SIC 28)	8,548	9,405	85,260	84,293	−857	967	71,598	16,130
Petroleum (SIC 29)	3,003	5,189	21,519	22,358	−2,186	−838	11,601	1,123
Rubber (SIC 30)	2,475	3,688	22,188	23,186	−1,213	−998	26,961	14,312

NOTE.—Units are millions of 1972 dollars squared. Data are described in the text. Variances and covariances are calculated around a time trend, as described in the text.

TABLE 2

BASIC VARIANCES AND COVARIANCES (Raw Data Seasonally Unadjusted)

	var(ΔS) (1)	var(ΔQ) (2)	var(S) (3)	var(Q) (4)	(1) − (2) (5)	(3) − (4) (6)	var(H) (7)	cov(H, S_{+1}) (8)
Aggregate nondurables	217,727	331,732	1,258,260	1,326,950	−114,004	−68,686	1,179,870	774,956
Chemicals	20,647	18,649	105,840	102,751	1,998	3,089	69,803	24,932
Petroleum	3,462	5,940	20,964	22,275	−2,447	−1,311	11,259	−6,225
Rubber	4,172	4,977	19,317	20,471	−805	−1,154	17,896	8,399

NOTE.—Units are millions of 1972 dollars squared. Data are described in the text. Variances and covariances are calculated around a time trend and seasonal dummies, as described in the text.

TABLE 3

STRUCTURAL PARAMETERS, MODEL WITHOUT A TARGET LEVEL

	$\hat{a}_0$	$\hat{a}_1$	$\hat{a}_2$	J*
	Raw Data Seasonally Adjusted			
Aggregate nondurables	.2443	.5126	.0129	12.69
	(.0453)	(.0903)	(.0188)	
Food	.3377	.3261	−.0000	6.61
	(.0585)	(.1167)	(.0175)	
Tobacco	.0373	.9256	.0311	4.27
	(.0828)	(.1652)	(.0510)	
Apparel	.3844	.2332	.0169	8.10
	(.0539)	(.1075)	(.0123)	
Chemicals	.4074	.1872	.0160	12.91
	(.0627)	(.1251)	(.0172)	
Petroleum	.1399	.7209	.0418	7.42
	(.0826)	(.1648)	(.0235)	
Rubber	−.0501	1.0999	−.0083	7.91
	(.1150)	(.2294)	(.0354)	
	Raw Data Seasonally Unadjusted			
Aggregate nondurables	−.1093	1.2182	−.0038	19.92
	(.0931)	(.1857)	(.0273)	
Chemicals	.3530	.2958	.0224	14.48
	(.0839)	(.1674)	(.0198)	
Petroleum	.3300	.3417	.0395	4.85
	(.0555)	(.1107)	(.0127)	
Rubber	.4204	.1612	−.0117	4.17
	(.1058)	(.2111)	(.0254)	

NOTE.—Variables are defined in the text. Asymptotic standard errors are in parentheses. Standard error on $a_1 = 1 - (1 + d)a_0 = 1 - 1.995a_0$ calculated as 1.995 times the standard error on a_0.

* J distributed as χ^2 with 4 df; critical levels: 9.49 at .05, 13.28 at .01, and 14.86 at .005.

target level, respectively. Almost all the parameter estimates are indeed positive. Consider the model without a target level first. With seasonally adjusted data 11 of 14 free signs on the a_i are correct, and with unadjusted the figure is 5 of 8. The number of free signs is 14 and 8 rather than 21 and 12 because the normalization rule $a_1 + (1 + d)a_0 = 1$ constrains either a_0 or a_1 to be positive in each equation. The comparable figures for the model with a target level are 19 of 21 and 9 of 12. Only two of the wrong-signed coefficients are significant at the .05 level (a_0 in the model with a target level for both seasonally adjusted rubber and seasonally unadjusted aggregate nondurables). In most equations the production cost a_1 and the cost of changing production a_0 are significant. Somewhat puzzling is the imprecision of the estimates of the inventory holding cost a_2 and the target level parameter a_3, which are rarely significant at the .05 level. They are, however, almost always positive and stand here in about the same

