

The Population Dynamics of Black-White-Mulatto Racial Systems

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June 17, 2010

Abstract Building on the two-sex model of intergenerational transmission developed by Preston and Campbell (*AJS* 1993), this paper provides a theoretical analysis of the dynamics of the racial distribution in black-white-mulatto racial systems (with black-white systems as a special case). We introduce the concept of “boundedness” for racial classification and racial switching, and show that bounded classification and switching imply long-run racial homogeneity. We also show that differential reproduction (even with bounded classification and switching) not only permits long-run racial heterogeneity but also more complicated outcomes such as limit cycles. Beyond our theoretical analysis, we also address the empirical case of Puerto Rico in the early 20th century, attempting to account for the dramatic growth of the white population share. While the nature of our (discrete-generations) model limits our ability to make precise short-run population projections, we show that the observed social norms governing the racial classification of children would have promoted the growth of the mulatto (not the white) share. Because the effects of racial classification and differential reproduction were roughly offsetting, the observed growth of the white share can be attributed almost entirely to racial switching.

*For helpful comments and suggestions, I am grateful to Phil Bonacich, Mara Loveman, John Martin, Jenna Nobles, and participants in the demography seminar at UW-Madison.

1 Introduction

The racial composition of Puerto Rico changed dramatically during the first half of the 20th century. The white population share rose from under 62% in 1899 to 65% in 1910 to 73% in 1920, reaching nearly 80% by 1950 (Duany 2002, p 248). Because external migration was negligible during the first decades of the century, these dynamics must have been driven primarily by racial “boundary crossing” (individuals switching between racial classes) or by racial “boundary shifting” (changes in the social norms governing racial classification) or by more mundane demographic forces such as differential reproduction (Loveman and Muniz 2007). In their analysis, Loveman and Muniz (2007) emphasize observed changes in racial classification practices (regarding children of mixed couples), though also find some indirect evidence that suggests racial switching (by non-white females upon marriage to a white male).

Reflection on this empirical case prompts a more general analysis of the population dynamics of black-white-mulatto racial systems.¹ Several theoretical questions arise immediately. How do the social norms governing the racial classification of children affect the dynamics of the racial distribution? Especially in light of the claim that “biological and cultural homogenization” is “inevitable” in these racial systems (Hoetink 1967, p 150), should we anticipate racial homogeneity or heterogeneity in the long run? Will the qualitative nature of the long-run equilibrium depend also on the patterns of racial mixing, racial switching, and differential reproduction? Of course, given a model of population dynamics, we might also wish to address particular empirical cases, attempting to quantify the underlying sources of observed change in the racial distribution.

Unfortunately, conventional demographic models are poorly suited for addressing these theoretical and empirical issues. In making population projections, demographers commonly use “one-sex” models which presume that the characteristics of children depend on the characteristics of only one parent. But in many societies, racial classification depends on the racial class of both the mother and father. To offer the simplest example, consider the “one-drop” rule used in the mainland US whereby a child is classified as black when the mother or father (or both) is black (see, e.g., Davis 1991). Population projections which assume that children simply inherit the mother’s (or, alternatively, the father’s) racial class would be flawed. Our present analysis thus requires a more flexible “two-sex” model.²

¹The mulatto category was used in Puerto Rico during the first decades of the 1900s. Our analysis also encompasses black-white systems, which are (from a formal perspective) merely a special case of black-white-mulatto systems.

²Demographers are well aware of the limitations of one-sex models. Indeed, in their discussion of the standard (female-dominant) projection method, Preston et al (2001, p 129) explicitly mention the present application: “Real situations are often too complex for the approach to be very satisfactory, e.g., when the classification of a child’s race is more likely to be associated with the father’s race than the mother’s.” We would merely add that male-dominant projection is also inadequate for the task at hand. The demographic literature on two-sex models includes Caswell and Weeks

In particular, we extend the two-sex framework originally developed by Preston and Campbell (1993) to study the intergenerational transmission of IQ scores. For our present application, their inheritance matrix (characterizing transmission of IQ from parents to child) becomes our racial classification matrix (characterizing the social norms of racial inheritance). Extending their framework, we also incorporate the possibility of racial switching (which has no analog in IQ transmission) and a flexible specification of racial mixing (to allow matching outcomes between the extremes of complete endogamy and random matching). The discrete-generations specification of the model – the current generation replaces the preceding generation in each period – necessarily limits our ability to make precise short-run population projections. Nevertheless, we will demonstrate that there is much to be learned both theoretically and empirically from even this “coarse-grained” model.

After specifying the model in Section 2, the next three sections comprise our theoretical analysis. In Section 3, we examine the effects of racial classification (ignoring racial switching and differential reproduction). Inspection of the observed classification matrices for Puerto Rico suggests the concept of “bounded” classification. Roughly, classification is bounded when the child’s racial class cannot lie outside the bounds established by the parents’ racial classes. Under this condition, we show that the population will become racially homogeneous in the long run (Proposition 1). We also demonstrate the potential for multiple equilibria (i.e., the particular race shared by the long-run population may depend on the initial racial distribution). In Section 4, we consider the effects of racial switching (assuming bounded racial classification and ignoring differential reproduction). If racial switching occurs only through racial mixing (i.e., some individuals adopt their partner’s racial class upon matching), then switching is also “bounded” in the sense that a couple’s post-switching racial classes cannot lie outside the bounds established by their pre-switching racial classes. Generalizing our first result, we show that bounded classification and bounded switching together imply long-run racial homogeneity (Proposition 2). In Section 5, we consider the effects of differential reproduction. We show that (even when classification and switching are bounded) differential reproduction may induce not only racial heterogeneity but also more complicated long-run dynamics such as limit cycles (in which the racial distribution never reaches a steady state).

It is important to recognize that the sorts of outcomes highlighted by our theoretical analysis – long-run racial homogeneity, multiple equilibria, limit cycles – cannot be generated by conventional one-sex models. From a mathematical perspective, the linearity of one-sex models dictates that the population distribution will always converge to a unique, heterogeneous equilibrium that is independent of initial conditions.³ In contrast, the non-linearity of two-sex models permit more complicated

(1986), Pollak (1986, 1990), Schoen (1988), Preston and Campbell (1993), and Pollard (1997).

³A (discrete-time dynamical system) model is *linear* when it can be written in the form $x_{t+1} = x_t A$ where x_{t+1} and x_t are $(1 \times n)$ vectors and A is an $(n \times n)$ matrix of (fixed) parameters. The qualitative behavior of linear systems is a consequence of the Perron-Frobenius Theorem (and

dynamics.⁴ To the extent that applied demographers (and sociologists more generally) remain unfamiliar with non-linear models, the present paper thus establishes one important motivation for these models (two-sex intergenerational transmission) and illustrates some of the types of dynamics they permit.⁵

Following our theoretical analysis, we return in Section 6 to the empirical case of Puerto Rico. Proceeding counterfactually, we first show that the observed racial classification practices (given no racial switching or differential reproduction) did not contribute to the growth of the white population share. Indeed, given the observed classification matrices for 1910 and 1920, the mystery is why Puerto Rico did not become mulatto. Continuing to proceed counterfactually, we next show that the observed pattern of differential reproduction (given observed racial classification and no racial switching) can help explain why the white share did not fall, but cannot account for its dramatic growth. Consequently, racial switching must also have played an important role. Our population projections suggest that bounded switching (when one partner adopts the other's racial class upon marriage) cannot by itself account for the observed growth of the white population share, and must have been accompanied by unbounded switching (by one or both partners who shared the same childhood racial class). Section 7 contains some concluding remarks.

2 The model

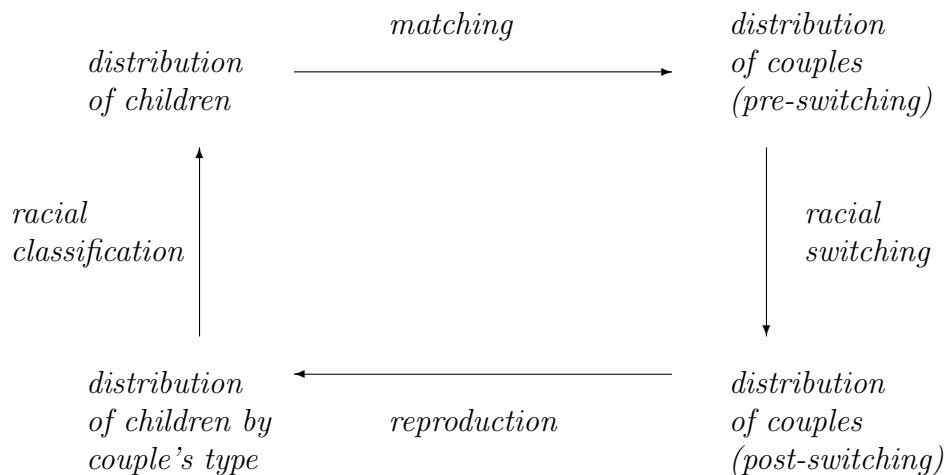
Before specifying the model formally, it may be helpful to begin informally with the schematic diagram presented in Figure 1. Starting in the upper-left corner, assume some initial (period t) distribution of children across the 3 racial classes (white, mulatto, black). Moving from the upper-left to upper-right corner, these children form couples of 9 different types (white-white, white-mulatto, white-black, mulatto-white, etc). The extent of racial mixing will be reflected by matching parameters (specified below) which permit a range of possibilities between the extremes of complete endogamy and random matching. Then, moving from the upper-right to lower-right corner, the distribution of couples may be altered through racial switching. This

associated regularity conditions on the A matrix). See Farina and Rinaldi (2000) for a mathematical treatment of linear models in general, and the standard demographic application (where A is the Leslie matrix) in particular.

⁴As discussed in the next section, our model is non-linear because interracial matching probabilities depend on the current racial distribution. Adopting the notation from the preceding footnote, the elements of the transition matrix A are not fixed, but instead depend on x_t . See Allman and Rhodes (2004) for an introduction to non-linear dynamical systems; more advanced textbooks include Drazin (1992) and Strogatz (1994).

⁵Demographers studying plant and animal (as opposed to human) populations have made much greater use of non-linear models (Lee 1987; Caswell 2008), as have formal (as opposed to applied) human demographers (see, e.g., Bonneuil 1994). In mathematical sociology, there is a long (if not rich) tradition of non-linear models, dating back at least to Simon (1952). See Fararo (1989) for an introduction to these models from the perspective of sociological theory.

Figure 1: Schematic diagram of the model



might take various forms. For instance, racial switching might occur when some non-white women adopt the racial classification of their white husbands (cf Loveman and Muniz 2007). Alternatively, racial switching might occur when some individuals or couples choose to “pass for white” (even when the couple shared the same non-white childhood racial classification). Then, moving from the lower-right to lower-left corner, couples have children (we allow average fertility to vary across types of couples), resulting in a distribution of children across the 9 types of couples. Finally, moving from the lower-left to the upper-left corner, children are assigned to racial classes based on the racial classification of their parents. While “official” classification rules (reflected by the instructions given to census takers) are sometimes deterministic (e.g., any child of a white-mulatto couple is necessarily mulatto), the model permits stochastic assignment to reflect the probabilities observed empirically. Having made one full loop around this diagram, we thus obtain the racial distribution for the next (period $t + 1$) generation of children; repeated iteration will yield population projections for subsequent generations.

Following Preston and Campbell (1993), our formal specification of the model employs matrix algebra. However, there is a close connection between the schematic diagram and the algebraic treatment. Once the racial distribution of children is specified as a (row) vector, each arrow in Figure 1 corresponds to (post)multiplication by a matrix. The population dynamics may thus be written as

$$x_{t+1} = x_t M_t S R C / \lambda_t \quad (1)$$

where x_t is the (1×3) *racial distribution* of children for period t

M_t is the (3×9 row stochastic) *matching matrix* for period t
 S is the (9×9 row stochastic) *switching matrix*
 R is the (9×9 diagonal) *reproduction matrix*
 C is the (9×3 row stochastic) *classification matrix*
 λ_t is the (scalar) *population growth factor* for period t

While the precise ordering is arbitrary, we construct these matrices with the racial classes – white (w), mulatto (m), black (b) – arranged in the sequence (w, m, b) . Similarly, we arrange the types of couples in the sequence $(ww, wm, wb, mw, mm, mb, bw, bm, bb)$, with the first element in each pair denoting the female’s race.^{6,7}

Because we are interested in the relative (rather than absolute) size of racial groups, we have cast the racial distribution as a probability (rather than frequency) distribution. This necessitates the final division by the population growth factor, which may be written as

$$\lambda_t = x_t M_t S R C \mathbf{1} \tag{2}$$

where $\mathbf{1}$ denotes an (3×1) column vector of 1’s. All non-diagonal elements of the reproduction matrix are zero, while diagonal element $R(ij, ij)$ indicates the average number of girls for each couple of type ij .⁸ In the special case where average fertility does not differ across types of couples (so that $R(ij, ij) = \lambda$ for all ij), equation (1) reduces simply to

$$x_{t+1} = x_t M_t S C \tag{3}$$

because the reproduction matrix is $R = \lambda I$ (where I is the identity matrix) while the population growth rate is $\lambda_t = \lambda$.

