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8 Influence Networks

Social psychologists have long been interested in social influence processes, and whether these processes will lead over time to the convergence (or divergence) of attitudes or opinions. Adopting a "structural" perspective on social influence, this chapter develops a simple model of *influence networks*. Given the (fixed) structure of the influence network, we assume that each individual gradually revises her opinion toward those of her contacts, depending on the relative strength of the social tie to each contact. Our analysis focuses on the conditions under which opinions converge in the long run. While this model is not a Markov chain process, the underlying mathematics will be familiar from previous chapters, and we will see another application of communication classes.

In the second section, which is based upon Friedkin and Johnsen (1997), we generalize the model so that each individual's initial opinion continues to exert some influence on the individual's later opinions even in the long run. In this version of the model, differences of opinion may persist even when every individual is influenced by every other. However, we will see that the opinions of individuals who hold very similar (formally, *structurally equivalent*) positions within the influence network do become more similar over time.

8.1 The basic model

We consider an influence network with n individuals. Each individual i holds an initial opinion $\mathbf{x}_0(i)$ which is a scalar (say between 0 and 10), and we arrange these initial opinions as an $(n \times 1)$ column vector $\mathbf{x}_0$. Interpersonal influence is characterized by a square $(n \times n)$ matrix W called the *influence matrix*. By convention, each row of W is a probability vector, and $W(i,j) > 0$ indicates that individual i is *influenced by* individual j. (We permit individuals to influence themselves, with i's "own" influence reflected by $W(i,i) > 0$.) Letting $\mathbf{x}_t$ denote the vector of opinions in period t, the dynamics of the opinion-formation process are given by the equation

$$\mathbf{x}_{t+1} = W\mathbf{x}_t$$

and hence

$$\mathbf{x}_t = W^t\mathbf{x}_0.$$

While these equations may seem quite familiar from previous chapters, and every row of W is a probability vector, it is important to recognize that this model is *not* a Markov chain process. Individuals do not transition between a finite set of states – there are n individuals rather than n states of a chain – and W should not be interpreted as a transition matrix. Rather, given that the W matrix is *post*multiplied by a *column* vector, individual i's opinion in period $t+1$ is a *weighted average* of the opinions held by i's contacts in period t. Nevertheless, as we'll see shortly, our experience with Markov chains will prove helpful in the analysis of influence networks.

8.1.1 An example

To illustrate, consider the following example with 9 individuals (adapted from Bonacich manscript, Chapter 10).

```
>> W   % influence matrix
W =
    0.8000    0.2000         0         0         0         0         0         0         0
    0.4000    0.6000         0         0         0         0         0         0         0
         0         0    0.3000    0.3500    0.3500         0         0         0         0
         0         0    0.5000    0.5000         0         0         0         0         0
         0         0         0    0.8000    0.2000         0         0         0         0
         0         0         0         0         0    1.0000         0         0         0
         0         0    0.7000         0         0         0    0.3000         0         0
         0         0         0         0    0.3000         0    0.3000    0.1000    0.3000
    0.2000         0         0         0         0    0.4000    0.1000    0.1000    0.2000
```

Further given a vector of initial opinions, we can compute the opinions held in the next period.

```
>> x0 = [8 4 6 2 9 10 7 5 1]'   % initial opinions in period 0
x0 =
     8
     4
     6
     2
     9
    10
     7
     5
     1

>> x1 = W * x0   % opinions held in period 1
x1 =
    7.2000
    5.6000
    5.6500
    4.0000
    3.4000
   10.0000
    6.3000
    5.6000
    7.0000
```

It is straightforward to verify that each individual's new (period $t+1$) opinions are a weighted average of the prior (period t) opinions held by the individual's contacts. For instance, for individual 1, we see that

$$\mathbf{x}_1(1) = W(1,1)\ \mathbf{x}_1(0) + W(1,2)\ \mathbf{x}_2(0) = (.8)(8) + (.2)(4) = 7.2$$

In this example, individual 1 is influenced by herself (reflected by the "weight" of 0.8 on her own prior opinion) and individual 2 (reflected by the weight of 0.2 on his prior opinion). Thus, individual 1's new opinion is a weighted average of these two prior opinions.