TABLE 4

STRUCTURAL PARAMETERS, MODEL WITH A TARGET LEVEL

	$\hat{a}_0$	$\hat{a}_1$	$\hat{a}_2$	$\hat{a}_3$	J*
	Raw Data Seasonally Adjusted				
Aggregate nondurables	.1759	.6489	.0228	1.1249	11.44
	(.1113)	(.2220)	(.0232)	(1.2178)	
Food	−.0786	1.1568	.0839	6.4669	2.62
	(.2914)	(.5813)	(.0868)	(3.4099)	
Tobacco	.0241	.9520	.0420	1.2325	3.76
	(.0854)	(.1704)	(.0540)	(2.0185)	
Apparel	.1117	.7271	.0257	4.8653	1.43
	(.1276)	(.2546)	(.0283)	(5.3242)	
Chemicals	.3990	.2041	.0171	.3256	12.83
	(.0671)	(.1339)	(.0177)	(.9832)	
Petroleum	.0775	.8453	.0367	1.1048	4.01
	(.0908)	(.1811)	(.0263)	(1.0980)	
Rubber	−.2456	1.4900	.0199	4.5217	1.79
	(.1189)	(.2372)	(.0494)	(10.6588)	
	Raw Data Seasonally Unadjusted				
Aggregate nondurables	−.2419	1.4827	.0617	2.0416	11.46
	(.1014)	(.2023)	(.0464)	(1.1844)	
Chemicals	.2092	.5827	.0375	.8601	13.50
	(.1392)	(.2777)	(.0282)	(.8514)	
Petroleum	.2232	.5546	.0253	.8504	3.53
	(.1029)	(.2053)	(.0206)	(1.3155)	
Rubber	.3100	.3816	−.0085	−3.0046	4.30
	(.1722)	(.3435)	(.0320)	(13.3695)	

NOTE.—See note to table 3.
* J distributed as χ^2 with 3 df; critical levels: 7.81 at .05, 11.34 at .01, and 12.84 at .005.

ratio to the other a_i and to each other as they did in Blanchard's (1983) estimates for the automobile industry.

However, these parameters, though positive and often significant, are not enough to make the model plausible. Results of the variance bounds test for the model without a target level are shown in table 5 and for the model with a target level in table 6. We noted above what would result for all data sets except possibly chemicals for the model without a target level and for the seasonally unadjusted petroleum industry in the model with a target level. Thus it is no surprise that tables 5 and 6 indicate that (7.a) and (7.b) were violated for all of these. However, the inequality for the model without a target level was violated for seasonally unadjusted chemicals as well, as was the inequality for the model with a target level for most of the data sets. Thus the inequalities were violated in 17 out of 22 instances, and nine of these were significant at the .05 level. The four data sets that did satisfy (7.b) did so insignificantly, with standard errors uniformly

TABLE 5

TEST STATISTICS, MODEL WITHOUT A TARGET LEVEL

	Equation (7.a) (1)	Equation (12) (2)	100 × (1)/(2) (3)
	Raw Data Seasonally Adjusted		
Aggregate nondurables	−8,074,590 (6,779,480)	146,256,000	−5.50
Food	−1,056,690 (1,881,690)	6,797,350	−15.54
Tobacco	−160,232 (38,669)	284,920	−56.23
Apparel	−659,426 (97,754)	1,762,890	−37.41
Chemicals	−262,668 (276,638)	4,151,380	−6.33
Petroleum	−279,082 (124,475)	3,465,590	−8.05
Rubber	−162,299 (161,445)	5,018,480	−3.23
	Raw Data Seasonally Unadjusted		
Aggregate nondurables	−13,324,700 (6,961,700)	315,102,000	−4.23
Chemicals	11,111 (335,832)	7,708,310	.14
Petroleum	−339,895 (81,276)	2,001,050	−16.98
Rubber	−63,054 (99,154)	1,036,880	−6.08

NOTE.—Units are millions of "normalized" dollars, obtained after measuring variables in 1972 dollars and normalizing $a_0 + a_1(1 + d) = 1$.

larger than the sizes of the inequality. Also, two of these four produced the only significantly wrong-signed parameter (a_0 for adjusted rubber and unadjusted aggregate nondurables). It therefore appears that the model does not explain well any of the data studied here.