Both the switching and classification matrices are row stochastic (i.e., each row is a probability distribution). For the switching matrix, the 9 rows correspond to the couple’s pre-switching type (based on the childhood race of each partner) while the 9 columns correspond to the couple’s post-switching type. Thus, element $S(ij, i'j')$

⁶While matrix algebra forces us to choose some ordering, we have tried to make our presentation easier to follow by avoiding numerical indices whenever possible. For instance, we will use $x_t(m)$ instead of $x_t(2)$ to denote the proportion of mulattos in period t , and use $C(mb, b)$ instead of $C(6, 3)$ to denote the probability that the child of a mulatto-black couple is classified as black.

⁷Beyond the two extensions of the original Preston-Campbell model mentioned in the introduction, it is also worth noting that we have slightly changed the specification of the model, characterizing the state of the process in period t by the distribution of children (the upper-left corner of Figure 1) rather than the distribution of couples (the upper-right corner). While this change might seem trivial (having no effect on the population projection), it usefully reduces the dimensionality of the state space from 9 (types of couples) to 3 (races of children). In fact, because the racial distribution is a probability distribution, our model is actually 2-dimensional, making possible our use of phase diagrams throughout this paper.

⁸Consequently, population size is stationary if $R(ij, ij) = 1$ for all types ij . Preston and Campbell (1993, p 1004) adopt a slightly different convention, interpreting $R(ij, ij)$ as the average number of *children* born to a type ij couple, which implies that population size is stationary if $R(ij, ij) = 2$. In turn, they multiply by $(1/2)$ to move from the number of children to the number of couples.

indicates the probability that a couple with pre-switching type ij transitions to post-switching type $i'j'$.⁹ In the simplest case with no racial switching, S is the identity matrix. For the classification matrix, the 9 rows correspond to the couple's post-switching type while the 3 columns correspond to child's race. Thus, element $C(ij, k)$ indicates the probability that a child born to a couple of type ij will be classified into race k . As already indicated, racial classification "rules" may be probabilistic to match empirical findings.

The absence of a time subscript on the S , R , and C matrices indicates that these matrices are assumed fixed over time. Obviously, as an empirical matter, the parameters in these matrices may be changing over time.¹⁰ But for the present paper, our central theoretical goal is to understand population dynamics holding fixed these (switching, reproduction, and classification) parameters. We will also assume that the parameters of the matching matrix are fixed over time. However, because opportunities for racial mixing vary with the racial distribution, the elements of the M_t matrix will change endogenously over time (hence the time subscript).

To motivate our specification of the matching matrix, it is helpful to view the racial distribution as the distribution of *girls* across racial classes.¹¹ Adopting this perspective, the 3 rows of the matching matrix correspond to the girl's racial class, the 9 columns of this matrix correspond to the types of couples, and element $M_t(k, ij)$ indicates the probability (for period t) that a girl of race k will form a couple of (pre-switching) type ij .¹² Fixing the girl's race (say w), 3 types of couples are feasible (ww , wm , wb) while the remaining 6 types are infeasible (because they presume a non- w female). Transitions to the infeasible types occur with zero probability. For the feasible transitions, we wish to permit flexibility in the degree of racial mixing, with complete endogamy and random matching as limiting cases. In particular, we assume

⁹The sequential structure of the model presumes that all switching occurs after matching. To permit switching also to occur before matching, we might incorporate another switching matrix (call it P) so that equation (1) becomes $x_{t+1} = x_t P M_t S R C / \lambda_t$. (Note that P would be 3×3 and hence not allow for gender differentials.) Anticipating our empirical application (section 6), we retain our original specification because we are especially interested in the effects of switching at the time of marriage (cf Loveman and Muniz 2007), and we further suspect that other forms of switching (e.g., switching associated with intragenerational social mobility) often occur after marriage.

¹⁰Indeed, much of Loveman's work on Puerto Rico (Loveman 2007, Loveman and Muniz 2007) focuses on the social and political forces that might have caused changes in the S and C matrices.

¹¹In our model, racial classification does not depend on the child's gender. Thus, boys and girls share the same racial distribution. We also assume equal numbers of boys and girls in each generation (i.e., the sex ratio is always 1:1). Nevertheless, for purposes of interpretation, it remains helpful to view x_t as the distribution for girls.

¹²For simplicity, we assume that all individuals are matched.

$$\begin{aligned}
M_t(w, ww) &= x_t(w) + (1 - \mu) x_t(m) + (1 - \beta) x_t(b) \\
M_t(w, wm) &= \mu x_t(m) \\
M_t(w, wb) &= \beta x_t(b) \\
M_t(m, mw) &= \mu x_t(w) \\
M_t(m, mm) &= (1 - \mu) x_t(w) + x_t(m) + (1 - \gamma) x_t(b) \\
M_t(m, mb) &= \gamma x_t(b) \\
M_t(b, bw) &= \beta x_t(w) \\
M_t(b, bm) &= \gamma x_t(m) \\
M_t(b, bb) &= (1 - \beta) x_t(w) + (1 - \gamma) x_t(m) + x_t(b)
\end{aligned} \tag{4}$$

where μ is the *white-mulatto matching parameter*

β is the *white-black matching parameter*

γ is the *mulatto-black matching parameter*

Each of the matching parameters is scaled between 0 and 1, with 0 indicating no racial mixing between the relevant groups (i.e., complete pairwise homophily bias), and 1 indicating unbiased racial mixing between these groups (i.e., no pairwise homophily bias). Thus, the outcome of the matching process is complete endogamy when $\mu = \beta = \gamma = 0$, and random matching when $\mu = \beta = \gamma = 1$. Of course, we may set these parameters to intermediate values to reflect empirical patterns of racial mixing. Note that, except in the extreme case of complete endogamy, matching outcomes are influenced by opportunities for contact (as emphasized by Blau 1977, Blau and Schwartz 1984). Consequently, the matching matrix (unlike the S , R , and C matrices) depends on the current racial distribution.

Perhaps it is also helpful to consider the distribution of couples induced by our matching process. Using matrix algebra, we multiply the racial distribution x_t by the matching matrix M_t to obtain the couples distribution $x_t M_t$. Listing the elements of this (1×9) vector,

$$\begin{aligned}
(x_t M_t)(ww) &= [x_t(w)]^2 + (1 - \mu) x_t(w) x_t(m) + (1 - \beta) x_t(w) x_t(b) \\
(x_t M_t)(mm) &= [x_t(m)]^2 + (1 - \mu) x_t(w) x_t(m) + (1 - \gamma) x_t(m) x_t(b) \\
(x_t M_t)(bb) &= [x_t(b)]^2 + (1 - \beta) x_t(w) x_t(b) + (1 - \gamma) x_t(m) x_t(b) \\
(x_t M_t)(wm) &= (x_t M_t)(mw) = \mu x_t(w) x_t(m) \\
(x_t M_t)(wb) &= (x_t M_t)(bw) = \beta x_t(w) x_t(b) \\
(x_t M_t)(mb) &= (x_t M_t)(bm) = \gamma x_t(m) x_t(b)
\end{aligned} \tag{5}$$

While other parameterizations of the matching matrix are possible, our specification

is simple, flexible, and ensures that the induced pattern of matching is feasible for any realization of the racial distribution.¹³

3 The effects of racial classification

To begin our theoretical analysis, we consider a simple version of the model with no racial switching and no differential reproduction. This will allow us to examine how population dynamics are affected by racial classification alone. Given no switching (so that $S = I$), equation (3) reduces to

$$x_{t+1} = x_t M_t C \tag{6}$$

The relative simplicity of (the matrix specification of) this equation might suggest that analysis would be trivial. However, equation (6) is in fact a non-linear system of 3 equations which involves the 3 matching parameters and the 27 parameters of the classification matrix.¹⁴

3.1 Bounded classification matrices

For purposes of analysis, it will thus be useful to impose some simplifying assumptions on the classification matrix. In doing so, we are closely guided by the empirical findings on racial classification reported by Loveman and Muniz (2007) based on census data from Puerto Rico in 1910 and 1920. Rearranging their data, we obtain the two classification matrices reported in Table 1.¹⁵ While Loveman and Muniz

¹³Arranging the elements of the $x_t M_t$ vector into the 3×3 matrix

$$\begin{bmatrix} (x_t M_t)(ww) & (x_t M_t)(wm) & (x_t M_t)(wb) \\ (x_t M_t)(mw) & (x_t M_t)(mm) & (x_t M_t)(mb) \\ (x_t M_t)(bw) & (x_t M_t)(bm) & (x_t M_t)(bb) \end{bmatrix},$$

any parameterization of the matching matrix must guarantee that the row marginals match the racial distribution (of girls) and that the column marginals match the racial distribution (of boys). This is easily verified for our present specification by substituting equation 5 into this matrix.

¹⁴Because x_t is a probability distribution, we can reduce the system to 2 equations by noting that $x_t(m) = 1 - x_t(w) - x_t(b)$ for all t . Further, because each row of C is a probability distribution, there are only 18 degrees of freedom in setting those parameters. Nevertheless, even in this simple case, the sheer number of parameters would complicate an exhaustive analysis.

¹⁵The symmetry of these classification probabilities (e.g., $C(mw, w) = C(wm, w)$) arises because Loveman and Muniz (2007) do not distinguish between “mirror-image” couples (e.g., mw and wm). Note that this mirror-image symmetry is not imposed by our model, which allows classification matrices to be given arbitrarily. We have slightly adjusted several entries (so that each row is precisely a probability vector) in order to facilitate our numerical analysis in Section 6. Beyond these (raw) classification probabilities, Loveman and Muniz (2007, Table 4, p 933) also report predicted classification probabilities based on a multinomial logit model (which allows them to control for couple’s attributes), obtaining similar results.

Table 1: Classification Matrices for Puerto Rico, 1910 and 1920

| | <i>w</i> | <i>m</i> | <i>b</i> | | <i>w</i> | <i>m</i> | <i>b</i> |
|-----------|--------------------|-------------|-------------|-----------|--------------------|-------------|-------------|
| <i>ww</i> | <i>.998</i> | <i>.002</i> | <i>.000</i> | <i>ww</i> | <i>.997</i> | <i>.003</i> | <i>.000</i> |
| <i>wm</i> | <i>.168</i> | <i>.830</i> | <i>.002</i> | <i>wm</i> | <i>.259</i> | <i>.741</i> | <i>.000</i> |
| <i>wb</i> | <i>.057</i> | <i>.832</i> | <i>.111</i> | <i>wb</i> | <i>.092</i> | <i>.827</i> | <i>.081</i> |
| <i>mw</i> | <i>.168</i> | <i>.830</i> | <i>.002</i> | <i>mw</i> | <i>.259</i> | <i>.741</i> | <i>.000</i> |
| <i>mm</i> | <i>.007</i> | <i>.992</i> | <i>.001</i> | <i>mm</i> | <i>.013</i> | <i>.985</i> | <i>.002</i> |
| <i>mb</i> | <i>.001</i> | <i>.727</i> | <i>.272</i> | <i>mb</i> | <i>.010</i> | <i>.758</i> | <i>.232</i> |
| <i>bw</i> | <i>.057</i> | <i>.832</i> | <i>.111</i> | <i>bw</i> | <i>.092</i> | <i>.827</i> | <i>.081</i> |
| <i>bm</i> | <i>.001</i> | <i>.727</i> | <i>.272</i> | <i>bm</i> | <i>.010</i> | <i>.758</i> | <i>.232</i> |
| <i>bb</i> | <i>.004</i> | <i>.023</i> | <i>.973</i> | <i>bb</i> | <i>.000</i> | <i>.013</i> | <i>.987</i> |
| | <i>1910 census</i> | | | | <i>1920 census</i> | | |

Source: Loveman and Muniz (2007, Table 3, p 931). Racial classes are white (*w*), mulatto (*m*), and black (*b*). Element $C(ij, k)$ in each matrix indicates the probability that a child whose mother was in racial class i and whose father was in racial class j was classified as race k . If these matrices were perfectly bounded (satisfying equation 7), the italicized elements would be equal to 0.

emphasize the differences between these matrices (and we address this issue below), we wish here to note two regularities that hold for either matrix.