To determine the long-run outcome of this opinion-formation process, we continue to iterate for the next 15 periods.

```
>> for t = 1:15; disp((W^t * x0)'); end

    7.2000    5.6000    5.6500    4.0000    3.4000   10.0000    6.3000    5.6000    7.0000
    6.8800    6.2400    4.2850    4.8250    3.8800   10.0000    5.8450    5.5700    8.0300
    6.7520    6.4960    4.3323    4.5550    4.6360   10.0000    4.7530    5.8835    8.1235
    6.7008    6.5984    4.5165    4.4436    4.5712   10.0000    4.4585    5.8421    8.0388
    6.6803    6.6394    4.5101    4.4801    4.4691   10.0000    4.4991    5.7047    7.9780
    6.6721    6.6557    4.4853    4.4951    4.4779   10.0000    4.5068    5.6543    7.9520
    6.6689    6.6623    4.4861    4.4902    4.4917   10.0000    4.4917    5.6465    7.9410
    6.6675    6.6649    4.4895    4.4882    4.4905   10.0000    4.4878    5.6420    7.9358
    6.6670    6.6660    4.4894    4.4888    4.4886   10.0000    4.4890    5.6384    7.9336
    6.6668    6.6664    4.4889    4.4891    4.4888   10.0000    4.4893    5.6372    7.9329
    6.6667    6.6666    4.4889    4.4890    4.4890   10.0000    4.4890    5.6370    7.9326
    6.6667    6.6666    4.4890    4.4890    4.4890   10.0000    4.4890    5.6369    7.9325
    6.6667    6.6666    4.4890    4.4890    4.4890   10.0000    4.4890    5.6368    7.9324
    6.6667    6.6667    4.4890    4.4890    4.4890   10.0000    4.4890    5.6368    7.9324
    6.6667    6.6667    4.4890    4.4890    4.4890   10.0000    4.4890    5.6368    7.9324
```

Note that the column vectors of opinions have been transposed so that the rows of this table correspond to time periods while the 9 columns correspond to the 9 individuals. For the present example, we see that opinions have reached an equilibrium by period 15. Further, we see that some individuals eventually share common opinions: $\mathbf{x}_{15}(i) = 6.6667$ for $i \in \{1,2\}$ and $\mathbf{x}_{15}(i) = 4.4890$ for $i \in \{3,4,5,7\}$. However, at least for this example, not everyone holds the same opinion even in the long run.

8.1.2 Direct influence vs. total influence

Following Friedkin and Johnsen (1997), it is useful to distinguish *direct* influence from *total* influence. Direct influence is reflected by the equation

$$\mathbf{x}_{t+1} \ = \ W\mathbf{x}_t$$

Thus, the direct influence matrix (W itself) maps old (period t) opinions into new (period $t+1$) opinions. In contrast, total influence is reflected by the equation

$$\mathbf{x}_\infty \ = \ W^\infty \mathbf{x}_0$$

Thus, the total influence matrix (W^∞) maps initial (period 0) opinions into long-run (period ∞) opinions. For the present example (where equilibrium is reached within 15 periods), the total influence matrix is given below.

```
>> W^15
ans =
    0.6667    0.3333         0         0         0         0         0         0         0
    0.6667    0.3333         0         0         0         0         0         0         0
         0         0    0.3524    0.4934    0.1542         0         0         0         0
         0         0    0.3524    0.4934    0.1542         0         0         0         0
         0         0    0.3524    0.4934    0.1542         0         0         0         0
         0         0         0         0         0    1.0000         0         0         0
         0         0    0.3524    0.4934    0.1542         0    0.0000         0         0
    0.0580    0.0290    0.2605    0.3647    0.1140    0.1739    0.0000    0.0000    0.0000
    0.1739    0.0870    0.0766    0.1073    0.0335    0.5217    0.0000    0.0000    0.0000
```