Moreover, the increase in deviations of costs from trend attributable to the optimal policy would appear to be economically as well as statistically noticeable. Column 2 in tables 5 and 6 contains total deviations of costs from trend (again, in "normalized" dollars, $a_1 + [1 + d]a_0 = 1$):

$$(1 - d)^{-1}[a_0 \operatorname{var}(\Delta Q) + a_1 \operatorname{var}(Q) + a_2 \operatorname{var}(H) - 2a_2a_3 \operatorname{cov}(H, S_{+1}) + a_2a_3 \operatorname{var}(S)]. \tag{12}$$

When (7.a) or (7.b) is divided by (12) (possibly with $a_3 = 0$ imposed in [12]), the result is a dimensionless measure of the extent to which the

TABLE 6

TEST STATISTICS, MODEL WITH A TARGET LEVEL

	Equation (7.b) (1)	Equation (12) (2)	100 × (1)/(2) (3)
	Raw Data Seasonally Adjusted		
Aggregate nondurables	−3,339,800 (6,904,450)	182,428,000	−1.83
Food	2,398,440 (3,050,810)	40,638,700	5.90
Tobacco	−158,798 (39,817)	293,430	−54.11
Apparel	−525,333 (97,687)	4,896,480	−10.73
Chemicals	−238,359 (279,689)	4,431,220	−5.37
Petroleum	−242,594 (137,816)	4,120,130	−5.89
Rubber	169,716 (382,009)	8,124,210	2.09
	Raw Data Seasonally Unadjusted		
Aggregate nondurables	9,642,140 (18,963,400)	417,671,000	2.31
Chemicals	241,510 (386,956)	13,544,900	1.78
Petroleum	−366,608 (187,799)	2,923,310	−12.50
Rubber	−21,256 (149,736)	1,456,590	−1.45

NOTE.—See note to table 5.

optimal policy increases or decreases deviations of costs from trend relative to the static policy. This is shown in column 3 of tables 5 and 6. The optimal policy *increases* expected cost deviations by up to 56 percent. If this increase were to be believed it would mean that deviations of profit margins from trend, and therefore presumably profit margins themselves, are substantially reduced.

It is of some interest to compare the results of the inequality tests with those of a common test of specification, the Hansen (1982) test of overidentifying restrictions that is reported in the columns labeled J (tables 3 and 4). This was accepted at the .05 level for about two-thirds of the data sets (food, tobacco, apparel, petroleum, and rubber) and was rejected at the .05 but accepted at the .005 level for the two other data sets. This compares favorably with the tests of the overidentifying restrictions in other recent studies (Eichenbaum 1982; Blanchard 1983). Thus it is perhaps fair to say that this traditional test is supportive of the model. It would appear, then, that the variance bounds test was an essential element in assessing the reasonableness of this model for these data.

D. *Additional Empirical Results*

The robustness of the conclusions of the previous subsection was checked by calculating two additional sets of estimates. The first related to some variance inequalities applied to deterministic seasonal components, the second to quarterly (instead of monthly) data.

To explain the first, let a j superscript denote the deterministic seasonal component of a variable in month j; $\bar{X}^s$ = the mean deterministic seasonal component of variable X, $\bar{X}^s = 1/12\ \Sigma_{j=1}^{12} X^j$; $\mathrm{var}(X^s)$ = the "variance" of the deterministic seasonal component, $\mathrm{var}(X^s) = 1/12\ \Sigma_{j=1}^{12} (X^j - \bar{X}^s)^2$, with $\mathrm{var}(\Delta X^s)$ and $\mathrm{cov}(X^s, Y^s)$ defined in the obvious way.