First, the child’s racial class is almost always “bounded” between the racial classes of the mother and father. More precisely, given our numerical indexing of classes ($w = 1, m = 2, b = 3$), this condition can be written as

$$C(ij, k) > 0 \quad \text{only if} \quad \min(i, j) \leq k \leq \max(i, j) \quad (7)$$

where i is the mother’s race, j is the father’s race, and k is the child’s race. Equivalently, this condition can be rewritten as

$$C(ij, k) = 0 \quad \text{if} \quad k < \min(i, j) \quad \text{or} \quad k > \max(i, j) \quad (7')$$

Each of the probabilities indicated by condition (7’) have been italicized in Table 1. Of course, in any empirical data, we might expect to find a few violations of any theoretical assumption (certainly due to measurement error and perhaps also for substantive reasons). Nevertheless, inspection of both matrices reveals that all of the italicized probabilities are quite small, with most negligibly larger than 0 while the largest ($C(bb, m)$ for 1910) is just over 2%.

A second condition is more subtle, but holds perfectly in the empirical data. Holding fixed the racial class of one parent, we assume that an increase (“darkening”) in the other parent’s class cannot increase the probability that their child is classified

as white. Similarly, holding fixed the racial class of one parent, we assume that a decrease (“lightening”) in the other parent’s class cannot increase the probability that their child is classified as black. Formally, this requires

$$\begin{aligned}
C(wm, w) &\geq C(wb, w) \\
C(mw, w) &\geq C(bw, w) \\
C(mb, b) &\geq C(wb, b) \\
C(bm, b) &\geq C(bw, b)
\end{aligned}
\tag{8}$$

Returning to the data, inspection of Table 1 reveals

$$\begin{aligned}
C(wm, w) = C(mw, w) = .168 &\geq .057 = C(wb, w) = C(bw, w) \\
C(mb, b) = C(bm, b) = .272 &\geq .111 = C(wb, b) = C(bw, b)
\end{aligned}$$

for 1910, and

$$\begin{aligned}
C(wm, w) = C(mw, w) = .259 &\geq .092 = C(wb, w) = C(bw, w) \\
C(mb, b) = C(bm, b) = .232 &\geq .081 = C(wb, b) = C(bw, b)
\end{aligned}$$

for 1920.¹⁶ Thus, as already indicated, the second condition holds precisely for both matrices.

As a matter of terminology, we will say that the social norms governing racial classification are *bounded* when the racial classification matrix satisfies both conditions (7) and (8). Although condition (7) holds only imperfectly for the matrices in Table 1, it is apparent that classification norms were approximately bounded in Puerto Rico in the early 20th century. Having motivated the boundedness condition with the observed classification matrices, it is worth noting that the “official” rules of racial classification were also bounded. While census takers seem in practice to have followed local norms, they had been instructed (by their superiors in the mainland US) to classify children of white-white couples as white, the children of black-black couples as black, and the children of all other (7 types of) couples as mulatto (see Loveman 2007; Loveman and Muniz 2007). It is straightforward to show that these deterministic rules are bounded, satisfying both equations (7) and (8).

For black-white-mulatto systems in other times and places, the boundedness condition may hold less well. For instance, in contemporary Brazil, where census racial classes are white (*branco*), brown (*pardo*), and black (*preto*), Schwartzman (2007) reports that over 11% of the children of black-black couples are classified as brown, and that over 9% of the children of brown-brown couples are classified as white.¹⁷

¹⁶Of course, the symmetry in the data (e.g., $C(wm, w) = C(mw, w)$) stems from the manner in which the data is reported by Loveman and Muniz (2007); see footnote 15. On the other hand, our key theoretical result below does not actually require condition (8), but merely the weaker pair of conditions (20) which essentially average together the “mirror image” probabilities.

¹⁷Schwartzman (2007, Table 4, p 952) reports data from the 2005 PNAD household survey. Telles (2004, Figure 4.4, p 94) reports similar results from the 1991 Brazilian census.

In Brazil, racial classification depends primarily on the child’s physical appearance (“color”) rather than descent (i.e., the parent’s racial classes *per se*). Consequently, fixing the parents’ type, variation in the child’s race may occur between and even within families (Harris 1963; Telles 2004). In itself, racial classification based on physical appearance would not vitiate the boundedness condition, as violations would arise only to the extent that biology permits the child’s color to lie outside the bounds of the parents’ colors. But the magnitude and asymmetry of observed violations suggest that the boundedness condition is less appropriate for contemporary Brazil.¹⁸

Controlling for parents’ social class, Schwartzman (2007) finds further that parents from higher social classes are more likely to assign “lighter” racial classes to their children. Colloquially, “money whitens” while “poverty darkens.” Essentially, on Schwartzman’s (2007) interpretation of the evidence, racial switching does not occur in adulthood (after matching, as assumed in our present model) but instead occurs entirely through the racial classification of children, which depends upon the social as well as racial classes of the parents. A rigorous treatment of the Brazilian case would thus seem to require a model in which individuals are characterized by both social and racial classes (allowing the model to address the interplay between social and racial intergenerational mobility). Consequently, we must bracket this brief discussion of Brazil as a topic for future work. Nevertheless, it helps indicate a potential scope condition on our present theoretical analysis.¹⁹

3.2 Population dynamics given bounded classification

Assuming that the classification matrix satisfies condition (7), the number of non-zero parameters in the classification matrix is reduced to 17 (with only 8 degrees of freedom). This allows us to rewrite equation (6) as

$$\begin{aligned} x_{t+1}(w) &= x_t(w) [1 + \phi_1 \mu x_t(m) + \phi_2 \beta x_t(b)] \\ x_{t+1}(b) &= x_t(b) [1 + \phi_3 \beta x_t(w) + \phi_4 \gamma x_t(m)] \end{aligned} \tag{9}$$

where

$$x_t(m) = 1 - x_t(w) - x_t(b) \tag{10}$$

¹⁸Consider brown-brown (type *mm*) couples. If departures from boundedness were induced purely by biology, one would expect the probability of a white child ($C(mm, w)$) and the probability of a black child ($C(mm, b)$) to be small and roughly equal. In fact, Schwartzman (2007, Table 4, p 952) reports $C(mm, w) = .096$ while $C(mm, b) = .005$.

¹⁹To address scope conditions on our theoretical analysis in a different fashion, we might also recall Hoetink’s (1967) distinction between two types of black-white-mulatto racial systems in the Caribbean: “North-West European” systems (with clear descent rules) and “Iberian” systems (where racial mobility may be related to economic mobility). While our present theoretical analysis may be more immediately applicable to former, we hope that it also provides a foundation for future work on the latter. Of course, for empirical purposes, our general model (equation 1) can be used to make population projections given arbitrary classification matrices (bounded or otherwise), as we illustrate in section 6 below.

and

$$\begin{aligned}
\phi_1 &= C(wm, w) + C(mw, w) - 1 \\
\phi_2 &= C(wb, w) + C(bw, w) - 1 \\
\phi_3 &= C(wb, b) + C(bw, b) - 1 \\
\phi_4 &= C(mb, b) + C(bm, b) - 1
\end{aligned}
\tag{11}$$

Having specified the racial distribution as a probability distribution, the model thus becomes a 2-equation system. While equation (9) might also include a third equation reflecting the mulatto share in period $t + 1$, that equation is made redundant by equation (10). Note that the ϕ parameters defined in equation (11) involve the remaining 8 classification probabilities.

For convenience, we can restate these dynamics to reflect changes (rather than levels) in the proportions of the groups. Subtracting $x_t(w)$ from both sides of the first line of equation (9), and subtracting $x_t(b)$ from both sides of the second line, we obtain

$$\begin{aligned}
x_{t+1}(w) - x_t(w) &= x_t(w) [\phi_1 \mu x_t(m) + \phi_2 \beta x_t(b)] \\
x_{t+1}(b) - x_t(b) &= x_t(b) [\phi_3 \beta x_t(w) + \phi_4 \gamma x_t(m)]
\end{aligned}
\tag{12}$$

Further suppressing time subscripts, these equations become

$$\begin{aligned}
\Delta x(w) &= x(w) [\phi_1 \mu x(m) + \phi_2 \beta x(b)] \\
\Delta x(b) &= x(b) [\phi_3 \beta x(w) + \phi_4 \gamma x(m)]
\end{aligned}
\tag{13}$$

where $\Delta x(i)$ denotes the change in the proportion of race i between periods.

We are interested in population dynamics over both the short and long run. Both issues can be addressed graphically using a phase diagram.²⁰ We orient this diagram so that the proportion of whites $x(w)$ is placed on the horizontal axis, while the proportion of blacks $x(b)$ is placed on the vertical axis. Any racial distribution corresponds to a point within the triangle (“simplex”) determined by the conditions $x(w) \geq 0$, $x(b) \geq 0$, and $x(w) + x(b) \leq 1$.²¹ On this diagram, we first plot the curves determined by the equations

$$\Delta x(w) = 0 \quad \text{and} \quad \Delta x(b) = 0
\tag{14}$$

Mathematicians call these curves “nullclines.” The first nullcline indicates the locus of points (racial distributions) for which there is no change in the proportion of whites; the second indicates the locus of points for which there is no change in the proportion of blacks. An intersection of these nullclines thus indicates a steady

²⁰See Allman and Rhodes (2004) for an accessible (undergraduate-level) introduction to the mathematical (dynamical systems) methods employed here.

²¹Note that the proportion of mulattos $x(m)$ is determined residually by $1 - x(w) - x(b)$. Thus, $x(m) = 1$ at the origin of the simplex (where $x(w) + x(b) = 0$), while $x(m) = 0$ everywhere along the hypotenuse of the simplex (where $x(w) + x(b) = 1$).

state (where $\Delta x(w) = \Delta x(b) = \Delta x(m) = 0$). Beyond identifying steady states, the nullclines also partition the phase diagram into regions. For each region, we can use equation (13) to determine whether the white and black population shares are rising or falling. This not only reveals short-run dynamics within each region but also allows us to assess the stability of steady states.²²

Using equation (13) to derive the nullclines, we find that the proportion of whites is unchanging if $x(w) = 0$ or

$$\phi_1 \mu x(m) + \phi_2 \beta x(b) = 0 \quad (15)$$

which (using equation 10) can be rewritten as

$$x(w) = 1 - x(b) \left(1 - \frac{\phi_2 \beta}{\phi_1 \mu}\right) \quad (16)$$

Similarly, the proportion of blacks is unchanging if $x(b) = 0$ or

$$\phi_3 \beta x(w) + \phi_4 \gamma x(m) = 0 \quad (17)$$

which can be rewritten as

$$x(b) = 1 - x(w) \left(1 - \frac{\phi_3 \beta}{\phi_4 \gamma}\right) \quad (18)$$

Graphically, both nullclines are linear, with the white nullcline (16) passing through the point $(x(w) = 1, x(b) = 0)$ and the black nullcline (18) passing through $(x(w) = 0, x(b) = 1)$. To assess population dynamics at points not on these nullclines, note further that the proportion of whites is rising (falling) if $x(w) > 0$ and the left-hand side of (15) is positive (negative), while the proportion of blacks is rising (falling) if $x(b) > 0$ and the left-hand side of (17) is positive (negative).

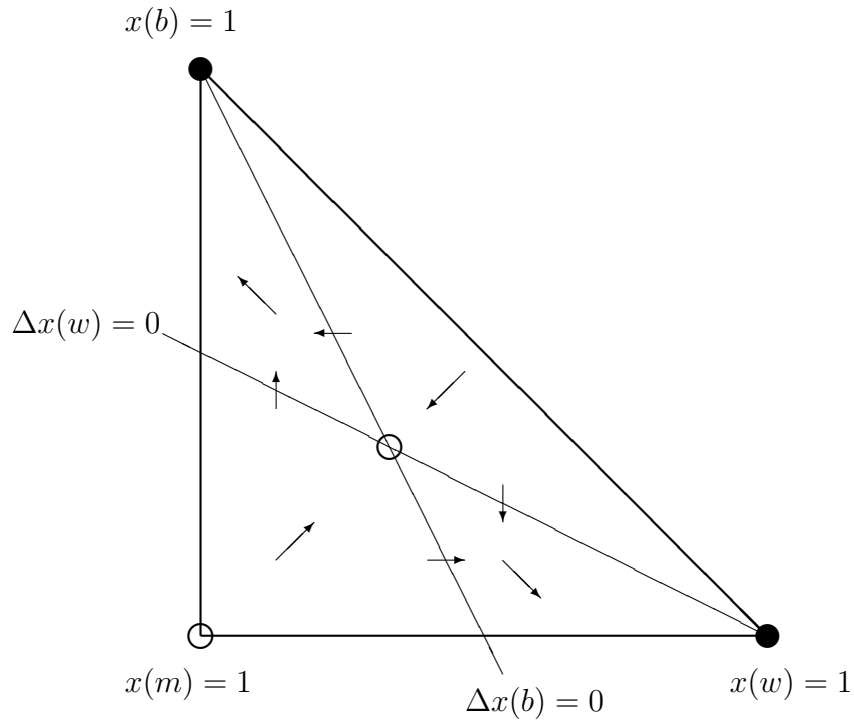
To illustrate the use of nullclines, consider the case where the classification probabilities satisfy the conditions

$$\phi_1 > 0 > \phi_2 \quad \text{and} \quad \phi_4 > 0 > \phi_3 \quad (19)$$

which arise when most children of wm and mw couples are classified as white, most children of mb and bm couples are classified as black, and when most children of wb and bw couples are classified as mulatto. Further suppose that all of the matching parameters are positive. Generically (without making more precise numerical assumptions on the classification or matching parameters), this case generates a phase diagram qualitatively similar to the one shown in Figure 2. On this diagram, the nullclines are denoted by thin lines, the qualitative dynamics for each region (and

²²Loosely, a steady state is stable when, following a small shock, the state (i.e., the current racial distribution) returns to the steady state (i.e., the long-run racial distribution).