To help interpret this matrix, consider individual 9. As we have already seen, individual 9's new (period $t+1$) opinion can be determined by computing a weighted average of the previous (period t) opinions using weights given in row 9 of the direct influence (W) matrix. The positive weights in this row indicate that individual 9 is directly influenced by individuals 1, 6, 7, 8, and 9. In contrast, we can determine this individual's long-run (period ∞) opinion by computing a weighted average of the initial (period 0) opinions using the weights given in row 9 of the total influence (W^∞) matrix. Intuitively, these weights do not reflect the proximate ("direct") influences on individual 9, but rather the ultimate ("total") influences. The positive weights in row 9 of this matrix reveal that 9's long-run opinion can be derived from the initial opinions held by individuals 1, 2, 3, 4, 5, and 6.

We have emphasized that an influence matrix should not be interpreted as a transition matrix for a Markov chain. Nevertheless, because each row of W is a probability vector, we can now make use of Theorem 1 from Chapter 1. Namely, if W is primitive, then every row of the total influence matrix W^∞ equals $\mathbf{v}$, the unique probability vector determined by the equation $\mathbf{v} = \mathbf{v}W$. Consequently, all opinions converge in the long run: $\mathbf{x}_\infty(i) = \mathbf{v}\mathbf{x}_0$ for all i. That is, every individual's long-run opinion is determined by the same weighted average of the initial opinions, where $\mathbf{v}(i)$ is the weight placed on i's initial opinion. Of course, in our example, given variation in the long-run opinions, it is evident that the W matrix is not primitive.

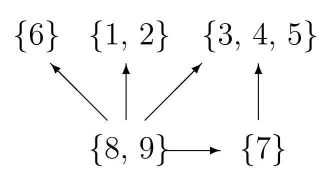
8.1.3 Communication classes in the influence network

In our example, we saw that opinions eventally converge for some subsets of individuals but not others. To understand why, it is useful to determine the communication classes of the influence network. Following the recipe presented in Chapter 7, we obtain the communication classes and image matrix below.

```
>> Z = double(W > 0);   % zero pattern of W
```

```
>> R = (eye(9) + Z)^8 > 0;   % reachability

>> C = R & R';   % can reach and be reached by

>> U = unique(C, 'rows')   % communication classes
U =
     0     0     0     0     0     0     0     1     1
     0     0     0     0     0     0     1     0     0
     0     0     0     0     0     1     0     0     0
     0     0     1     1     1     0     0     0     0
     1     1     0     0     0     0     0     0     0

>> M = U * Z * U' > 0;

>> M = M & ~eye(5)   % image matrix
M =
     0     1     1     1     1
     0     0     0     1     0
     0     0     0     0     0
     0     0     0     0     0
     0     0     0     0     0
```

On the directed graph below, an edge from communication class $[i]$ to class $[j]$ indicates that class $[i]$ is *influenced by* class $[j]$.[1] Note that the classes {1, 2}, {3, 4, 5}, and {6} are closed, while the classes {7} and {8, 9} are open.



The convention adopted in this diagram – arrows denote *influenced by* rather than *influence* – might initially seem counterintuitive. However, this convention helps rationalize our terminology because the closed classes are indeed "closed" to outside influence, while the open classes are "open" to outside influence. Moreover, because the arrows reflect mathematical dependence between opinions, this diagram shows how we could have "reduced" the problem of finding the equilibrium opinion vector to a series of simpler subproblems. More specifically, we could begin by finding the equilibrium opinion within each of the closed classes, and then use those results to solve for equilibrium opinions within the open classes.

[1] Using the terminology developed in Chapter 7, this graph is a reduced influence diagram. We do not need to reverse the edges of this diagram because this reversal is already implicit in our *post*multiplication of the W matrix by the *column* vector $\mathbf{x}_t$.