Consider comparing costs under the optimal policy with costs that result under the alternative policy that suppresses all deterministic seasonal variation in inventories but otherwise allows inventories to grow at their trend rate, $H_t^A = EH_t + (\bar{H}^s - H^j)$, where j is the month corresponding to time period t. It may be shown by an argument analogous to that in Section II that the model (1) implies that

$$
\begin{aligned}
0 < (1 - d)^{-1}(\{&a_0[\mathrm{var}(\Delta S) - \mathrm{var}(\Delta Q)] + a_1[\mathrm{var}(S) \\
&- \mathrm{var}(Q)] - a_2\,\mathrm{var}(H)\} + \{a_0[\mathrm{var}(\Delta S^s) \\
&- \mathrm{var}(\Delta Q^s)] + a_1[\mathrm{var}(S^s) - \mathrm{var}(Q^s)] \\
&- a_2\,\mathrm{var}(H^s)\}),
\end{aligned}
\tag{13.a}
$$

$$
\begin{aligned}
0 < (1 - d)^{-1}(\{&a_0[\mathrm{var}(\Delta S) - \mathrm{var}(\Delta Q)] + a_1[\mathrm{var}(S) \\
&- \mathrm{var}(Q)] - a_2\,\mathrm{var}(H) + 2a_2a_3\,\mathrm{cov}(H, S_{+1})\} \\
&+ \{a_0[\mathrm{var}(\Delta S^s) - \mathrm{var}(\Delta Q^s)] + a_1[\mathrm{var}(S^s) - \mathrm{var}(Q^s)] \\
&- a_2\,\mathrm{var}(H^s) + 2a_2a_3\,\mathrm{cov}(H^s, S^s_{+1})\}).
\end{aligned}
\tag{13.b}
$$

Inequality (13.a) applies to a model without a target level, (13.b) to a model with a target level. Inequality (13.a) in conjunction with inequality (7.a) says that when firms allow deterministic inventory seasonals to depart from their mean level, costs must not be increased to such an extent that the cost savings detailed in (7.a) are more than offset. Further, these departures will cut costs only insofar as they make $\mathrm{var}(Q^s)$ and $\mathrm{var}(\Delta Q^s)$, the deterministic seasonal costs of production and changing production, smaller than $\mathrm{var}(S^s)$ and $\mathrm{var}(\Delta S^s)$, the deterministic seasonal cost that obtains when there are no departures of inventory seasonals from their mean levels. Inequality (13.b) in conjunction with inequality (7.b) has a comparable interpretation.

It is of interest, then, to calculate the relevant variances and covariances, as well as to estimate the size and standard errors of (13.a) and (13.b). The relevant second moments for the four seasonally unadjusted data sets are displayed in table 7. For two of the four data sets (aggregate nondurables and rubber), we can conclude without calculating any parameter estimates that (13.a) will be rejected. (This follows since cols. 5 and 6 are negative for these two data sets in both table 7 and table 2.) For the other two data sets, parameters do have to be estimated to sign (13.a), and, for all four data sets, parameter estimates are needed to sign (13.b). The model, then, seems to be qualitatively consistent with (13.a) to a slightly greater degree than with (7.a), in that two data sets rather than one have second moments that are consistent with one relevant inequality.

In a more formal, quantitative sense, however, the model performs as poorly with respect to (13.a) and (13.b) as it did with respect to (7.a) and (7.b). Once again, almost all the inequalities are wrong signed, about half of them significantly so (see tables 8 and 9).[12] The only exception, once again, is chemicals, which does, however, satisfy (13.a) and (13.b) in a statistically significant fashion.

For these data, as for the automobile data studied by Blanchard (1983), then, the seasonals appear to contain little evidence to suggest that manufacturers are selecting their inventories in accord with (1).

The second additional set of estimates calculated inequalities (7.a) and (7.b) for quarterly, seasonally adjusted data. These data were constructed from the monthly data for sales by adding the figures for the relevant three months and for inventories by selecting the last month of the quarter.

