Economics Letters 24 (1987) 249–252
North-Holland

A NOTE ON THE POWER OF LEAST SQUARES TESTS FOR A UNIT ROOT

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Received 29 April 1987

Two least squares tests for a unit autoregressive root are inconsistent if the process being studied is stationary around a time trend, and a time trend is not included as a regressor.

1. Introduction

Recent research in macroeconomics has emphasized the possibility that there are unit autoregressive roots in macroeconomic time series [e.g., Nelson and Plosser (1982)]. This has led to increased interest in tests for the presence of such roots. Despite the development of some relatively general and powerful procedures [e.g., Bhargava (1986) and Phillips and Perron (1986)], much research uses the simple ordinary least squares tests suggested in the pioneering research of Fuller (1976) and Dickey and Fuller (1979). Some applications use versions of these tests that do not include a time trend as a regressor [e.g., Kleidon (1986) and Mattey and Meese (1986)]. The purpose of this note is to point out that these versions are inconsistent if the series is stationary around a time trend, an alternative that often [e.g., Kleidon (1986) and Mattey and Meese (1986)] is the relevant one. Indeed, the asymptotic distribution of the test statistics is more concentrated around zero under this alternative than under the null.

2. Model and tests

The researcher assumes that the covariance stationary first difference of a series y_t follows a finite-order autoregression order $k-1$ ($k \geqslant 1$),

$$\Delta y_t = m + \phi_1 \Delta y_{t-1} + \cdots + \phi_{k-1} \Delta y_{t-k+1} + e_t, \tag{1}$$

where e_t is serially uncorrelated and the roots of $1 - \phi_1 z - \dots \phi_k z^k$ are outside the unit circle. One way to test for the hypothesized unit root is to rewrite (1) as

$$y_t = m + \phi_1 \Delta y_{t-1} + \cdots + \phi_{k-1} \Delta y_{t-k+1} + \gamma y_{t-1} + e_t. \tag{2}$$

Let $\hat{\gamma}$ be the OLS estimate of γ. One can compare either $T(\hat{\gamma}-1)$, or the t-statistic for H_0: $\gamma = 1$, to the values in the tables in Fuller (1976) to test the null. One can also include a time trend as a

* This research was supported by the National Science Foundation. This paper was first written while the author was a National Fellow at the Hoover Institution.

0165-1765/87/$3.50

regressor in (2), and use some other tables in Fuller (1976). In either case, values of $T(\hat{\gamma}-1)$, or of the t-statistic, that are far from zero call for rejection of the null.

It seems intuitively obvious that one should include the time trend if the alternative is that y_t is stationary around a time trend. This clearly is the intention of Dickey and Fuller (1979) and Fuller (1976), and it is often done by applied researchers [e.g., Nelson and Plosser (1982)]. Sometimes, however, the time trend is omitted even when trend stationarity is the relevant alternative. In tests for a unit root in aggregate stock prices and dividends, for example, the term is omitted in some (but not all) of Kleidon's (1986) tests and in all of Mattey and Meese's (1986) tests; Shiller (1981), in a paper cited by Mattey and Meese (1986) and, especially, by Kleidon (1986), emphasizes that trend stationarity is the alternative of interest. Kleidon (1986, table 6) finds considerably less evidence against the null of a unit root when the time trend is omitted, and indicates in his footnote 29 that this is unsurprising if the series with a unit root has a non-zero drift. Theorem 1 establishes that this is also to be expected if the trend stationary alternative is correct.

Theorem 1. Suppose that

$$y_t = m + \alpha_1 y_{t-1} + \cdots + \alpha_k y_{t-k} + \delta t + e_t, \tag{3}$$

where $k \geqslant 1$; e_t is zero mean and serially uncorrelated; the roots of $1-\alpha_1 z - \ldots \alpha_k z^k$ are outside the unit circle; Δy_t is stationary to fourth order and ergodic; Δy_t has absolutely summable fourth cumulants; $y_0, \ldots, y_{-k+1}$ are fixed; and, finally, $\delta \neq 0$. Rewrite (3) as

$$y_t = m + \phi_1 \Delta y_{t-1} + \cdots + \phi_{k-1}\Delta y_{t-k+1} + \gamma y_{t-1} + \delta t + e_t, \tag{4}$$

with $\gamma \equiv \Sigma_1^k \alpha_i$, $|\gamma| < 1$ [Fuller (1976)], and $\phi_j \equiv -\Sigma_{i=j+1}^k \alpha_i$. Let γ be estimated by ordinary least squares by regressing y_t on $X_{1t}' \equiv (1, \Delta y_{t-1}, \ldots, \Delta y_{t-k+1})$ and y_{t-1}. Let T be the sample size. Then $T(\hat{\gamma}-1)$ and the usual t-statistic for H_0: $\gamma = 1$ each converge in probability to zero.