8.1.4 Solving sequentially for the long-run opinions

To implement this sequential solution procedure, we begin with class {1, 2}. Because this class is closed to outside influence, the long-run opinions of this class can be determined entirely from the corresponding (2×2) submatrix of W and (2×1) subvector of $\mathbf{x}_0$.

```
>> W(1:2, 1:2)   % submatrix for closed class {1,2}
ans =
    0.8000    0.2000
    0.4000    0.6000

>> W(1:2, 1:2)^15   % total influence
ans =
    0.6667    0.3333
    0.6667    0.3333

>> x0(1:2)   % initial opinions
ans =
     8
     4

>> W(1:2, 1:2)^15 * y0(1:2)   % long-run opinions
ans =
    6.6667
    6.6667
```

Note that the convergence of the opinions within a closed class follows from the primitivity of the corresponding submatrix. It is also interesting to note that, while the opinions of individuals 1 and 2 are mutually determined, individual 1 exerts more "total" influence than individual 2, with the long-run opinions of both individuals determined by the weights $\mathbf{v}(1) = 2/3$ and $\mathbf{v}(2) = 1/3$. Intuitively, the greater relative weight on 1's initial opinion is due to the relative strength of 1's "own" influence parameter, which causes individual 2 to move toward 1's opinion faster than individual 1 moves toward 2's opinion.[2]

Similar analysis reveals the long-run opinions for closed class {3, 4, 5}.

```
>> W(3:5,3:5)   % submatrix for closed class {3, 4, 5}
ans =
    0.3000    0.3500    0.3500
```

[2] More generally, suppose that the influence matrix is given by

$$W = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}$$

Given the condition $\mathbf{v} = \mathbf{v}W$ and the requirement that $\mathbf{v}$ is a probability vector, we obtain the weights $\mathbf{v}(1) = \beta/(\alpha+\beta)$ and $\mathbf{v}(2) = \alpha/(\alpha+\beta)$. Increasing both α and β by the same factor would have thus have no effect on the long-run equilibrium, but merely slow the speed of convergence.

```
    0.5000    0.5000         0
         0    0.8000    0.2000

>> W(3:5,3:5)^15   % total influence
ans =
    0.3524    0.4934    0.1542
    0.3524    0.4934    0.1542
    0.3524    0.4934    0.1542

>> y0(3:5)   % initial opinions
ans =
     6
     2
     9

>> W(3:5,3:5)^15 * y0(3:5)   % long-run opinions
ans =
    4.4890
    4.4890
    4.4890
```

Again, convergence of opinions within this (closed) class follows from the primitivity of the submatrix.

While we could follow this same procedure again for the final closed class $\{6\}$, the problem is trivial because there is only one individual in this class. Because individual 6 is not influenced by any other individual ($W(6,6) = 1$), her opinion never changes over time, and her initial opinion remains her long-run opinion ($\mathbf{x}_\infty(6) = \mathbf{x}_0(6) = 10$).

Having solved for the long-run opinions within each closed class, we may now turn to the open classes. For the sole individual in class $\{7\}$, the long-run opinion $\mathbf{x}_\infty(7)$ is determined by the equation

$$\mathbf{x}_\infty(7) = W(7,3)\ \mathbf{x}_\infty(3) + W(7,7)\ \mathbf{x}_\infty(7)$$

We have already solved for individual 3's long-run opinion, and could further substitute for the values of $W(7,3)$ and $W(7,7)$ to obtain individual 7's long-run opinion. But it may be more instructive to note that, because row 7 of W is a probability vector, the preceding equation may be rewritten as

$$\mathbf{x}_\infty(7) = (1 - W(7,7))\ \mathbf{x}_\infty(3) + W(7,7)\ \mathbf{x}_\infty(7)$$

and hence we would obtain

$$\mathbf{x}_\infty(7) = \mathbf{x}_\infty(3)$$

regardless of the strength of individual 7's own influence effect (assuming $W(7,7) < 1$). Intuitively, because individual 3 is the sole "outside" influence on individual 7, individual 7's opinion must converge to 3's opinion in the long run, and the strength of 7's own effect merely determines the speed of convergence.

