## The Hicks-Marshall Rules of Derived Demand: An Expository Note

## John Kennan University of Wisconsin-Madison

October, 1998

- 1. "The demand for anything is likely to be more elastic, the more readily substitutes for the thing can be obtained."
- 2. "The demand for anything is likely to be less elastic, the less important is the part played by the cost of that thing in the total cost of some other thing, in the production of which it is employed."
- 3. "The demand for anything is likely to be more elastic, the more elastic is the supply of co-operant agents of production."
- 4. "The demand for anything is likely to be more elastic, the more elastic is the demand for any further thing which it contributes to produce."

(Hicks, J.R.: The Theory of Wages. London: Macmillan, 1932, page 242, quoting Pigou)

2. (Marshall corrected by Hicks) It is "important to be unimportant" only when the consumer can substitute more easily than the entrepreneur. Hicks p. 246

Hicks gives a mathematical derivation of these rules. The purpose of this note is merely to explain this derivation. Those who find Hicks easy to follow need read no further.

## The Elasticity of Substitution

This is best explained in terms of the effect of an increase in capital on labor's share of output. Note that in the data labor share is roughly constant (about 2/3) while the capital labor ratio has been rising.

Under constant returns, pQ = vK + wL (where K is capital, with price v, and L is labor, with price w, and Q is output, with price p).

Labor share is given by:

$$s = \frac{wL}{py} = \frac{wL}{vK + wL} = \frac{1}{\frac{vK}{wL} + 1}$$

Therefore the change in s depends on the elasticity of K/L with respect to v/w: this is  $\sigma$ , the elasticity of substitution.

$$\sigma = -\frac{d\log\frac{K}{(L)}}{d\log\frac{v}{(w)}}$$

This is exactly like asking what happens to revenue when quantity and price move along a demand curve. In the inelastic case, an increase in quantity reduces revenue. Here the relatively inelastic case means that an increase in the capital-labor ratio reduces the product of the capital-labor and factor-price ratio, and this increases labor share. If  $\sigma = 0$ , then labor and capital cannot be substituted at all, so when capital increases labor gets a larger share: this is true whenever  $0 < \sigma < 1$ . If  $\sigma = \infty$ , then labor and capital are perfect substitutes, and s falls when K increases.

If  $\sigma = 1$ , then a change in the capital-labor ratio leaves labor share unchanged. This was an important justification for the Cobb-Douglas production function.

Also, each factor's share is the ratio of its marginal to its average product:

$$s = \frac{\frac{w}{p}}{\frac{y}{L}} = \frac{MPL}{APL}$$

# **Derivation of the Hicks-Marshall Rules**

Start with a CES production function, a constant elasticity demand function, and a constant elasticity supply function for capital. The result will hold locally even if these are not exact.

The logic is this. A wage is set (for example by a union), and each firm in a competitive industry maximizes profit, taking the output price and the price of capital as given. The output price is determined by free entry, at average cost. In order to break even, the firm must minimize cost. When the amount of capital changes, the price of capital changes along the supply curve.

Notation

Given:

σ the elasticity of substitution,

η the elasticity of product demand,

e the elasticity of supply of capital

s labor share.

Find:

 $\lambda$  the elasticity of labor demand

μ the cross-elasticity of demand for capital with respect to the wage

τ the elasticity of output with respect to the wage

Each elasticity is taken as a positive number, so for example,  $\sigma = -d\log(K/L)/d\log(v/w)$ . Since Q is expected to fall when w rises, define the elasticity of output as  $\tau = -d\log(Q)/d\log(w)$ . Similarly, since K is expected to fall when w rises (on the basis that consumers have better substitution possibilities than producers), define the cross-elasticity of demand for capital as  $\mu = -d\log(K)/d\log(w)$ .

Write the production function as

$$Q^{\rho} = \alpha K^{\rho} + \beta L^{\rho}$$
;  $\alpha + \beta = 1, \rho < 1$ 

Differentiate this to get the marginal products:

$$\rho Q^{\rho-1}MPK = \rho \alpha K^{\rho-1}$$
$$\rho Q^{\rho-1}MPL = \rho \beta L^{\rho-1}$$

Then

$$MPK^{\sigma} = \alpha^{\sigma}APK \; ; \; \sigma \equiv \frac{1}{1-\rho}$$
  
 $MPL^{\sigma} = \beta^{\sigma}APL$ 

This loglinear relationship between marginal and average products in the CES production function gives a

closed form solution for the labor demand elasticity.

Steps

1. Since factors are paid their marginal products,  $MPL^{\sigma} = \beta^{\sigma}APL$  implies

$$\sigma[\log(w) - \log(p)] = \sigma\log(\beta) + \log(Q) - \log(L)$$

Differentiate this with respect to log(w), allowing Q to change, and allowing for the resulting movement along the product demand curve:

$$\sigma[1-\frac{\tau}{\eta}] = \lambda - \tau$$

Rearrange this as

$$\tau = \left[\frac{\eta}{\eta - \sigma}\right](\lambda - \sigma)$$

2. The relationship between the capital-labor ratio and the factor-price ratio for the CES can be written as

$$\log(K) - \log(L) = \sigma \log(\alpha/\beta) - \sigma \log(\nu) + \sigma \log(w)$$