Proof. Let X_1 be the $T \times k$ matrix whose tth row is X_{1t}'. Define $M_1 = I - X_1(X_1'X_1)^{-1}X_1'$. For any scalar random variable z_t, let $[z_t]$ be the $T \times 1$ vector with z_t as its tth element. We have

$$\hat{\gamma} = \{[y_{t-1}]'M_1[y_{t-1}]\}^{-1}\{[y_{t-1}]'M_1[y_t]\}$$

$$\Rightarrow T^{3/2}(\hat{\gamma}-1) = \{T^{-3}[y_{t-1}]'M_1[y_{t-1}]\}^{-1}\{T^{-3/2}[y_{t-1}]'M_1[\Delta y_t]\}.$$

It will be shown that $T^{3/2}(\hat{\gamma}-1) \xrightarrow{\mathrm{P}} 0$ by showing that (i) $T^{-3}[y_{t-1}]'M_1[y_{t-1}]$ converges in probability to a non-zero constant, and (ii) $T^{-3/2}[y_{t-1}]'M_1[\Delta y_t] \xrightarrow{\mathrm{P}} 0$. This will establish that $T(\hat{\gamma}-1) \xrightarrow{\mathrm{P}} 0$. Then the fact that $T^{3/2}(\hat{\gamma}-1) \xrightarrow{\mathrm{P}} 0$ will be used to show that (iii) the t-statistic for $\mathrm{H}_0: \gamma = 1$ converges in probability to zero.

(i) By the definition of M_1, $\{T^{-3}[y_{t-1}]'M_1[y_{t-1}]\} = T^{-3}[y_{t-1}]'[y_{t-1}] + \{T^{-2}[y_{t-1}]'X_1\}\{T^{-1}X_1'X_1\}^{-1}\{T^{-2}X_1'[y_{t-1}]\}$. Let $\mu = \mathrm{E}\Delta y_t = \delta/(1-\alpha_1 - \cdots - \alpha_k)$, $\mu \neq 0$, since $\delta \neq 0$ by hypothesis. Recall that a stationary finite-order autoregressive process has an absolutely summable autocovariance function. By Lemma 2.2 in West (1987), then, $T^{-3}[y_{t-1}]'[y_{t-1}] \xrightarrow{\mathrm{P}} (1/3)\mu^2$. By Lemma A.3 in West (1987), $\{T^{-2}[y_{t-1}]'X_1\} \xrightarrow{\mathrm{P}} (1/2)\mu \mathrm{E}X_{1t}'$. It follows that

$$\begin{aligned}\mathrm{p}\lim T^{-3}\{[y_{t-1}]'M_1[y_{t-1}]\} &= (1/3)\mu^2 - (1/2)\mu\mathrm{E}X_{1t}'(\mathrm{E}X_{1t}X_{1t}')^{-1}(1/2)\mu\mathrm{E}X_{1t} \\ &= (1/3)\mu^2 - (1/4)\mu^2 \neq 0.\end{aligned}$$

The second equality follows since $\mathrm{E}X_{1t}'(\mathrm{E}X_{1t}X_{1t}')^{-1}\mathrm{E}X_{1t}=1$: let [1] be a $T\times 1$ vector of ones. Then

$$0=M_1[1]\Rightarrow[1]=X_1(X_1'X_1)^{-1}X_1'[1]=X_1(X_1'X_1)^{-1}\Sigma X_{1t}$$

$$\Rightarrow[1]'[1]=\Sigma X_{1t}'(\Sigma X_{1t}X_{1t}')^{-1}\Sigma X_{1t}\Rightarrow 1=(T^{-1}\Sigma X_{1t}')(T^{-1}\Sigma X_{1t}X_{1t}')^{-1}(T^{-1}\Sigma X_{1t}). \tag{5}$$