We turn finally to the open class {8, 9}. The long-run opinions for individuals 8 and 9 are jointly determined by the equations

$$\begin{aligned}\mathbf{x}_\infty(8) &= W(8,5)\ \mathbf{x}_\infty(5) + W(8,7)\ \mathbf{x}_\infty(7) + W(8,8)\ \mathbf{x}_\infty(8) + W(8,9)\ \mathbf{x}_\infty(9) \\ \mathbf{x}_\infty(9) &= W(9,1)\ \mathbf{x}_\infty(1) + W(9,6)\ \mathbf{x}_\infty(6) + W(9,7)\ \mathbf{x}_\infty(7) + W(9,8)\ \mathbf{x}_\infty(8) \\ &\quad + W(9,9)\ \mathbf{x}_\infty(9)\end{aligned}$$

After substitution and simplification, we obtain

$$\begin{aligned}\mathbf{x}_\infty(8) &= 2.9927 + 0.3333\ \mathbf{x}_\infty(9) \\ \mathbf{x}_\infty(9) &= 7.2278 + 0.125\ \mathbf{x}_\infty(8)\end{aligned}$$

and thus

$$\begin{aligned}\mathbf{x}_\infty(8) &= 5.6386 \\ \mathbf{x}_\infty(9) &= 7.9324\end{aligned}$$

While opinions converge within closed classes (given primitive submatrices), it is interesting to note that opinions do not generally converge within open classes. Intuitively, while the opinions of 8 and 9 are mutually determined, these individuals place different weights on “outsiders” who themselves hold different long-run opinions and are uninfluenced by 8 or 9.

8.2 The persistence of initial opinions

Having presented the basic model, we now generalize the model so that each individual’s initial opinion may have a permanent effect on their long-run opinion. More precisely, the dynamics of opinion formation are now given by

$$\mathbf{x}_{t+1} = \alpha W \mathbf{x}_t + (1-\alpha)\mathbf{x}_0$$

where $\alpha \in [0,1]$ is a coefficient of social influence. Implicitly, the basic model fixed α equal to 1.[3] To obtain a non-recursive version of this formula, we may substitute the equation for period-1 opinions,

$$\mathbf{x}_1 = \alpha W \mathbf{x}_0 + (1-\alpha)\mathbf{x}_0$$

into the formula for period-2 opinions to obtain

$$\begin{aligned}\mathbf{x}_2 &= \alpha W \mathbf{x}_1 + (1-\alpha)\mathbf{x}_0 \\ &= \alpha W(\alpha W \mathbf{x}_1 + (1-\alpha)\mathbf{x}_0) + (1-\alpha)\mathbf{x}_0 \\ &= [\alpha^2 W^2 + \alpha(1-\alpha)W + (1-\alpha)]\mathbf{x}_0\end{aligned}$$

[3] To generalize even further, we might specify the model as $\mathbf{x}_{t+1} = AW\mathbf{x}_t + (I-A)\mathbf{x}_0$ where A is a diagonal matrix with (diagonal) elements $\alpha_1, \ldots, \alpha_n$. In this way, we would permit individuals to be differentially susceptible to social influence. Note the similarity to the mover-stayer model discussed in Chapter 2.