Figure 2: Phase diagram given $\phi_1 > 0 > \phi_2$ and $\phi_4 > 0 > \phi_3$



Explanation: On this diagram, the white population share ($x(w)$) is placed on the horizontal axis, the black population share ($x(b)$) is placed on the vertical axis, and the mulatto population share ($x(m) = 1 - x(w) - x(b)$) can be determined residually. Any racial distribution thus corresponds to a point in the triangle. The white nullcline ($\Delta x(w) = 0$) indicates points (racial distributions) for which the white share is stable; the black nullcline ($\Delta x(b) = 0$) indicates points for which the black share is stable. The arrows indicate qualitative dynamics in each region of the diagram. The circles indicate steady states (points at which all racial shares are stable), which may be stable (solid circles) or unstable (open circles).

along each nullcline) are denoted by arrows, and long-run equilibria are denoted by circles (with solid circles indicating stability and open circles indicating instability).

To understand the placement of the arrows in Figure 2, recall that the white population share is rising (falling) when the left-hand side of equation (15) is positive (negative). Graphically, this implies that the white share is rising at points below the white nullcline, and falling at points above the white nullcline. Similarly, the black population share is rising (falling) when the left-hand side of equation (17) is positive (negative). Graphically, this implies that the black share is rising at points to the left of the black nullcline, and falling at points to the right of the black nullcline. Together, these results allow us to determine (qualitatively) the direction of change for every region of the phase diagram and along the nullclines.²³

As indicated by the arrows, if the proportions of both whites and blacks are low (near the origin of the simplex, where $x(m) = 1$), then both of these groups are growing. Conversely, when the sum of these proportions is relatively high (near the hypotenuse, where $x(m) = 0$), then both groups are shrinking. Intuition might thus suggest a stable long-run outcome in which all 3 races have positive shares. However, upon closer inspection of the phase diagram, it becomes apparent that the only interior equilibrium is unstable. For instance, suppose we begin with a population composed mostly of mulattos with a few whites and almost no blacks (so that the initial racial distribution corresponds to a point slightly above the horizontal axis, but left of the black nullcline). The population will follow a trajectory along which both white and black shares initially rise.²⁴ However, the black share will eventually cease to grow (as the trajectory crosses the black nullcline) and then begin to fall, while the white share continues to grow until the entire population is white. Alternatively, starting from a point closer to the vertical axis, the population might follow a trajectory that initially appears to head toward the unstable interior equilibrium but eventually “veers off” toward the all-black equilibrium. In the long run, the population will be either all white (so that $x(w) = 1$) or else all black (so that $x(b) = 1$) depending on the initial racial distribution.

Having examined one generic case (corresponding to condition 19), would other assumptions on the classification matrix would generate qualitatively different results, or is the population always racially homogeneous in the long run? To under-

²³Note that the arrows in Figure 2 provide merely qualitative indications of the direction of change. Making stronger numerical assumptions on the classification and matching parameters, we could compute the precise magnitude and direction of change at every point on the phase diagram. Indeed, we will adopt this procedure in our empirical application below (see Figures 6 and 7). Here, we are heading toward a theoretical result (Proposition 1 below) which relies merely on the qualitative nature of the dynamics (and hence holds generically for any numerical specification of the parameters).

²⁴In the present context, a *trajectory* is the time sequence of racial distributions $\{(x_0(w), x_0(b)), (x_1(w), x_1(b)), (x_2(w), x_2(b)), \dots\}$ generated by equation (9). Making precise numerical assumptions on the parameters of the model, these trajectories can be plotted on the phase diagram (see Figures 4 and 5 below).

take an exhaustive analysis, we need first to identify all of the generic cases. The boundedness condition (8) requires

$$\phi_1 \geq \phi_2 \quad \text{and} \quad \phi_4 \geq \phi_3 \quad (20)$$

From the first of these inequalities, we obtain the 3 possibilities

$$\begin{aligned} (i) \quad & \phi_1 > \phi_2 > 0 \\ (ii) \quad & \phi_1 > 0 > \phi_2 \\ (iii) \quad & 0 > \phi_1 > \phi_2 \end{aligned} \quad (21)$$

Similarly, the second of the inequalities in (20) yields the 3 possibilities

$$\begin{aligned} (i) \quad & \phi_4 > \phi_3 > 0 \\ (ii) \quad & \phi_4 > 0 > \phi_3 \\ (iii) \quad & 0 > \phi_4 > \phi_3 \end{aligned} \quad (22)$$

Taking the cross-product of these possibilities, we obtain 9 different cases.²⁵ However, because conditions (21*i*) and (22*i*) are mutually inconsistent, only 8 cases need actually be considered.²⁶ Note that the case we have already considered (condition 19) corresponds to the combination of (21*ii*) and (22*ii*).

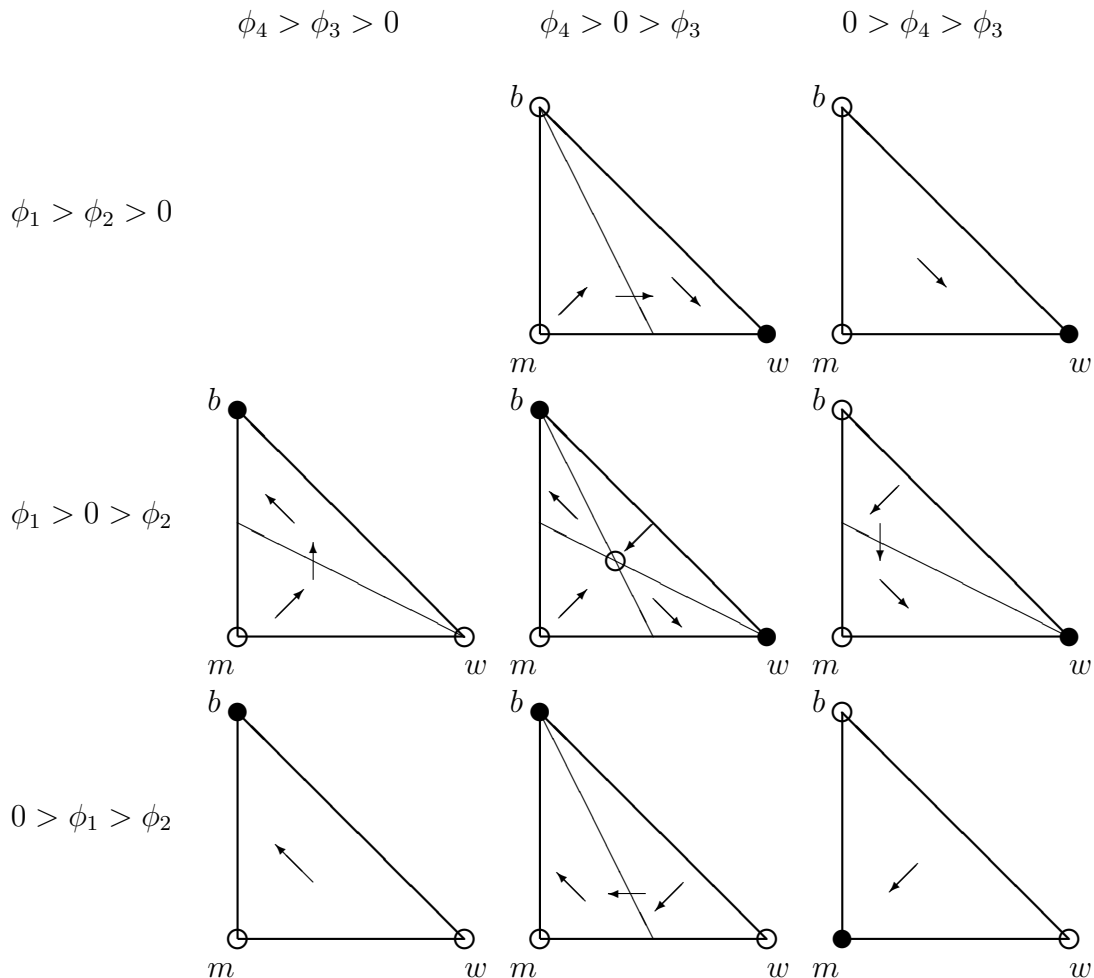
For each of these 8 generic cases, both the short-run dynamics and long-run outcomes can be gleaned from inspection of the phase diagrams in Figure 3. For instance, consider the lower-right case (where $0 > \phi_1 > \phi_2$ and $0 > \phi_4 > \phi_3$). Because both the white and black shares fall at every point in the diagram, everyone in the population will be mulatto in the long run. Next, consider the lower-middle case (where $0 > \phi_1 > \phi_2$ and $\phi_4 > 0 > \phi_3$). If the initial racial distribution corresponds to a point to the right of the black nullcline, then both the black and white shares would initially fall. But eventually, the black share would begin to rise, and continue rising until the entire population is black. Alternatively, if the initial distribution corresponds to a point to the left of the black nullcline, then the black share would immediately rise until everyone becomes black. Proceeding in this way for every case shown in Figure 3, we obtain the answer to the homogeneity question posed above. Our key finding may be summarized informally by

Proposition 1. *Given bounded racial classification (with no differential reproduction or racial switching), the population will be racially homogeneous in the long run.*

²⁵We are ignoring non-generic (knife-edge) cases where any of the strict inequalities in (21) or (22) is replaced with an equality. Outcomes in these non-generic cases are implicit in the phase diagrams in Figure 3; one or both nullclines coincide with an edge of the triangular simplex. The non-generic cases may be of mathematical interest because they may generate a continuum of steady states. However, the empirical relevance of these cases is limited because arbitrarily small changes in the parameter values lead to a generic case.

²⁶Together, conditions (21*i*) and (22*i*) imply $\phi_2 + \phi_3 > 0$ and thus $C(wb, m) + C(bw, m) < 0$, which is impossible because these classification probabilities are non-negative.

Figure 3: Phase diagrams for all 8 generic cases



Explanation: Assuming that racial classification is bounded, the dynamics of the racial distribution are illustrated by one of these 8 diagrams (depending on the precise relationship among the classification parameters). Each diagram can be interpreted in the same manner as Figure 2. To make the figure less cluttered, nullclines are not labeled, and the points at which the population is racially homogeneous are denoted simply by w , m , and b . Note that, for some of the cases, one or both of the nullclines lie outside the triangular simplex. Inspection of these diagrams reveals that, for all 8 cases, the population will be racially homogeneous in the long run.

As we have already seen, multiple equilibria (both homogeneous) are possible for the case corresponding to condition (19). The precise long-run outcome (all-black or all-white) will depend on the initial racial distribution. In contrast, for every other case, there is a unique, homogeneous equilibrium that will be obtained in the long run regardless of the initial racial distribution.

Proposition 1 implicitly assumes that some racial mixing occurs (i.e., the 3 matching parameters are positive). Intuitively, in the absence of racial mixing, the racial distribution would never change; a population that is initially racially heterogeneous would remain so forever. However, it is interesting to note that Proposition 1 requires no further assumption on the matching parameters. While the matching parameters are important for determining the *rate* at which the population moves toward racial homogeneity, the qualitative nature of this long-run outcome depends entirely on the relationships between the classification parameters (equations 21 and 22).