(ii) Write

$$y_t=\mu_0+\mu t+\Theta_t,\qquad \Delta y_t=\mu+\Delta\Theta_t;\quad \Theta_t\equiv y_t-\mathrm{E}y_t.$$

Since one of the columns of X_1 contains the constant term,

$$\begin{aligned}
&T^{-3/2}[y_{t-1}]'M_1[\Delta y_t]\\
&\quad=T^{-3/2}([\Theta_{t-1}]+[\mu t])'M_1[\Delta\Theta_t] \qquad\qquad (6)\\
&\quad=T^{-3/2}[\Theta_{t-1}]'[\Delta\Theta_t]-T^{-1/2}(T^{-1}[\Theta_{t-1}]'X_1)(T^{-1}X_1'X_1)^{-1}(T^{-1}X_1'[\Delta\Theta_t])\\
&\qquad+T^{-3/2}[\mu t]'[\Delta\Theta_t]-T^{-3/2}([\mu t]'X_1)(X_1'X_1)^{-1}(X_1'[\Delta\Theta_t]).
\end{aligned}$$

Since $\Delta\Theta_t$ and Θ_t are stationary and ergodic, the first two of the four terms after the second equality in (6) converge in probability to zero. Apart from a factor of μ, the third term is $T^{-3/2}\Sigma t\Delta\Theta_t=T^{-3/2}[-\Theta_0-\Theta_1-\cdots-\Theta_{T-1}+T\Theta_T]$. This has expectation zero. Since y_t has an absolutely summable autocovariance function, it is easy to show that $\lim_{T\to\infty}\operatorname{var}\{T^{-3/2}[-\Theta_0-\Theta_1-\cdots-\Theta_{T-1}+T\Theta_T]\}=0$. By Chebyschev's inequality, then, this term converges in probability to zero.

Let C_T and D_T be $k\times k$ diagonal matrices, C_T having T^{-2} in its (1, 1) element and $T^{-3/2}$ elsewhere on the diagonal, D_T having $T^{-1/2}$ in its (1, 1) element and T^{-1} elsewhere. Let X_1^* be the $T\times k$ matrix with its tth row

$$(1\Delta y_{t-1}-\mu\Delta y_{t-2}-\mu\ldots\Delta y_{t-k+1}-\mu)\equiv(1\Delta\Theta_{t-1}\Delta\Theta_{t-2}\ldots\Delta\Theta_{t-k+1})\equiv(1Z_t')\equiv X_{1t}^{*\prime}. \tag{7}$$

Note that X_1^* may be obtained from X_1 by a non-singular linear transformation. Apart from a factor of μ, therefore, the fourth and final term after the second equality in (6) may be written

$$[t]'X_1^*C_T(T^{3/2}D_TX_1^{*\prime}X_1^*C_T)^{-1}D_TX_1^{*\prime}[\Delta\Theta_t]. \tag{8}$$

Now, $T^{-1/2}\Sigma\Delta\Theta_t=T^{-1/2}(-\Theta_0+\Theta_T)\xrightarrow{\mathrm{P}}0$ by Chebyschev's inequality, since y_t has an absolutely summable autocovariance function. It was just shown that $T^{-3/2}\Sigma t\Delta\Theta_t\xrightarrow{\mathrm{P}}0$. It follows that

$$[t]'X_1^*C_T\xrightarrow{\mathrm{P}}(1/2,0\ldots0),$$

$$D_TX_1^{*\prime}[\Delta\Theta_t]\xrightarrow{\mathrm{P}}(0\ \mathrm{E}\Delta\Theta_t\Delta\Theta_{t-1}\ldots\mathrm{E}\Delta\Theta_t\Delta\Theta_{t-k+1})',$$

and that $T^{3/2}D_TX_1^{*\prime}C_T$ converges in probability to a block diagonal matrix with 1 in its (1, 1) position and $\mathrm{E}Z_tZ_t'$ in its $(k-1)\times(k-1)$ lower right-hand block. This implies that (8) converges in probability to zero.