Further substitution yields

$$\begin{aligned}\mathbf{x}_3 &= \alpha W\mathbf{x}_2 + (1-\alpha)\mathbf{x}_0 \\ &= \alpha W(\alpha^2W^2 + \alpha(1-\alpha)W + (1-\alpha))\mathbf{x}_0) + (1-\alpha)\mathbf{x}_0 \\ &= [\alpha^3W^3 + \alpha^2(1-\alpha)W^2 + \alpha(1-\alpha)W + (1-\alpha)]\mathbf{x}_0\end{aligned}$$

By induction, we thus obtain the general formula

$$\mathbf{x}_t = \left(\alpha^tW^t + (1-\alpha)\sum_{i=0}^{t-1}\alpha^iW^i\right)\mathbf{x}_0$$

This equation is more complicated than we might have anticipated.[4] Nevertheless, it is straightforward to solve for the long-run opinion vector given that the equilibrium condition

$$\mathbf{x}_\infty = \alpha W\mathbf{x}_\infty + (1-\alpha)\mathbf{x}_0$$

can be rewritten as

$$\mathbf{x}_\infty = (1-\alpha)(I-\alpha W)^{-1}\mathbf{x}_0$$

Thus, $(1-\alpha)(I-\alpha W)^{-1}$ is the total influence matrix for the generalized model.

8.2.1 An example

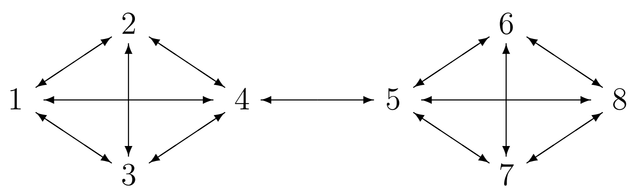
In the basic model (with $\alpha = 1$), we saw that all opinions converge if the influence matrix is primitive. This is no longer true in the generalized model (with $\alpha < 1$). To illustrate, consider the following example (drawn from Friedkin and Johnsen 1997).

```
>> W   % influence matrix
W =
    0.7000    0.1000    0.1000    0.1000         0         0         0         0
    0.5500    0.1500    0.1500    0.1500         0         0         0         0
    0.5500    0.1500    0.1500    0.1500         0         0         0         0
    0.4000    0.1500    0.1500    0.1500    0.1500         0         0         0
         0         0         0    0.1500    0.1500    0.1500    0.1500    0.4000
         0         0         0         0    0.1500    0.1500    0.1500    0.5500
         0         0         0         0    0.1500    0.1500    0.1500    0.5500
         0         0         0         0    0.1000    0.1000    0.1000    0.7000
```

Using the zero pattern of this matrix to obtain the influence diagram, it is evident that there is a single communication class due to the "bridge" between individuals 4 and 5.[5]

[4] In particular, note that this formula *cannot* be written as $\mathbf{x}_t = \alpha W^t\mathbf{x}_0 + (1-\alpha)\mathbf{x}_0$.

[5] We have omitted loops to simplify the influence diagram.



However, we also observe two "influence cliques" given by the sets {1, 2, 3, 4} and {5, 6, 7, 8}.[6] Further setting the vector of initial opinions and the social influence parameter we can obtain the long-run opinions through iteration.

```
>> x0   % initial opinions
x0 =
     0
    25
    45
    50
    50
    55
    75
   100

>> alpha = .5;   % social influence parameter

>> x = x0; for t = 1:20; x = alpha * W * x + (1-alpha) * x0; disp(x'); end

    6.0000   21.5000   31.5000   37.7500   62.2500   68.5000   78.5000   94.0000
    6.6375   20.9563   30.9563   37.6750   62.3250   69.0438   79.0438   93.3625
    6.8025   21.0444   31.0444   37.7209   62.2791   68.9556   78.9556   93.1975
    6.8714   21.1064   31.1064   37.7672   62.2328   68.8936   78.8936   93.1286
    6.9040   21.1381   31.1381   37.7902   62.2098   68.8619   78.8619   93.0960
    6.9197   21.1536   31.1536   37.8015   62.1985   68.8464   78.8464   93.0803
    6.9273   21.1611   31.1611   37.8070   62.1930   68.8389   78.8389   93.0727
    6.9310   21.1647   31.1647   37.8096   62.1904   68.8353   78.8353   93.0690
    6.9328   21.1665   31.1665   37.8109   62.1891   68.8335   78.8335   93.0672
    6.9337   21.1673   31.1673   37.8115   62.1885   68.8327   78.8327   93.0663
    6.9341   21.1677   31.1677   37.8118   62.1882   68.8323   78.8323   93.0659
    6.9343   21.1679   31.1679   37.8120   62.1880   68.8321   78.8321   93.0657
    6.9344   21.1680   31.1680   37.8120   62.1880   68.8320   78.8320   93.0656
    6.9344   21.1681   31.1681   37.8121   62.1879   68.8319   78.8319   93.0656
    6.9345   21.1681   31.1681   37.8121   62.1879   68.8319   78.8319   93.0655
    6.9345   21.1681   31.1681   37.8121   62.1879   68.8319   78.8319   93.0655
    6.9345   21.1681   31.1681   37.8121   62.1879   68.8319   78.8319   93.0655
    6.9345   21.1681   31.1681   37.8121   62.1879   68.8319   78.8319   93.0655
```