Our proposition may recall Hoetink's (1967) claim that "biological homogenization" is inevitable in racial systems. However, recognizing that racial classification is a social (not biological) process, the racial distribution should not be conflated with the distribution of the population by physical appearance. Indeed, the long-run racial distribution may be independent of the long-run distribution of physical appearance. To illustrate this point, suppose that physical appearance is simply a matter of "color," and that each individual's color can be summarized by a number between 0 and 1. Further assume (ignoring complications from genetics) that each child's color is given deterministically by the average of his or her parents' colors. In the absence of differential reproduction, this implies that the average color of the population does not change across generations. Given Proposition 1, all members of the population will eventually belong to the same racial class. Because this implies random matching of couples within each generation, it is obvious that any variation in color will gradually diminish over time. In the long run, all members of the population will share the same color (which, given "conservation of pigment" across generations, is determined by the average color of individuals in the initial generation). But importantly, this one-color population could be either all-black or all-white or all-mulatto depending on the (social) rules of racial classification.²⁷

4 The effects of racial switching

To extend our theoretical analysis, we now permit racial switching while continuing to ignore the possibility of differential reproduction. As already indicated above,

²⁷In the context of Puerto Rico, the increase in white population in the early 1900s was apparently accompanied by compositional change of physical attributes – "darkening" – within this racial category. Given the divergence between the physical traits of individuals classified as white in Puerto Rico and the mainland US, census officials regarded Puerto Rican racial data as "misreported" and stopped collecting it altogether after 1950 until 2000. See Duany (2002, esp p 252).

population dynamics for this case are given by

$$x_{t+1} = x_t M_t S C \tag{23}$$

The relative simplicity of this matrix equation once again belies the wide range of outcomes that might be generated by this non-linear system. While we continue to assume that the classification matrix is bounded, the switching matrix alone has 81 parameters (with 72 degrees of freedom). However, inspection of this equation also immediately suggests one direction for theoretical investigation. Multiplication of the switching and classification matrices yields the (9×3) matrix SC where element $(SC)(ij, k)$ indicates the probability that a child of a couple with pre-switching type ij is assigned to racial class k . If this compound matrix satisfies the same boundedness conditions as the classification matrix – if conditions (7) and (8) continue to hold when we substitute SC for C – then long-run racial homogeneity remains a necessary implication of the model. But what conditions must be placed on the switching matrix in order for the SC matrix to be bounded? This question prompts our discussion of bounded switching matrices in the next subsection.

4.1 Bounded switching matrices

In our discussion of Figure 1, we mentioned two types of racial switching. The first type occurs when, upon matching, one individual adopts the partner’s racial class. The second type occurs when an individual or couple switch racial classes even when both partners shared the same childhood racial class. For the first type of switching, the individual’s post-switching racial class is bounded by the pre-switching racial classes of the couple. More formally, recalling our numerical indexing of racial classes ($w = 1, m = 2, b = 3$), we may state the condition

$$S(ij, i'j') > 0 \quad \text{only if} \quad j \leq j' \leq i' \leq i \quad \text{or} \quad i \leq i' \leq j' \leq j \tag{24}$$

where i and j denote pre-switching racial classes while i' and j' denote post-switching racial classes. While this condition captures the possibility that one partner adopts the other’s childhood class, it also allows “partial” adjustments (e.g., a couple could switch from bw to mw).²⁸ More concretely, recalling that the switching matrix is constructed with pre-switching types on the rows and post-switching types on the columns, condition (24) permits positive switching probabilities only for the

²⁸On the other hand, we might have specified condition (25) even more weakly as

$$S(ij, i'j') > 0 \quad \text{only if} \quad \min(i, j) \leq i' \leq \max(i, j) \quad \text{and} \quad \min(i, j) \leq j' \leq \max(i, j)$$

which more closely resembles condition (7). However, this would permit “flipping” of the couple’s races (e.g., a couple could switch from bw to wb).

The proof is given in the Appendix. Given our analysis of bounded classification in the preceding section, an immediate consequence of this result is

Proposition 2. *Given bounded racial classification and bounded racial switching (with no differential reproduction), the population will be racially homogenous in the long run.*

Thus, restricting attention to the particular form of racial switching highlighted by Loveman and Muniz (2007), we obtain the same long-run homogeneity result as before.²⁹ However, it is important to realize that the 8 cases depicted in Figure 3 are now determined by conditions on the compound matrix SC rather than the classification matrix C alone.³⁰ More substantively, while population projections based on bounded classification rules alone (with no switching) might imply that the population is heading toward one homogeneous equilibrium (say all-mulatto), population projections which also incorporate bounded switching might imply convergence toward a different homogeneous equilibrium (say all-white).

4.2 An example with a black-white racial system

We now construct a simple example to illustrate Proposition 2. To make this example as transparent as possible, we consider a black-white racial system (with no mulatto category) in which racial classification follows the deterministic “one-drop” rule, and there is some black-to-white racial switching (colloquially, some blacks “pass for white”). Roughly, this describes much of the mainland US during the early 20th century (Davis 1991).³¹ Intuitively, if there is any racial mixing (i.e., $\beta > 0$), the one-drop rule creates a bias toward black population growth because all children of wb or bw couples will be classified as black. On the other hand, any black-to-white racial switching would create a flow in the opposite direction. Intuition might thus suggest the existence of a heterogeneous long-run outcome – with positive population shares for both blacks and whites – in which these flows are exactly balanced. However, Proposition 2 indicates that this heterogeneous equilibrium exists only if some couples transition from bb to ww . That is, if racial switching is confined to post-matching transitions from wb or bw to ww , then the population will be racially homogenous

²⁹Formally, Proposition 2 generalizes Proposition 1. Proposition 1 assumed the (bounded) switching matrix $S = I$, while Proposition 2 allows any bounded switching matrix.

³⁰More precisely, the ϕ parameters defined in equation (11) must be revised so that SC is substituted uniformly for C . For instance, ϕ_1 becomes $(SC)(wm, w) + (SC)(mw, w) - 1$.

³¹We ignore the possibility of white-to-black switching for both empirical reasons (such flows were presumably small during this era) and mathematical reasons (to simplify equations 30-33). But note that Proposition 2 guarantees long-run racial homogeneity even if bounded switching occurs in both directions. For empirical estimates of black-to-white switching during this era, see Burma (1946) and Eckard (1947).

in the long run.³²

To demonstrate these claims formally, note that the matching matrix for this example is given by

$$\begin{array}{c} w \\ b \end{array} \left[\begin{array}{cccc} ww & wb & bw & bb \\ x_t(w) + (1 - \beta)x_t(b) & \beta x_t(b) & 0 & 0 \\ 0 & 0 & \beta x_t(w) & (1 - \beta)x_t(w) + x_t(b) \end{array} \right] \quad (27)$$

while the switching matrix is given by

$$\begin{array}{c} ww \\ wb \\ bw \\ bb \end{array} \left[\begin{array}{cccc} ww & wb & bw & bb \\ 1 & 0 & 0 & 0 \\ S(wb, ww) & S(wb, wb) & 0 & 0 \\ S(bw, ww) & 0 & S(bw, bw) & 0 \\ S(bb, ww) & S(bb, wb) & S(bb, bw) & S(bb, bb) \end{array} \right] \quad (28)$$

and the classification matrix is given by

$$\begin{array}{c} w \\ ww \\ wb \\ bw \\ bb \end{array} \left[\begin{array}{cc} b \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{array} \right] \quad (29)$$

Population dynamics are thus determined by the one-equation system

$$\begin{aligned} x_{t+1}(w) &= [x_t(w)]^2 + (1 - \beta) x_t(w) x_t(b) \\ &\quad + \beta x_t(w) x_t(b) [S(wb, ww) + S(bw, ww)] \\ &\quad + \{[x_t(b)]^2 + (1 - \beta) x_t(w) x_t(b)\} S(bb, ww) \end{aligned} \quad (30)$$

which can be rewritten as

$$\Delta x(w) = -x(b)\{\beta x(w)[1 - S(wb, ww) - S(bw, ww) + S(bb, ww)] - S(bb, ww)\} \quad (31)$$

Assuming that no couples switch from bb to ww , this simplifies further to

$$\Delta x(w) = -x(w) x(b) \beta [1 - S(wb, ww) - S(bw, ww)] \quad (32)$$

Note that the matching parameter β affects the rate (but not direction) of change in the white population share. If the final bracketed term is positive, the white share falls until the entire population is black. Conversely, if the final bracketed

³²Of course, in the empirical context of the US in the early 20th century, even if classification and switching were bounded, implying long-run homogeneity, changes in the racial distribution would have been very slow due to the low level of racial mixing (see, e.g., Davis 1991, p 66).

term is negative, the white share rises until the entire population is white. Thus, if couples never switch from bb to ww , the long-run outcome is racially homogeneous. In contrast, if some couples do switch from bb to ww , then there is a heterogeneous equilibrium with the white share given by

$$x^*(w) = \frac{S(bb, ww)}{\beta[1 - S(wb, ww) - S(bw, ww) + S(bb, ww)]} \quad (33)$$

whenever parameter values are such that $x^*(w)$ is between 0 and 1. In that case, because the white share is rising when $x(w)$ is small ($0 < x(w) < x^*(w)$) and falling when $x(w)$ is large ($x^*(w) < x(w) < 1$), equation (33) describes the unique stable equilibrium.

5 The effects of differential reproduction

We now permit differential reproduction, so that population dynamics are given by

$$x_{t+1} = x_t M_t S R C / \lambda_t \quad (34)$$

where R is the diagonal reproduction matrix and λ_t is the population growth factor in period t . Proposition 2 has already indicated that, even if racial classification and switching are bounded, long-run racial homogeneity is not guaranteed if average fertility rates vary across types of couples. Beyond long-run racial heterogeneity, differential reproduction can also induce more complicated dynamics, including limit cycles in which the racial distribution never converges to a fixed point. This brief section offers a numerical example to illustrate the possibility of these types of long-run outcomes.

For this example, we assume that the average number of girls is 1 for white women, 1.5 for mulatto women, and 2 for black women. We further assume that children of wm and mw couples are classified as white, while children of all other mixed couples are classified as mulatto.³³ The reproduction and classification matrices are thus

³³Of course, our model permits reproduction rates to differ across all 9 types of couples (not merely by mother's race), and also permits probabilistic (as opposed to deterministic) classification rules. For this hypothetical example, we keep the parameter specification simple to make it easier for readers to develop intuition for the dynamics to be illustrated in Figures 4 and 5. Future research might attempt to provide a full characterization of the conditions (sets of parameter values) under which different types of dynamics (e.g., limit cycles) arise.

given by

$$\begin{array}{l}
 \begin{array}{c} ww \\ wm \\ wb \\ mw \\ mm \\ mb \\ bw \\ bm \\ bb \end{array} \begin{bmatrix} ww & wm & wb & mw & mm & mb & bw & bm & bb \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix} \\
 \begin{array}{c} ww \\ wm \\ wb \\ mw \\ mm \\ mb \\ bw \\ bm \\ bb \end{array} \begin{bmatrix} w & m & b \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
 \end{array} \tag{35}$$

For simplicity, we further assume no racial switching (so that $S = I$). Thus, for this example, both classification and switching are bounded but there is differential reproduction. Further setting the matching parameters so that

$$\beta = 1/3, \mu = 1/2, \text{ and } \gamma = 1/2, \tag{36}$$

one can show that the racial distribution eventually converges to the equilibrium

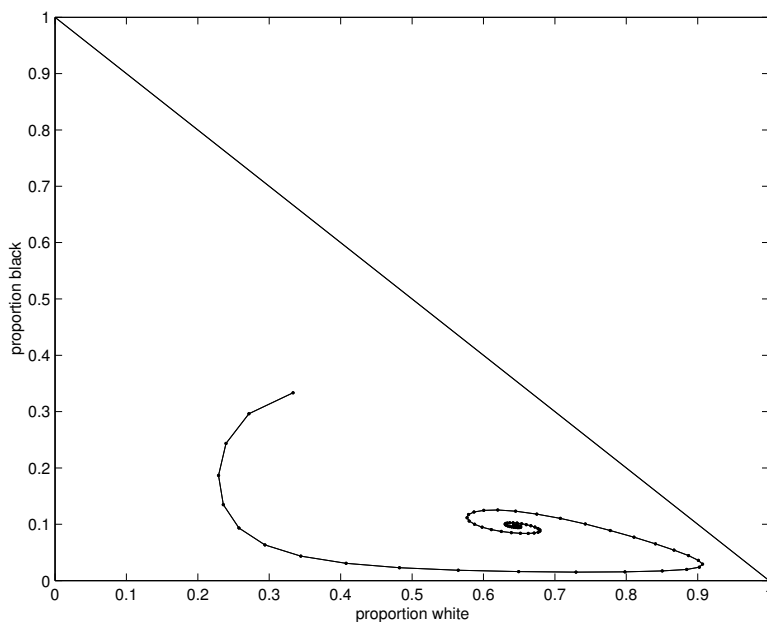
$$x^*(w) = 0.6452, x^*(m) = 0.2581, \text{ and } x^*(b) = 0.0968 \tag{37}$$

for any initial racial distribution.