(iii) Write the square of the t-statistic for H_0: $\gamma = 1$ as

$$\hat{\tau}^2 = \left\{T^{3/2}(\hat{\gamma}-1)\right\}^2 / \left\{\hat{\sigma}^2\left(T^{-3}[y_{t-1}]'M_1[y_{t-1}]\right)^{-1}\right\},$$

where $\hat{\sigma}^2$ is T^{-1} times the sum of squared OLS residuals. It was just shown that $T^{3/2}(\hat{\gamma}-1)$ converges in probability to zero, $T^{-3}[y_{t-1}]'M_1[y_{t-1}]$ to a non-zero constant. To establish that $\hat{\tau} \xrightarrow{p} 0$, then, it suffices to show that $\hat{\sigma}^2$ converges in probability to a non-zero constant.

Let F_T be a $(k+1)\times(k+1)$ diagonal matrix with $T^{-3/2}$ in its $(k+1,\ k+1)$ position, $T^{-1/2}$ elsewhere on the diagonal; let X be the $T\times(k+1)$ regression matrix; let X^* be the $T\times(k+1)$ matrix with $(X_{1t}^{*\prime} y_{t-1}) \equiv (1\ Z_t',\ y_{t-1})$ in its tth row [X_{1t}^* and Z_t are defined in (8)]; let $M = I - X(X'X)^{-1}X' = I - X^*(X^{*\prime}X^*)^{-1}X^{*\prime}$. Since $y_{t-1} = \mu_0 + \mu(t-1) + \Theta_{t-1}$, $t = \mu^{-1}(y_{t-1} + \mu - \mu_0 - \Theta_{t-1}) \Rightarrow M[\delta t] = M[-\delta\mu^{-1}\Theta_{t-1}]$ since y_{t-1} and a constant are among the columns of X. Define $\rho_t = e_t - \delta\mu^{-1}\Theta_{t-1}$. Then since $\hat{\sigma}^2 = T^{-1}[\delta t + e_t]'M[\delta t + e_t]$,

$$\hat{\sigma}^2 = T^{-1}[\rho_t]'[\rho_t] - T^{-1}[\rho_t]'X^*F_T\left(F_TX^{*\prime}X^*F_T\right)^{-1}F_TX^{*\prime}[\rho_t].$$

Now, $T^{-1/2}[\rho_t]'X^*F_T \xrightarrow{p} (\mathrm{E}X_{1t}^{*\prime}\rho_t(\mu/2)\mathrm{E}\rho_t) = (\mathrm{E}X_{1t}^{*\prime}\rho_t 0)$ by ergodicity, Lemma A.2 in West (1987), and the fact that $\mathrm{E}\rho_t = 0$. By Lemmas 2.2 and A.2 in West (1987), $T^{-2}\Sigma y_{t-1}\mathrm{E}X_{1t}^* \xrightarrow{p} (\mu/2)\mathrm{E}X_{1t}^*$, $T^{-3}\Sigma y_{t-1}^2 \xrightarrow{p} \mu^2/3$, and by ergodicity $T^{-1}\Sigma X_{1t}^*X_{1t}^{*\prime} \xrightarrow{p} \mathrm{E}X_{1t}^*X_{1t}^{*\prime}$; this means that the $(k\times k)$ upper left-hand block of $(F_TX^{*\prime}X^*F_T)^{-1}$ converges in probability to $\{\mathrm{E}X_{1t}^*X_{1t}^{*\prime} - (3/4)\mathrm{E}X_{1t}^*\mathrm{E}X_{1t}^{*\prime}\}^{-1}$. It follows that $\hat{\sigma}^2 \xrightarrow{p} \mathrm{E}\rho_t^2 - (\mathrm{E}X_{1t}^*\rho_t)'\{\mathrm{E}X_{1t}^*X_{1t}^{*\prime} - (3/4)\mathrm{E}X_{1t}^*\mathrm{E}X_{1t}^{*\prime}\}^{-1}\mathrm{E}X_{1t}^*\rho_t = \mathrm{E}\rho_t^2 - (\mathrm{E}Z_t\rho_t)'(\mathrm{E}Z_tZ_t')^{-1}\mathrm{E}Z_t\rho_t$, which is non-zero since $(\rho_t,\ Z_t')'$ has a variance–covariance matrix of full rank. The last equality follows since ρ_t and Z_t each have unconditional mean zero.

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