[6] In graph theory, a *clique* is a set of nodes such that (i) every node within this set is directly linked to every other node within this set, and (ii) this set is not strictly contained within any larger clique. More tersely, a clique is a maximal complete subgraph. Thus, even in the present example where all individuals belong to the same communication class (because all individuals exert indirect influence on all others), we may still identify two "influence cliques" based on direct influence.

```
    6.9345   21.1681   31.1681   37.8121   62.1879   68.8319   78.8319   93.0655
    6.9345   21.1681   31.1681   37.8121   62.1879   68.8319   78.8319   93.0655
```

The column vectors of opinions have again been transposed so that the rows on this table correspond to time periods while the columns correspond to individuals. In this example, we see that opinions have reached an equilibrium by period 20. Of course, we could also have obtained this equilibrium outcome using the total influence matrix.

```
>> inv(eye(8) - alpha * W) * (1-alpha)   % total influence matrix
ans =
    0.8313    0.0537    0.0537    0.0540    0.0046    0.0005    0.0005    0.0018
    0.2891    0.5671    0.0671    0.0675    0.0057    0.0006    0.0006    0.0023
    0.2891    0.0671    0.5671    0.0675    0.0057    0.0006    0.0006    0.0023
    0.2282    0.0635    0.0635    0.5670    0.0479    0.0054    0.0054    0.0193
    0.0193    0.0054    0.0054    0.0479    0.5670    0.0635    0.0635    0.2282
    0.0023    0.0006    0.0006    0.0057    0.0675    0.5671    0.0671    0.2891
    0.0023    0.0006    0.0006    0.0057    0.0675    0.0671    0.5671    0.2891
    0.0018    0.0005    0.0005    0.0046    0.0540    0.0537    0.0537    0.8313

>> ans * x0   % long-run opinions
ans =
    6.9345
   21.1681
   31.1681
   37.8121
   62.1879
   68.8319
   78.8319
   93.0655
```

It is interesting to note that, while individuals 4 and 5 hold the same initial opinion, their opinions diverge over time. Intuitively, 4 and 5 are "pulled" toward the other members of their respective cliques. At the same time, we also observe some convergence of opinions within cliques.

8.2.2 Structural equivalence in influence networks

Social network analysts have proposed a variety of methods for identifying "roles" or "positions" within networks. For instance, the set of individuals may be partitioned into *structural equivalence* classes, with individuals i and j assigned to the same class if and only if they hold precisely the same pattern of social ties (to and from every individual k). In the context of influence networks, this definition may be modified slightly so that individual i and j are regarded as *structurally equivalent in* W when

$$W(i,k) = W(j,k) \text{ for all } k \neq i, j.$$

Thus, in our example, individuals 2 and 3 are structurally equivalent in W (as are individuals 6 and 7). Given this definition, Friedkin and Johnsen (1997) prove the following result. If i and j are structurally equivalent in W, then the difference between their long-run opinions is proportional to the difference between their initial opinions. More precisely,