Since the estimates were very similar to those for monthly data, only a summary of the final results seems worth reporting. Inequality (7.a) was signed wrong for six of seven data sets (the exception was tobacco and resulted from wrong-signed estimates of a_0 and a_2). Inequality (7.b) was signed wrong for all seven data sets. Four of the 14 wrong signs were significant at the 5 percent level; the correct sign for tobacco was not.

These additional tests, then, support the results reported above.[13]

[12] Column 2 in tables 8 and 9 reports total deviations of expected costs from trend when the deterministic seasonal component is accounted for:

$$\begin{aligned}(1 - d)^{-1}\{[a_0 \operatorname{var}(\Delta Q) + a_1 \operatorname{var}(Q) + a_2 \operatorname{var}(H) - 2a_2a_3 \operatorname{cov}(H, S_{+1})\\ + a_2a_3 \operatorname{var}(S)] + [a_0 \operatorname{var}(\Delta Q^s) + a_1 \operatorname{var}(Q^s) + a_2 \operatorname{var}(H^s)\\ - 2a_2a_3 \operatorname{cov}(H^s, S^s_{+1}) + a_2a_3 \operatorname{var}(S^s)]\}.\end{aligned} \tag{14}$$

[13] One further set of estimates was obtained, but since the results were meaningless they do not appear to warrant reporting in the text. An independent measure of production was obtained by using the Federal Reserve Board's (FRB) index of indus-

TABLE 7

BASIC VARIANCES AND COVARIANCES—SEASONAL COMPONENTS

	$\mathrm{var}(\Delta S^s)$ (1)	$\mathrm{var}(\Delta Q^s)$ (2)	$\mathrm{var}(S^s)$ (3)	$\mathrm{var}(Q^s)$ (4)	(1) − (2) (5)	(3) − (4) (6)	$\mathrm{var}(H^s)$ (7)	$\mathrm{cov}(H^s, S^s_{+1})$ (8)
Aggregate nondurables	2,109,340	2,641,980	1,115,950	1,238,090	−532,640	−122,136	50,935	23,749
Chemicals	57,917	48,951	40,877	29,718	8,967	11,161	3,133	2,727
Petroleum	4,129	2,017	1,371	816	2,111	555	780	559
Rubber	16,812	19,141	7,269	8,348	−2,330	−1,079	479	−25

NOTE.—Units are millions of 1972 dollars squared. Data and calculation are described in the text.

TABLE 8

TEST STATISTICS, SEASONAL MODEL WITHOUT A TARGET LEVEL

	Equation (13.a) (1)	Equation (14)* (2)	100 × (1)/(2) (3)
Aggregate nondurables	−31,394,100 (8,448,700)	558,919,000	−5.61
Chemicals	1,290,390 (327,677)	12,936,200	9.98
Petroleum	−292,635 (97,243)	2,914,500	−10.04
Rubber	−168,585 (54,951)	2,195,920	−7.67

NOTE.—See note to table 5.
* Equation (14) is defined in n. 12.

TABLE 9

TEST STATISTICS, SEASONAL MODEL WITH A TARGET LEVEL

	Equation (13.b) (1)	Equation (14) (2)	100 × (1)/(2) (3)
Aggregate nondurables	−234,130 (21,052,500)	713,810,000	−.00
Chemicals	1,929,020 (636,967)	1,927,400	10.1
Petroleum	−209,849 (162,745)	3,108,010	−6.75
Rubber	−247,514 (151,183)	3,167,820	−7.81

NOTE.—See note to table 8.

IV. Conclusions

This section summarizes the basic conclusions of this paper. It would seem that the linear quadratic model does a poor job of rationalizing these inventory data. In effect, a contradiction results when it is assumed that the actual inventory path chosen is the one that is optimal

trial production. This is available seasonally adjusted, and (7.a) and (7.b) were estimated for five of the seasonally adjusted data sets (aggregate nondurables, chemicals, food, petroleum, and rubber). Parameter estimates, unfortunately, were uniformly nonsensical, with about three-fourths of them signed wrong. The tests of (7.a) and (7.b), therefore, do not seem worth reporting. But it is perhaps worth noting that var Q < var S for all five data sets. Apparently the FRB index of industrial production does not jibe with the the Department of Commerce figures on sales and inventories. Alan Blinder has suggested to me that this is because the FRB measure includes production that adds value to inventories of works in progress.

according to the model. The allegedly optimal path is dominated by a naive alternative path.