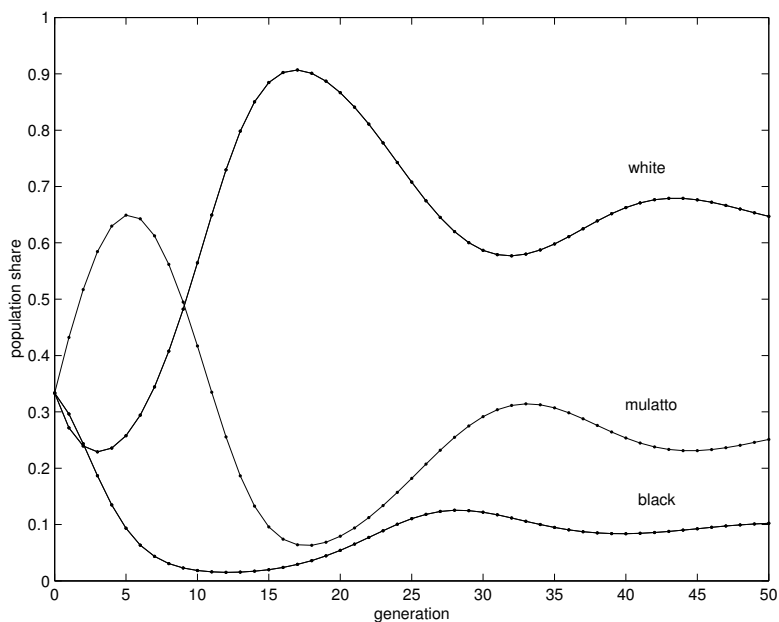
Having illustrated the possibility of a heterogenous equilibrium, we may also consider the short-run dynamics as the population approaches this long-run outcome. Assuming that the population is initially split evenly between the 3 racial groups (so that $x_0(w) = x_0(m) = x_0(b) = 1/3$), the top panel of Figure 4 shows the trajectory (“orbit”) followed over subsequent generations, while the bottom panel shows the time paths of the population shares (for the first 50 generations).³⁴ It is interesting to note that the trajectory does not approach the long-run equilibrium directly, but instead “overshoots” this outcome, forming a counterclockwise spiral. Over the first few generations, the mulatto share rises (at the expense of both blacks and whites). But because the children of wm and mw couples are classified as white, the white share begins to rise (after generation 3) and the mulatto share begins to fall (after generation 5). As the population becomes increasingly white, the higher fertility among blacks and mulattos eventually increases the population shares of blacks (after generation 12) and mulattos (after generation 18). But the growth of these groups is self-limiting: the black and mulatto shares eventually fall (after generations 28 and 33) while the white share eventually rises (after generation 32).

³⁴When the trajectory is plotted on the triangular simplex (upper panel of Figure 4), time is implicit (though readers can see from the “dots” along the trajectory how much the racial distribution changed each generation). To show more explicitly how the racial shares vary over time, we have also plotted time paths (bottom panel of Figure 4). Formally, the time path for racial group i is the sequence $(t, x_t(i))$ for $t \in \{0, 1, 2, \dots\}$. Essentially, the two panels of Figure 4 provide two different ways to depict the same trajectory.

Figure 4: Orbit diagram given $\beta = 1/3$

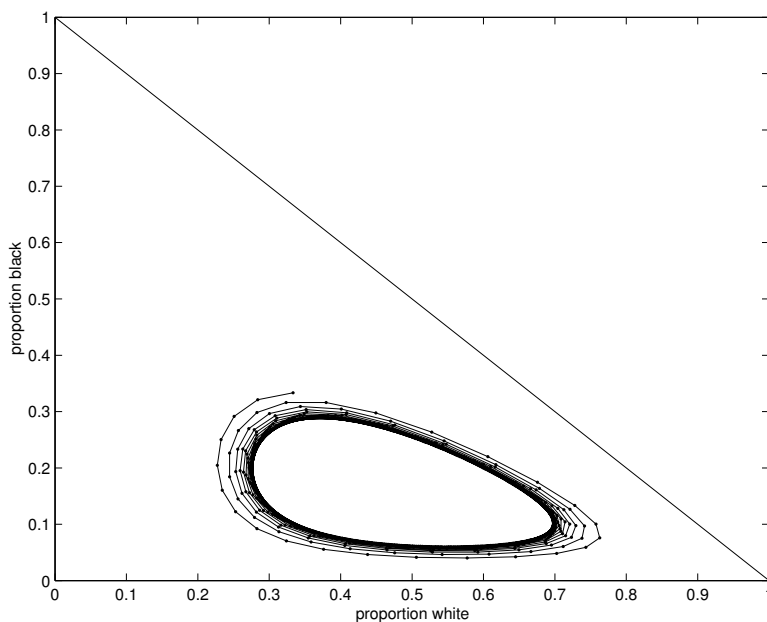


Time paths of population shares

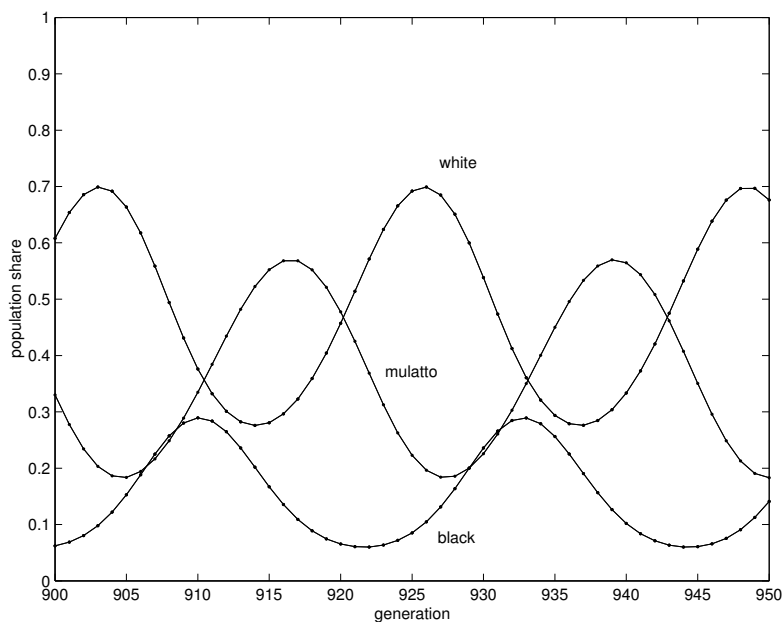


Explanation: The top panel displays the trajectory of the racial distribution – the sequence $(x_t(w), x_t(b))$ for $t \in \{0, 1, 2, \dots\}$. The bottom panel displays the initial time path for each racial group i – the sequence $(t, x_t(i))$ for $t \in \{0, \dots, 50\}$. Both diagrams assume the parameter values given in equations (35) and (36), along with the initial condition $x_0(w) = x_0(m) = x_0(b) = 1/3$.

Figure 5: Orbit diagram given $\beta = 1/6$



Time paths of population shares



Explanation: The top panel displays the trajectory of the racial distribution – the sequence $(x_t(w), x_t(b))$ for $t \in \{0, 1, 2, \dots\}$. The bottom panel displays the time path for each racial group i after the trajectory has converged to the limit cycle – the sequence $(t, x_t(i))$ for $t \in \{900, \dots, 950\}$. Both diagrams assume the parameter values given in equations (35) and (38), along with the initial condition $x_0(w) = x_0(m) = x_0(b) = 1/3$.

Of course, as shown in Figure 4, these oscillations dampen over time, so that the population eventually reaches the steady state given in equation (37). However, given a further decrease in black-white mixing, it is possible that these oscillations never dampen. In particular, given

$$\beta = 1/6, \mu = 1/2, \text{ and } \gamma = 1/2, \quad (38)$$

the long-run outcome is a limit cycle. Assuming once again that the population is initially split evenly across the 3 racial groups, the top panel of Figure 5 shows that the trajectory spirals inward toward this cycle.³⁵ The bottom panel shows the time paths of population shares after many generations (when the trajectory has converged to the limit cycle). For this example, we see that the duration of the cycle is about 25 generations. Over the course of the cycle, the white share varies between 27% and 70%; the mulatto share varies between 18% and 57%; the black share varies between 6% and 29%.³⁶

6 Why didn't Puerto Rico become mulatto?

Having completed our theoretical analysis, we now return to the empirical case of Puerto Rico in the early 20th century. Given the nature of our (discrete-generations) model, we cannot provide fine-grained population projections of the sort made by Loveman and Muniz (2007, p 920 ff and online supplement). In particular, our model assumes that time periods are very long (corresponding to generation length) and that the population has no age structure (one generation is immediately replaced by the next). Nevertheless, even this coarse-grained model can shed some light on the dynamics of the racial distribution in the empirical case at hand.

6.1 Racial classification

We start by addressing the following counterfactual question: In the absence of racial switching and differential reproduction, would the observed classification rules (reported in Table 1) have caused the white population share to rise? As emphasized by Loveman and Muniz (2007), there is a noticeable shift in racial classification norms between 1910 and 1920. For each type of couple, children in the 1920 census are somewhat more likely to be assigned to lower (“lighter”) categories. But is this racial “boundary shifting” a sufficient (or even partial) explanation for the observed growth of the white population share?

³⁵Given an initial condition “inside” of the cycle (e.g., $x_0(w) = x_0(m) = .4$ and $x_0(b) = .2$), the trajectory would have spiraled outward towards the limit cycle.

³⁶Holding fixed all of the other parameters in this example, limit cycles arise when β falls below a threshold which is approximately equal to .2; further decreases in β increase both the length and magnitude of the cycle.

To answer this question, we draw first on our analysis from Section 3. Comparing the classification matrices in Table 1 to conditions (21) and (22), we find that both matrices satisfy conditions (21iii) and (22iii). In particular, comparing the 1910 matrix to equation (11), we find

$$\begin{aligned}
\phi_1 &= .168 + .168 - 1 = -.664 \\
\phi_2 &= .057 + .057 - 1 = -.886 \\
\phi_3 &= .111 + .111 - 1 = -.778 \\
\phi_4 &= .272 + .272 - 1 = -.456
\end{aligned}
\tag{39}$$

which implies

$$0 > \phi_1 > \phi_2 \quad \text{and} \quad 0 > \phi_4 > \phi_3 \tag{40}$$

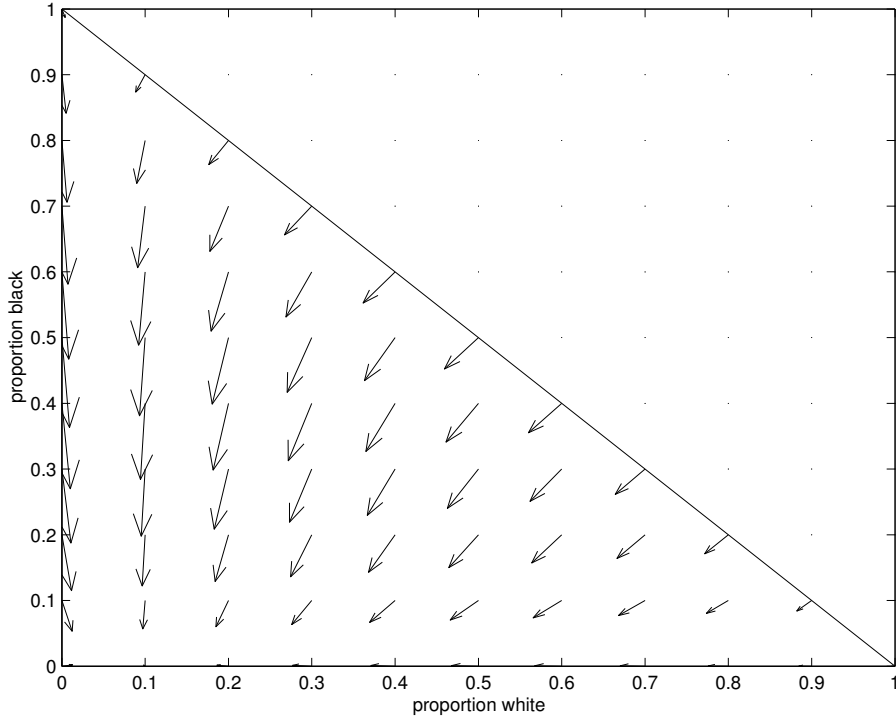
Readers may verify that the 1920 classification matrix also generates the inequalities in equation (40). Thus, for these decades, the population dynamics are those indicated by the phase diagram in the lower-right corner of Figure 3; the model predicts that (in the absence of racial switching or differential reproduction) the white and black population shares should be falling while the mulatto share should be rising. Indeed, as indicated by the phase diagram, these short-run dynamics should hold regardless of the initial racial distribution. Against the title of the Loveman and Muniz (2007) paper, the salient question thus becomes: Why didn't Puerto Rico become mulatto?

The intuition behind this prediction is straightforward. Consider a population composed initially of whites and mulattos. Given no differential reproduction, we may suppose that every couple has 2 children. Without racial mixing, there would be no change over time in the racial distribution; every couple would perfectly “reproduce itself.” Thus, any change in racial shares hinges on the classification of children of mixed (wm or mw) couples. Intuitively, if both $C(wm, w)$ and $C(mw, w)$ exceed $1/2$, then mixed couples produce more white than mulatto children, and the white share rises. Conversely, if both $C(wm, w)$ and $C(mw, w)$ are less than $1/2$, then mixed couples produce more mulatto than white children, and the mulatto share rises. Inspection of Table 1 reveals that these classification probabilities are well below $1/2$ for both 1910 (where $C(wm, w) = C(mw, w) = .168$) and 1920 (where $C(wm, w) = C(mw, w) = .259$).³⁷ Thus, for these years, the racial classification rules promoted the growth of the *mulatto* share. While the direction of the dynamics is determined entirely by the classification parameters, the rate depends on the level of racial mixing (μ). More racial mixing between whites and mulattos would imply more mulatto children, which would imply faster growth of the mulatto share.