$$\mathbf{x}_\infty(i) - \mathbf{x}_\infty(j) = \left(\frac{1-\alpha}{1-\alpha\gamma}\right)(\mathbf{x}_0(i) - \mathbf{x}_0(j))$$

where, by definition,

$$\gamma = W(i,i) - W(j,i) = W(j,j) - W(i,j)$$

Because the rows of W are probability vectors, we see that $\gamma \in [-1,1]$ and thus $((1-\alpha)/(1-\alpha\gamma)) \leq 1$. Substantively, if individuals i and j are structurally equivalent in W, their difference of opinion shrinks over time.[7]

To illustrate, consider the long-run opinions for different values of α.

```
>> for alpha = 0:.1:.9; x = inv(eye(8)-alpha * W)*(1-alpha)*x0; disp([alpha x']); end

    0          0          25         45         50         50         55         75         100
    0.1000     1.2269    24.2791    42.2791    47.5474    52.4526    57.7209    75.7209    98.7731
    0.2000     2.5147    23.5206    39.5206    45.0924    54.9076    60.4794    76.4794    97.4853
    0.3000     3.8776    22.7348    36.7348    42.6414    57.3586    63.2652    77.2652    96.1224
    0.4000     5.3381    21.9395    33.9395    40.2068    59.7932    66.0605    78.0605    94.6619
    0.5000     6.9345    21.1681    31.1681    37.8121    62.1879    68.8319    78.8319    93.0655
    0.6000     8.7392    20.4871    28.4871    35.5051    64.4949    71.5129    79.5129    91.2608
    0.7000    10.9101    20.0466    26.0466    33.3946    66.6054    73.9534    79.9534    89.0899
    0.8000    13.8733    20.2607    24.2607    31.7820    68.2180    75.7393    79.7393    86.1267
    0.9000    19.3366    22.8035    24.8035    31.8881    68.1119    75.1965    77.1965    80.6634
```

This table is arranged so that each row corresponds to a different value of α (given in the first column) while the remaining columns correspond to the individuals.[8] Focusing on (structurally equivalent) individuals 2 and 3, note that the preceding formula reduces to

$$\mathbf{x}_\infty(3) - \mathbf{x}_\infty(2) = (1-\alpha)(\mathbf{x}_0(3) - \mathbf{x}_0(2)) = (1-\alpha)(20)$$

and we may verify from the table that their long-run opinions differ by 20 when $\alpha = 0$, by 12 when $\alpha = 0.4$, and by 4 when $\alpha = 0.8$.

[7]To be more precise, their opinions will grow closer for any $\gamma \in (-1,1)$. When $\gamma = 1$, both i and j are uninfluenced by others, and retain their initial opinions forever. When $\gamma = -1$, individuals i and j exchange opinions every period, alternating between $\mathbf{x}_0(i)$ and $\mathbf{x}_0(j)$.

[8]Compare to Friedkin and Johnsen (1997), Table 1, p 214.

8.3 Further reading

Most social psychology textbooks contain extensive discussion of social influence. See, for example, Chapters 7-9 in Aronson, Wilson, and Akert, *Social Psychology: The Heart and the Mind*, HarperCollins, 1994. The "structural" approach featured in this chapter has a long history, with contributions by authors including French (1956), Harary (1959), and DeGroot (1974). But within contemporary sociology, this approach is probably most closely associated with Noah Friedkin. For a book-length treatment, see Friedkin's *Structural Theory of Social Influence*, Cambridge, 1998. The first section of the present chapter draws on the presentation of influence networks in Bonacich, manuscript, Chapter 10. The second section is based on Friedkin and Johnsen, *Social Networks*, 1997. See Wasserman and Faust, *Social Network Analysis*, Cambridge, 1994, Chapters 9-12, for a more general discussion of structural equivalence and related concepts in social network analysis.