In the model without a target level for inventories, this follows simply because production is more variable than sales. Inventories therefore cannot be chosen simply to perform their putative function, smoothing production.[14] For the model with a target level, the matter is slightly more complicated. Inventories do usually track their target level (except in the petroleum industry). But this makes production and inventories so variable that inventories cannot be chosen as hypothesized to minimize quadratic inventory, production, and target-level costs.

The basic implication of this is that inventories appear to serve some role other than production smoothing. The inventory literature suggests two possible explanations for the excess volatility of production. The first is backlog costs. Now, as we have seen, the typical formulation—a simple cost of having inventories deviate from a target level—is inadequate, at least for these data. But this does not rule out more sophisticated formulations. Some encouraging evidence from a model that includes such a formulation may be found in West (1983*a*).

The second possible explanation relates to stochastic cost variability. It is possible that inventories serve mainly to smooth production not in the face of random varying demand, but in the face of randomly varying costs. In this case production may be more variable than sales (as noted by Topel [1982]). Stochastic cost variability has been crudely allowed for in some recent work by calling the unobservable disturbances "cost shocks" (e.g., Blanchard 1983). But if cost variability is an important determinant of optimal inventory stocks, it clearly is essential to model the cost variations explicitly. Some encouraging evidence from a model that does such modeling may be found in Blinder (1983).

It seems fair to say, however, that a convincing explanation of the excess volatility of production has yet to be made (see Blinder 1983).

Appendix

This Appendix briefly outlines the procedure used to derive the asymptotic covariance matrix of the parameters needed to calculate inequalities (7.a) and (7.b). Much more detail may be found in West (1983*a*).

Write the three-equation system consisting of (10) and (11) as

$$\mathbf{y}_1 = \mathbf{X}\mathbf{b}_1 + \mathbf{u}_1,$$

$$\mathbf{y}_2 = \mathbf{Z}\mathbf{b}_2 + \mathbf{u}_2,$$

$$\mathbf{y}_3 = \mathbf{Z}\mathbf{b}_3 + \mathbf{u}_3,$$

[14] This has been conjectured by Blinder (1981*b*) and Blanchard (1983).

where $\mathbf{y}_1$ is the vector of observations of the left-hand side of (10); $\mathbf{y}_2$ and $\mathbf{y}_3$ contain vectors of inventories and sales; $\mathbf{X}$ contains the right-hand-side variables in (10), and $\mathbf{Z}$ the right-hand-side variables in (11). The elements of $\mathbf{u}_1$ are MA(2) (see n. 10); those of $\mathbf{u}_2$ and $\mathbf{u}_3$ are independently and identically distributed.

Vector $\mathbf{b}_1$ was estimated by two-step, two-stage least squares (2SLS), $\hat{\mathbf{b}}_1 = (\hat{\mathbf{A}}\mathbf{Z}'\mathbf{X})^{-1}\hat{\mathbf{A}}\mathbf{Z}'\mathbf{y}_1$; $\hat{\mathbf{A}}$ is Hansen's (1982) optimal weighting matrix (no heteroscedasticity correction), $\hat{\mathbf{A}} = \mathbf{X}'\mathbf{Z}(\mathbf{Z}'\hat{\mathbf{\Omega}}\mathbf{Z})^{-1}$; $\hat{\mathbf{\Omega}}$, the variance-covariance matrix of $\mathbf{u}_1$, was calculated from a 2SLS estimate of $\mathbf{u}_1$. The numerical simulations in West (1984) suggest that $\mathbf{b}_1$ is likely to be estimated only slightly less efficiently than it would have been had it been estimated by a "full-information" technique that specified the demand side of the market, solved for the equilibrium of the model, and imposed cross-equation constraints. Vectors $\hat{\mathbf{b}}_2$ and $\hat{\mathbf{b}}_3$ were estimated by ordinary least squares (OLS).