Nevertheless, one might conjecture that our theoretical prediction diverges from the empirical findings because the classification matrices in Table 1 are only approxi-

³⁷Note that the condition $\phi_1 < 0$ can be rewritten as $[C(wm, w) + C(mw, w)]/2 < 1/2$. Thus, as indicated in footnote 16, the direction of the dynamics is actually determined by the *average* of the mirror-image probabilities (which is essentially what was reported by Loveman and Muniz 2007, Table 3, p 931, the source for our Table 1).

Figure 6: Phase diagram using 1920 classification matrix



Explanation: Each arrow indicates the direction and magnitude of change in the racial distribution over the course of one generation. These projections (based on equation 1) assume no switching and no differential reproduction (i.e., S and R are set equal to the identity matrix). They further assume the observed 1920 classification matrix (from Table 1) along with the estimated matching parameters $\mu = 0.3$, $\beta = 0.2$, and $\gamma = 1.0$ (see footnote 38).

mately bounded (i.e., condition 7 holds only imperfectly). To investigate, we use the observed 1920 classification matrix to construct the phase diagram shown in Figure 6. Note that this diagram omits nullclines but shows more precisely the vectorfield governing population dynamics.³⁸ In contrast to the lower-right phase diagram in Figure 3, we see that there is now a region of the diagram (near the vertical axis) in which the white population share is (slightly) rising. Similarly, while the relevant region is too small to see on the diagram, the black share is (slightly) rising near the horizontal axis. Consequently, neither the black and white share would ever

³⁸Beyond the classification matrix from Table 1, the projections reported here (and those in the remainder of this section) also require estimates of the matching parameters. From Loveman and Muniz (2007), we obtained the racial distribution (Table 1, p 921) and the distribution of types of couples (Table 2, p 927) for 1920. Substitution of these proportions into the last 3 lines of equation (5) yields the (rough) estimates $\mu = 0.3$, $\beta = 0.2$, and $\gamma = 1.0$. The 1910 distributions yield similar estimates of the matching parameters.

fall completely to zero; further computations reveal a unique long-run equilibrium with 8.6% white, 91% mulatto, and 0.4% black. Intuitively, if the population was initially all mulatto, some children of mm couples would be classified as w or b , causing those groups to grow. Nevertheless, while explaining why the white population would never completely vanish, these results clearly cannot account for its observed growth. Using the racial distribution for either 1910 (65% white, 5% black) or 1920 (74% white, 4% black) as an initial condition, the white population share should have fallen.³⁹

6.2 Differential Reproduction

Even though our model does not permit fine-grained population projections, it thus seems clear that the norms governing racial classification of children did not contribute to the growth of the white population share. Differential reproduction or racial switching (or both) must have been responsible.⁴⁰

Continuing to proceed counterfactually, we now consider the joint effect of racial classification and differential reproduction given no racial switching. Estimates of the reproduction parameters can be drawn from the data reported in Table 2.⁴¹ The gross reproduction rate (GRR) is computed for each racial class simply by summing the mean number of girls over all age groups. Comparing the GRR across racial classes, it might appear at first that differential reproduction cannot help explain the growth of the white population share. Indeed, the GRR does not vary much across classes, and is actually slightly lower for whites (2.42) than mulattas (2.44).

However, in accounting for population growth, the more relevant measure is the net reproduction rate (NRR), which reflects the fact that some women die before reaching the end of their child-bearing years.⁴² While the difference between GRR and NRR is often slight in contemporary data from developed countries, the distinc-

³⁹Using the 1910 classification matrix to recompute the vectorfield in Figure 6, the results change only slightly, with the predicted decline in the white share being somewhat more rapid.

⁴⁰A priori, external migration would also be a likely suspect. However, as an empirical matter, both in-migration and out-migration were negligible during the 1910s (see Loveman and Muniz 2007, online supplement, for estimates). Widespread out-migration to the US mainland did not occur until the 1940s (Senior 1947, Table 3, p 7).

⁴¹Because Loveman and Muniz (2007) report fertility by female’s racial class (rather than couple’s type), we will assume that the male’s racial class does not affect the couple’s fertility. Note that this simplification is imposed by their data, not our model (which permits fertility to vary across all 9 types of couples).

⁴²In demography, the probability of survival from birth to age x is denoted by l_x/l_0 and the mean number of girls by age group $[x, x + 4]$ is denoted by ${}_5F_x^F$. Using the data in Table 2, a good approximation of the net reproduction rate is given by the formula

$$\text{NRR} = \sum_x \frac{1}{2} \left(\frac{l_x}{l_0} + \frac{l_{x+5}}{l_0} \right) {}_5F_x^F$$

where the summation is taken over $x \in \{15, 20, \dots, 45\}$.

Table 2: Female survival and fertility by race, Puerto Rico, 1910

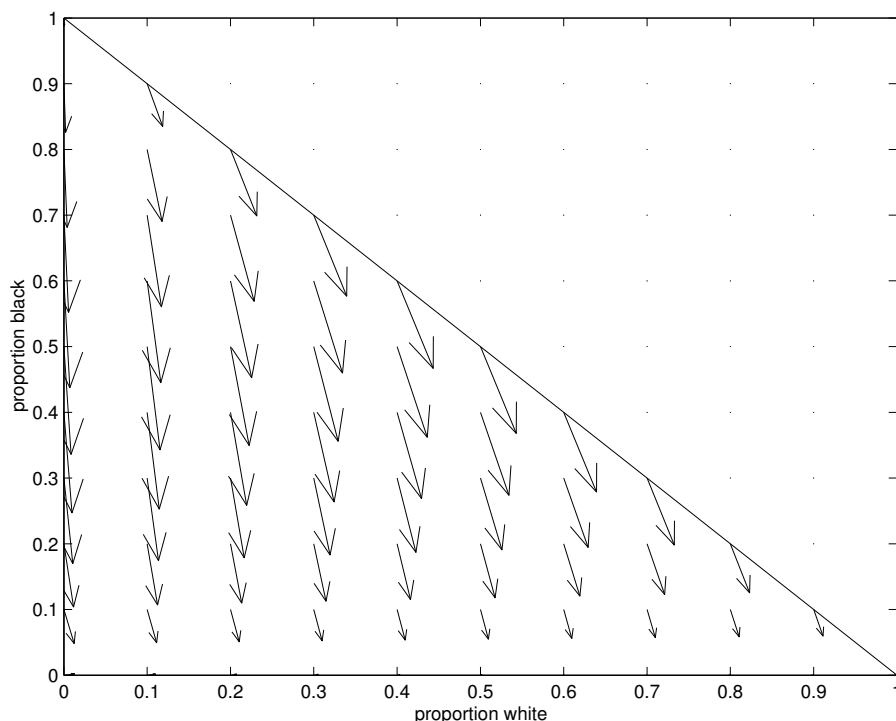
| age group | <i>white women</i> | | <i>mulatta women</i> | | <i>black women</i> | |
|-----------|---|--|---|--|---|--|
| | <i>probability of survival (from birth)</i> | <i>mean number of girls (by age group)</i> | <i>probability of survival (from birth)</i> | <i>mean number of girls (by age group)</i> | <i>probability of survival (from birth)</i> | <i>mean number of girls (by age group)</i> |
| 0-4 | 1.0000 | 0 | 1.0000 | 0 | 1.0000 | 0 |
| 5-9 | 0.7568 | 0 | 0.7024 | 0 | 0.6265 | 0 |
| 10-14 | 0.7225 | 0 | 0.6588 | 0 | 0.5704 | 0 |
| 15-19 | 0.6999 | 0.234 | 0.6307 | 0.237 | 0.5353 | 0.246 |
| 20-24 | 0.6657 | 0.627 | 0.5883 | 0.632 | 0.4832 | 0.659 |
| 25-29 | 0.6140 | 0.629 | 0.5259 | 0.637 | 0.4101 | 0.663 |
| 30-34 | 0.5636 | 0.505 | 0.4681 | 0.510 | 0.3472 | 0.532 |
| 35-39 | 0.5155 | 0.317 | 0.4155 | 0.320 | 0.2937 | 0.334 |
| 40-44 | 0.4686 | 0.093 | 0.3671 | 0.093 | 0.2486 | 0.098 |
| 45-49 | 0.4220 | 0.017 | 0.3223 | 0.017 | 0.2101 | 0.017 |
| 50-54 | 0.3744 | 0 | 0.2789 | 0 | 0.1754 | 0 |
| 55-59 | 0.3259 | 0 | 0.2367 | 0 | 0.1436 | 0 |
| 60-65 | 0.2782 | 0 | 0.1969 | 0 | 0.1153 | 0 |
| 65+ | 0.2299 | 0 | 0.1583 | 0 | 0.0893 | 0 |
| | <i>GRR = 2.42</i> <i>NRR = 1.41</i> | | <i>GRR = 2.44</i> <i>NRR = 1.20</i> | | <i>GRR = 2.55</i> <i>NRR = 0.96</i> | |

Source: Loveman and Muniz (2007, online appendix, Tables S2-S4 and S10-S12). Loveman and Muniz (2007) report the total fertility rate (TFR) rather than the gross reproduction rate (GRR). In moving from TFR to GRR, we have assumed that the sex ratio at birth (SRB) is 1.05, though the precise ratio is irrelevant for assessing differential reproduction. See the text for the computation of GRR and the net reproduction rate (NRR).

tion between these measure is quite important in this data because of the relatively high mortality rates for young women. Indeed, Table 2 reveals that over half of white women died before reaching age 40; over half of mulatta women died before reaching age 30; over half of black women died before reaching age 20. Given this association between mortality and race, we find that NRR varies considerably across races, and is about 17% higher for whites (1.41) than mulattas (1.20). Thus, focusing on NRR rather than GRR, it appears that differential reproduction might well have played a significant role in the growth of the white population share.

To assess numerically the joint effect of racial classification and differential reproduction, we construct a new phase diagram (Figure 7) based on the 1920 classification matrix (from Table 1) and reproduction levels given by the NRRs (from Table 2). Obviously, the most striking feature of this figure is the rapid decrease in the black share (indicated by the long downward-pointing arrows) for most initial conditions.

Figure 7: Phase diagram given differential reproduction



Explanation: Each arrow indicates the direction and magnitude of change in the racial distribution over the course of one generation. These projections (based on equation 1) assume no switching (i.e., S is set equal to the identity matrix). They further assume the observed 1920 classification matrix (from Table 1) and reproduction levels given by the NRRs (from Table 2) along with the estimated matching parameters $\mu = 0.3$, $\beta = 0.2$, and $\gamma = 1.0$ (see footnote 38).

Substantively, if the black share had been large in 1910, then it would have fallen dramatically over the course of the decade. But more crucially for our present purposes, the white share is now (slightly) rising for all initial conditions. Graphically, the arrows are not only pointing (strongly) downward but also (slightly) to the right. However, we can also see that the increase in the white share is negligible when the black share is small (at points near the horizontal axis). Indeed, using either the observed 1910 or 1920 racial distribution as an initial condition, further computations reveal that the white share would have grown by less than one-half of a percentage point over the course of a generation. Thus, starting from the baseline provided by Figure 6, differential reproduction may explain why the white share did not fall. But we have not yet accounted for its dramatic rise.

