Supplemental Notes: Counting and Probability

Economics 310 **Spring 2015**

Counting and Probability

Early problems in the history of probability often involved games of chance where the probabilities for basic outcomes were clear but the probabilities of interesting events were difficult to calculate because of the large number of basic outcomes for events of interest. A number of these problems can be formulated as problems of drawing k objects with and without replacement out of a set of n while being or not being concerned with the ordering. Solving them requires counting the ways in which you can do this. As an example we consider the case where we have $n = 4$ objects, labeled A, B, C, and D, and wish to draw $k = 2$. Recall that $n! = n \times (n-1) \times (n-2) \times \cdots \times 1$ is called n factorial.

Result 1 *(ordered, with replacement) The total number of ways k objects can be drawn out of a set of n with replacement is n^k.*

For the first draw there are n choices, for the second one there are again n choices and so on. In the example, the set of outcomes is $\{AA, AB, AC, AD, BA, BB, BC, BD, CA, CB, CC, CD, DA, DB, DC, DD\}$, with sixteen elements.

Result 2 *(ordered, without replacement) The total number of ways k objects can be drawn out of a set of n without replacement is $n \times (n-1) \times (n-2) \times \ldots \times (n-k+1) = n!/(n-k)!$.*

For the first draw there are n choices, for the second one there are $n-1$ choices and so on. In the example the set of outcomes is $\{AB, AC, AD, BA, BC, BD, CA, CB, CD, DA, DB, DC\}$, with twelve elements.

Result 3 *(unordered, without replacement) The total number of ways k objects can be drawn out of a set of n without replacement is $n!/(k!(n-k)!)$.*

The total number of ways k objects can be drawn out of a set of n *in order* without replacement is $n!/(n-k)!$ by Result 2. If we do not care about the ordering, we have to take account of the number of different ways we can order k objects. This is $k!$, by Result 2 again, so we have to divide this into the $n!/(n-k)!$ to get

$$\binom{n}{k} \equiv \frac{n!}{k!(n-k)!}.$$

The left hand side notation is read as "n choose k." In our example, the set of outcomes is $\{AB, AC, AD, BC, BD, CD\}$, with six elements.

Example 1: How many bridge hands of 13 cards can be formed from a 52-card deck? Since the order does not matter, Result 3 is relevant. The number is

$$\binom{52}{13} = \frac{52!}{13!(52-13)!} = 635,013,559,600.$$

Result 4 *(unordered, with replacement) The total number of ways k objects can be drawn out of a set of n with replacement is $(n+k-1)!/(k!(n-1)!)$.*

This one is messier than the others. Reformulate the problem as follows: put k objects in n bins, allowing for more than one object per bin. We can describe the result by the k numbers or by a sequence of $n-1$ zeros and k ones, where a zero indicates that we have finished with one of the n bins, and one indicates one of the objects in that particular bin. For example, with $n=3$ bins and $k=2$ objects we could have the following outcome: (2,3) which would be coded as 0101: 0 (because the first bin is empty), 1 (because there is an object in the second bin), 0 (because there is no additional object in the second bin), 1 because there is an object in the third bin. We do not record the last 0 because it would always end in a zero. In the same example (2,2) would be coded as 0110: 0 because the first bin is empty, 1 because there is an object in the second bin, 1 because there is a second object in this bin, 0 because there is no additional object in this bin.

Now the problem is one of choosing a set of k ones out of a set of $n+k-1$ which can be done in $(n+k-1)!/(k!(n-1)!)$ different ways.

In the example the set is $\{AA, AB, AC, AD, BB, BC, BD, CC, CD, DD\}$, with ten elements.

Example 2: Isaac Newton and Samuel Pepys debated the following question: is the probability of tossing at least one six in six tosses with a fair die smaller than, equal to, or larger than the probability of tossing at least two sixes in twelve tosses?

Consider the probability of the first event. There are 6^6 different outcomes for the six tosses. Each has probability $1/6^6$. The question is how many of the 6^6 outcomes are <u>favorable</u>, that is, how many have at least one six. It is easier to answer the opposite: how many outcomes have no six at all. There are five possibilities for each toss in that case, so 5^6 have no six. Hence the probability of at least one six is

$$Pr(\text{at least one six}) = 1 - Pr(\text{no six}) = 1 - 5^6/6^6 \approx 0.665.$$

Consider the probability of the second event. There are 6^{12} different outcomes, again each with the same probability $1/6^{12}$. How many have at least two sixes is 6^{12} minus the number that have at most one six. At most one six is either no six or exactly one six. The number of outcomes with no six is 5^{12}. The event of exactly one six can be partitioned into twelve events depending on the location of the six. The number of outcomes with a six in the first toss is 5^{11}. Hence the number of outcomes with a single six is $12 \cdot 5^{11}$. Hence the probability of at least two sixes is

$$Pr(\text{at least two sixes}) = 1 - Pr(\text{no six}) - Pr(\text{one six})$$

$$= 1 - 5^{12}/6^{12} - 12 \cdot 5^{11}/6^{12} \approx 0.619.$$

Hence, at least two sixes in twelve tosses is less likely than at least one six in six tosses.

Example 3: Suppose there are 25 people with dogs at a park, and suppose they all choose randomly from a set of n breeds of dogs. How large should n be to ensure the chances of everybody having a different breed of dog is at least 0.5? (As n increases this probability clearly goes up.)

The number of ways 25 people can choose from n types of dogs is n^{25}. All these outcomes are equally likely, with probability $1/n^{25}$. The number of ways they can pick with a different dog breed for everybody is $n!/(n-25)!$. Now we want to know n such that

$$\frac{n!/(n-25)!}{n^{25}} \geq 0.5, \qquad \text{and } \frac{(n-1)!/(n-1-25)!}{(n-1)^{25}} < 0.5.$$

At $n = 442$:

$$\frac{n!/(n-25)!}{n^{25}} = 0.5008,$$

and

$$\frac{(n-1)!/(n-1-25)!}{(n-1)^{25}} = 0.49996.$$

Hence $n = 442$ is the solution.