Let $\boldsymbol{\theta}$ denote the parameter vector, $\boldsymbol{\theta} = (\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \sigma_{22}, \sigma_{23}, \sigma_{33})$. Variables $\sigma_{ij} = Eu_{it}u_{jt}$ and are needed to calculate (7.a) and (7.b) since they figure into the variances and covariances in these inequalities. (These second moments, again, were calculated as functions of $\mathbf{b}_2$, $\mathbf{b}_3$, and the σ_{ij} as described briefly in the text and in detail in West [1983*a*].) Now the $\hat{\mathbf{b}}_i$ were calculated as just described, the $\hat{\sigma}_{ij}$ from the moments of the OLS residuals. Thus the $\hat{\mathbf{b}}_i$ and $\hat{\sigma}_{ij}$ satisfy the orthogonality conditions:

$$0 = T^{-1}\Sigma\mathbf{h}_t(\hat{\boldsymbol{\theta}}) = \begin{bmatrix} T^{-1}\Sigma\hat{\mathbf{A}}\mathbf{Z}_t(y_{1t} - \mathbf{X}_t'\hat{\mathbf{b}}_1) \\ T^{-1}\Sigma\mathbf{Z}_t(y_{2t} - \mathbf{Z}_t'\hat{\mathbf{b}}_2) \\ T^{-1}\Sigma\mathbf{Z}_t(y_{3t} - \mathbf{Z}_t'\hat{\mathbf{b}}_3) \\ \hat{\sigma}_{22} - T^{-1}\Sigma(y_{2t} - \mathbf{Z}_t'\hat{\mathbf{b}}_2)^2 \\ \hat{\sigma}_{23} - T^{-1}\Sigma(y_{2t} - \mathbf{Z}_t'\hat{\mathbf{b}}_2)(y_{3t} - \mathbf{Z}_t'\hat{\mathbf{b}}_3) \\ \hat{\sigma}_{33} - T^{-1}\Sigma(y_{3t} - \mathbf{Z}_t'\hat{\mathbf{b}}_3)^2 \end{bmatrix}$$

$$= \begin{bmatrix} T^{-1}\hat{\mathbf{A}}\mathbf{Z}'(\mathbf{y}_1 - \mathbf{X}\hat{\mathbf{b}}_1) \\ T^{-1}\mathbf{Z}'(\mathbf{y}_2 - \mathbf{Z}\hat{\mathbf{b}}_2) \\ T^{-1}\mathbf{Z}'(\mathbf{y}_3 - \mathbf{Z}\hat{\mathbf{b}}_3) \\ \hat{\sigma}_{22} - T^{-1}\Sigma(y_{2t} - \mathbf{Z}_t'\hat{\mathbf{b}}_2)^2 \\ \hat{\sigma}_{23} - T^{-1}\Sigma(y_{2t} - \mathbf{Z}_t'\hat{\mathbf{b}}_2)(y_{3t} - \mathbf{Z}_t'\hat{\mathbf{b}}_3) \\ \hat{\sigma}_{33} - T^{-1}\Sigma(y_{3t} - \mathbf{Z}_t'\hat{\mathbf{b}}_3)^2 \end{bmatrix}.$$

As proved by Hansen, then, the asymptotic covariance matrix of $\sqrt{T}(\boldsymbol{\theta} - \boldsymbol{\theta}^*)$ is $(\text{plim } T^{-1}\Sigma\mathbf{h}_{t\boldsymbol{\theta}})^{-1}\mathbf{S}(\text{plim } T^{-1}\Sigma\mathbf{h}_{t\boldsymbol{\theta}}')^{-1}$, where $\boldsymbol{\theta}^*$ is the true but unknown $\boldsymbol{\theta}$ and $\mathbf{S} = \Sigma_{j=-2}^{2} E\mathbf{h}_t\mathbf{h}_{t-j}'$. Further details on how this covariance matrix was calculated may be found in West (1983*a*).

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