Table 3: Projected white population share after one generation

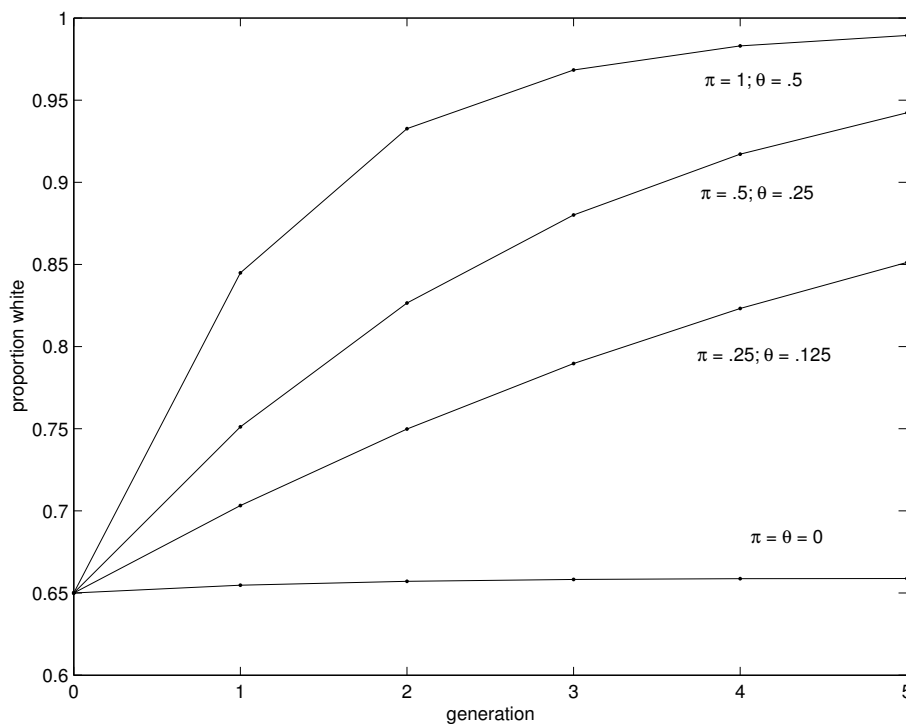
| | | <i>non-bounded switching</i> (θ) | | | | |
|------------------------------------|------|---|--------|--------|--------|--------|
| | | 0.00 | 0.25 | 0.50 | 0.75 | 1.00 |
| <i>bounded switching</i> (π) | 0.00 | 0.6547 | 0.7078 | 0.7600 | 0.8112 | 0.8615 |
| | 0.25 | 0.6767 | 0.7295 | 0.7813 | 0.8323 | 0.8823 |
| | 0.50 | 0.6987 | 0.7511 | 0.8026 | 0.8533 | 0.9030 |
| | 0.75 | 0.7205 | 0.7726 | 0.8238 | 0.8741 | 0.9236 |
| | 1.00 | 0.7422 | 0.7940 | 0.8449 | 0.8949 | 0.9441 |

Explanation: Using the observed 1910 racial distribution as an initial condition (so that $x_0(w) = 0.65$), this table reports the projected white share after one generation ($x_1(w)$) for different combinations of the switching parameters (see equation 41). These projections (based on equation 1) further assume the observed 1920 classification matrix (from Table 1) and reproduction levels given by the NRRs (from Table 2) along with the estimated matching parameters $\mu = 0.3$, $\beta = 0.2$, and $\gamma = 1.0$ (see footnote 38).

ww), we still cannot account for the observed growth of the white population share. The model predicts that the white share would have risen from 65% to 74% over the course of a generation. In fact, an increase of approximately this magnitude actually occurred within a *decade*.

While our coarse-grained model does not permit more a precise quantitative decomposition of the underlying causes, we have thus reached the conclusion that racial switching must have been an important factor in the growth of the white population share. Moreover, at least some of this switching must have been non-bounded, occurring without racial mixing. To expand upon the results from Table 3, we have chosen several different pairs of parameter values, and projected ahead the racial distribution for several more generations. Our results are shown in Figure 8. While we cannot judge the relative importance of the two types of switching (we have set $\pi = 2\theta$ simply for illustration), it appears that both types of switching must have been fairly common to account for the rapid growth of the white share that was actually observed.

Figure 8: Projections of the white population share



Explanation: These projections (based on equation 1) assume the observed 1920 classification matrix (from Table 1), the reproduction levels given by the NRRs (from Table 2), the estimated matching parameters $\mu = 0.3$, $\beta = 0.2$, and $\gamma = 1.0$ (see footnote 38), and the switching parameters (see equation 41) indicated on the diagram. The initial condition for each projection is given by the observed 1910 racial distribution (so that $x_0(w) = 0.65$).

7 Concluding remarks

Demographic projections of population dynamics are typically based on one-sex models which presume that the characteristics of children depend on the characteristics of only one parent (generally the mother). But to the extent that racial classification depends on the racial classes of both parents, demographic analysis of racial systems requires a two-sex model. This paper has extended the two-sex framework developed by Preston and Campbell (1993), and shown how it can be used to study the intergenerational dynamics of the racial distribution.

We have made several theoretical contributions to the literature on racial systems. First, we have introduced the concept of “boundedness” for both racial classification and racial switching in the context of black-white-mulatto systems. Roughly, classification is bounded when the child’s racial class cannot lie outside the bounds

established by the parent’s racial classes, and switching is bounded when a couple’s post-matching racial classes cannot lie outside the bounds established by their pre-matching racial classes. Second, we have shown that bounded classification and switching (with no variation in fertility across racial classes) cause the population to become racially homogeneous in the long run. Third, we have shown that differential reproduction (even with bounded classification and switching) may induce not only racial heterogeneity but also more complicated long-run dynamics such as limit cycles (in which the racial distribution never reaches a steady state).

These theoretical findings depend crucially on the non-linearity of our projection model. Applied demographers will be familiar with the claim that the long-run population distribution (i.e., the “stable population”) is unique, heterogeneous, and does not depend upon the initial distribution. But this is merely a mathematical consequence of the linearity of conventional one-sex models, not a necessary truth about real-world population dynamics governed by more complicated processes. Given a social process that is inherently non-linear, the misapplication of linear models may not only induce error in short-run projections, but also provide a completely misleading picture of long-run tendencies. Two-sex intergenerational transmission provides one (but not the only) motivation for non-linear models; future work might attempt to also incorporate into projection models other forms of “density dependence” (where model parameters depend upon the current population distribution).⁴⁴

While our discrete-generations framework does not permit fine-grained population projections, we have also provided some insights into the dramatic growth of the white population share in Puerto Rico in the early 1900s. First, against Loveman and Muniz (2007), we have shown that the social norms governing the racial classification of children did not contribute to the growth of the white share. Given the classification matrices observed for 1910 and 1920, these norms would actually have promoted the growth of the mulatto share. Second, we have shown that differential reproduction (given the observed classification matrices) can explain why the white share did not fall, but cannot account for the rapid growth of this share. Consequently, we conclude that racial switching must have also played an important role. To account for the dramatic growth in the white share, our projections suggest that significant levels of racial switching must have occurred not only upon matching (e.g., switching by a mulatta female upon marriage to a white male), but also independent of racial mixing (e.g., when a mulatto couple becomes a white couple).⁴⁵

⁴⁴See Lee (1987) and Caswell (2008) on density dependence in demographic models. Arguably, the mathematics of non-linear dynamical systems could play a more vital role in sociology more generally, providing a vocabulary and methodology for describing and analyzing social processes (cf Fararo 1989). We encourage readers who are averse to mathematical notation to see Abraham and Shaw (1992) for a graphical introduction to dynamical systems.

⁴⁵Against Loveman and Muniz (2007), we have argued that (changes in) the racial classification of children of mixed couples did not play a role in the growth of the white share. However, other apparent conflicts between our analysis and theirs seem largely semantic. In particular, while we have employed the term “racial classification” narrowly to describe the social norms governing the

Moving beyond our current empirical application, our model could be used immediately to generate projections of the racial distribution for other countries with black-white-mulatto systems (e.g., Brazil).⁴⁶ But it is important to recognize that this framework could also be easily modified to address other types of racial systems and other forms of intergenerational mobility. Though our present theoretical analysis (sections 3 to 5) was specialized to bounded black-white-mulatto systems, the general model (equation 1) can be applied to any racial system (bounded or otherwise) with three classes.⁴⁷ Moreover, recalling that Preston and Campbell (1993) originally developed their framework to study intergenerational transmission of IQ, our present model might be further adapted to address population dynamics in other contexts where the child’s trait depends on the traits of both parents. For instance, recent research on social stratification (Beller 2009) emphasizes the importance of mother’s class (as well as father’s class) in determining intergenerational social mobility. Alternatively, sociologists of religion have sometimes found effects of both mother’s and father’s religiosity on the child’s religiosity (see, e.g., Hayes and Pitelkew 1993).⁴⁸

Other extensions of our model may be less immediate but potentially important. Given the limitations of discrete-generation models for making short-run population projections, researchers might well consider age-structured two-sex models.⁴⁹ With or without an age structure, future versions of the present model might also elaborate the individual’s “type.” In our formulation, each individual is characterized simply by his or her racial class. However, given the important distinction between (socially

classification of children, Loveman and Muniz (2007) employ this term (as well as “racial boundary shifting”) much more broadly to encompass (changes in) the social norms by which adults classify themselves and others based on traits such as physical appearance or income. Given our terminology, any reclassification of adults would constitute “racial switching.” Once these semantic differences are recognized, we concur strongly with Loveman and Muniz (2007) that dramatic growth of the white share occurred largely through reclassification, and that net effect of demographic processes (they had already accounted for differential reproduction and differential mortality; we have also accounted for racial mixing) on the change in racial shares was small.

⁴⁶Telles (2004, p 94), commenting on the racial classification matrix in contemporary Brazil (his Table 4.4), speculates that “the clear tendency for whitening the racial classification of children would, by itself, eventually lead to the elimination of blacks after several generations.” Our framework would readily permit a more quantitative assessment.

⁴⁷Incorporation of more than three classes would require respecification of the matching matrix, though various parameterizations might be suggested by analogy to statistical models for two-way tables (where the baseline assumption of independence between rows and columns corresponds to our baseline assumption of random matching). See, e.g., Powers and Xie (2000, Chap 4).

⁴⁸More speculatively, Voas (2003) suggests that a shift in the intergenerational transmission pattern in England during the 1940s from “semi-dominant” (where children often become Anglican with only one Anglican parent) to “recessive” (where children become Anglican only if both parents are Anglican) helps account for secularization in that country. Our framework seems well-suited for addressing this hypothesis.

⁴⁹Beyond the references in footnote 2, relevant work in mathematical demography includes Hadelar (1989), Pruss and Schappacher (1994), Martcheva (1999), and Inaba (2000).

assigned) race and (biologically determined) physical appearance, some parameters of the model might arguably depend on the latter. For instance, racial homophily in matching could be the indirect result of selection based on physical appearance (even if no one cared about partner's racial class *per se*). Moreover, as emphasized by Lieberson and Waters (1988, p 164 ff), matching probabilities depend not only upon racial but also non-racial characteristics (e.g., social class). To the extent that racial switching is associated in some countries with economic mobility (cf Hoetink 1967; Schwartzman 2007), the switching probabilities in our model might be derived from more primitive assumptions on the social mobility process. Thus, future versions of the model might elaborate the individual's type to include his or her physical appearance or social class (or other non-racial characteristics).⁵⁰

⁵⁰Of course, the desirability of these extensions must be balanced against the increased complexity of the model and the attendant data requirements. For instance, if each individual has n characteristics, and each characteristic i comes in v_i varieties, then there will be $\prod_{i=1}^n v_i$ types of individuals and $(\prod_{i=1}^n v_i)^2$ types of couples.

8 Appendix

8.1 Proof of Lemma

We wish to show that the boundedness of the classification matrix (conditions 7 and 8) and the boundedness of the switching matrix (conditions 24 and 26) together imply boundedness of the compound matrix SC . That is,

$$(SC)(ij, k) = 0 \quad \text{if } k < \min(i, j) \text{ or } k > \max(i, j) \quad (42)$$

and

$$\begin{aligned} (SC)(wm, w) &\geq (SC)(wb, w) \\ (SC)(mw, w) &\geq (SC)(bw, w) \\ (SC)(mb, b) &\geq (SC)(wb, b) \\ (SC)(bm, b) &\geq (SC)(bw, b) \end{aligned} \quad (43)$$

Condition (42) follows immediately from (7) and (24). Schematically, letting the stars denote elements which may be positive, these conditions imply

$$\begin{array}{c} S \\ \left[\begin{array}{cccccc} * & & & & & \\ * & * & & * & & \\ * & * & * & * & * & * \\ * & & * & * & & \\ & & & * & & \\ & & & * & * & * \\ * & & * & * & * & * \\ & & * & & * & * \\ & & & & & * \end{array} \right] \end{array} \begin{array}{c} C \\ \left[\begin{array}{ccc} * & & \\ * & * & \\ * & * & * \\ * & * & \\ & * & \\ * & * & \\ * & * & * \\ & * & * \\ & & * \end{array} \right] \end{array} = \begin{array}{c} SC \\ \left[\begin{array}{ccc} * & & \\ * & * & \\ * & * & * \\ * & * & \\ & * & \\ * & * & \\ * & * & * \\ & * & * \\ & & * \end{array} \right] \end{array} \quad (44)$$

and thus (42) is satisfied. To verify the first inequality in (43), we may use the first two lines of (26) to obtain

$$S(wm, ww) [1 - C(wm, w)] \geq S(wb, ww) [1 - C(wm, w)] \quad (45)$$

and

$$\begin{aligned} [S(wm, ww) + S(wm, wm)] C(wm, w) \\ \geq [S(wb, ww) + S(wb, wm) + S(wb, bb)] C(wm, w) \end{aligned} \quad (46)$$

Summing the left-hand and right-hand sides of these inequalities, we obtain

$$\begin{aligned} S(wm, ww) + S(wm, wm)C(wm, w) \\ \geq S(wb, ww) + [S(wb, wm) + S(wb, ww)] C(wm, w) \\ \geq S(wb, ww) + S(wb, wm)C(wm, w) + S(wb, bb)C(wb, w) \end{aligned} \quad (47)$$

where the final inequality follows from the first line of (8). Each of the other lines of (43) can be established in the same manner.

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