Differentiate this with respect to log(w), allowing K to change, and allowing for the resulting movement along the capital supply curve:

$$\lambda - \mu = \sigma \frac{\mu}{e} + \sigma$$

Rearrange this as

$$\mu = \left[\frac{e}{\sigma + e}\right](\lambda - \sigma)$$

$$\lambda - \mu = \left[\frac{\sigma}{\sigma + e}\right](\lambda + e)$$

Then using the result from point 1 above,

$$\tau - \mu = \sigma \left[ \frac{\eta + e}{(\sigma + e)(\eta - \sigma)} \right] (\lambda - \sigma)$$

3. Differentiating the production function gives

$$\frac{dQ}{dw} = MPK \frac{dK}{dw} + MPL \frac{dL}{dw}$$

$$= (1-s)APK \frac{dK}{dw} + sAPL \frac{dL}{dw}$$

$$\frac{w}{Q} \frac{dQ}{dw} = (1-s)\frac{w}{K} \frac{dK}{dw} + s\frac{w}{L} \frac{dL}{dw}$$

So

$$\tau = (1-s)\mu + s\lambda$$

and

$$\tau - \mu = s(\lambda - \mu)$$

$$= s\sigma \left[ \frac{\lambda + e}{\sigma + e} \right]$$

Compare the two expressions for  $\tau$  -  $\mu$  to get

$$\tau - \mu = \sigma \left[ \frac{\eta + e}{(\sigma + e)(\eta - \sigma)} \right] (\lambda - \sigma)$$

$$= s\sigma \left[ \frac{\lambda + e}{\sigma + e} \right]$$

$$\frac{\lambda - \sigma}{\theta} = \lambda + e$$

where  $\theta$  is defined as

$$\theta = \frac{s(\eta - \sigma)}{\eta + e}$$

Then

$$\lambda = \frac{\sigma + e\theta}{1 - \theta}; \quad \theta = \frac{s(\eta - \sigma)}{\eta + e}$$

$$\lambda = \frac{\sigma(\eta + e) + es(\eta - \sigma)}{\eta + e - s(\eta - \sigma)}$$

It is clear that  $\lambda$  is increasing in  $\theta$ . So an increase in labor share increases the elasticity of labor supply if and only if  $\eta > \sigma$  – consumers are more elastic than producers. Also,  $\theta$  is increasing in  $\eta$ , so a higher elasticity of product demand means a higher elasticity of labor demand.

Note also

$$\mu = \frac{e\theta}{1-\theta}; \quad \tau = \frac{s\sigma + e\theta}{1-\theta}$$

$$\lambda = \frac{\sigma}{1-\theta} + \mu; \quad \tau = \frac{s\sigma}{1-\theta} + \mu$$

So Hicks's correction of 2 above is for the case where the cross-elasticity of demand for capital is positive – an increase in the wage increases the quantity of capital, because substitution in production outweighs the effect on output due to substitution by consumers.

The other two rules require a little more work.

#### Lemma

If  $f(x) = \frac{ax+b}{cx+d}$ , then f is an increasing function if and only if bc < ad.

Proof:

Let y = f(x) and write the bilinear equation

$$c xy + d y = a x + b$$

Let  $x_0, y_0$  be defined by  $cx_0+d=0$  and  $cy_0-a=0$ . Then

$$c xy - cx_0y = cy_0x + b$$

$$c (x-x_0)(y-y_0) = b + cx_0y_0 = b - ad/c$$

$$(x-x_0)(y-y_0) = [bc - ad]/c^2 = g$$

So if g is positive, x and y are reciprocals, relative to the origin  $(x_0,y_0)$ . In order for x and y to be positively related, g must be negative, meaning bc < ad.

Effect of  $\sigma$  on  $\lambda$ :

$$\lambda = \frac{[\eta + (1-s)e]\sigma + es\eta}{s\sigma + (1-s)\eta + e}$$

$$ad - bc = [\eta + (1-s)e][(1-s)\eta + e] - s^2e\eta$$

$$= (1-s)(\eta + e)^2 > 0$$

So  $\lambda$  is an increasing function of  $\sigma$  (Rule 1)

Effect of e on  $\lambda$ :

$$\lambda = \frac{[s\eta + (1-s)\sigma]e + \eta\sigma}{e + (1-s)\eta + s\sigma}$$

$$ad - bc = [s\eta + (1-s)\sigma][(1-s)\eta + s\sigma] - \eta\sigma$$

$$= s(1-s)(\eta - \sigma)^2 \ge 0$$

So  $\lambda$  is an increasing function of e (Rule 3)

## **Special Cases**

If  $\eta = \sigma$ , then e is irrelevant:  $\lambda = \sigma$ 

If e is infinite, then  $\lambda = s\eta + (1-s)\sigma$ , and if  $\eta = \sigma$  this result holds as well. The point here is that there are two ways to suppress the effect of changes in v: either K doesn't change (because the consumers' response and the producers' response are exactly offsetting), or else K does change, but the supply curve is flat, so v doesn't change.

NOTE: These results are local except in the Cobb-Douglas case, because the marginal and average products are not proportional in the CES except when  $\sigma=1$ , so s is not a parameter. But starting at some wage  $w_0$ , we can obtain the values of  $Q_0,p_0,K_0,v_0,L_0$  implied by that wage, and let  $s_0=(w_0L_0)/(p_0Q_0)$ . Then the results are valid locally, with  $s=s_0$ .