ROUGH NOTES: PREFERENCES AND UTILITY

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Overview.

Voluntary Exchange. If an airline mechanic gets paid $30,000 a year, it must be that the mechanic would rather have the money than the leisure time, while the airline would rather have the mechanic than have the money. Another example: apartment rentals.

That is, price is bounded below by what the seller will accept: the supply price, and bounded above by what the buyer is willing to pay: the demand price. If there is any room between these (as there usually is) then either the buyer or the seller (usually both) obtain a surplus on the transaction. For example, the mechanic might have accepted a job at $25000 a year, it that was the only job available; the airline might have been willing to pay $40000 a year, if no one would work for less.

Supply Price. Lowest price the seller would accept, rather than fail to trade.

Demand Price. Highest price the buyer would pay, rather than do without.

Consumer Surplus. Difference between demand price and price actually paid.

Producer Surplus (profit). Difference between price actually received and supply price.

Market Efficiency. Trade should occur whenever there is a potential surplus.

Allocate scarce resources to those who value them the most.

This is a convenient analytical scheme. There are many factors which influence the supply price and not the demand price, or vice versa, so if we want to study the wage rate for mechanics, it is useful to study first the supply side of the market, and then the demand side. For example, an oil embargo will decrease the demand for mechanics, so the wage will fall; the end of the Iraq war increased the supply of (civilian) mechanics, and this should also have reduced the wage. Increasing the drinking age has no obvious effect on either the supply or the demand for mechanics, so there is no reason to believe that the wage will be affected in either direction.

Law of Demand. What employers will pay for mechanics or college teachers or nurses or accountants depends on how many of these workers they already have. That is, there comes a time when another CPA in the accounting department at Google or Walmart would just get in the way (long before this point, the company will have stopped hiring CPAs). At the other extreme, if the company had no accountants at all, the financial side of the operation would be a shambles, and the first accountant hired would be enormously valuable.

A second car is nice, but not as valuable as the first car. The same is true for tv sets, or rooms in a house, or shoes, or vacations.

In other words there is a negative relationship between demand price and the quantity demanded: the *demand curve* (everything else constant).

The more I have already, the less I will be prepared to pay for another one.

The higher is the price, the less I will buy.

There is an exception to this (at least in theory): if there is a basic good (potatoes, according to Giffen), and a close substitute for that good which is better but more expensive (meat), it can happen that an increase in the price of the basic good leads to an increase in the quantity demanded.

Upward Sloping Supply. Similarly there is a positive relationship between supply price and quantity supplied: the *supply curve* (everything else constant).

The more I am already supplying, the more it takes to induce me to supply another one.

The higher is the price, the more I will sell.

Law of One Price. In a competitive market (a homogeneous product, many small buyers and sellers acting independently with full information) all transactions must occur at the same price. For example, if gas stations in Middleton are selling gas for $5 and gas stations in Madison are selling gas for $3 the market is not in equilibrium.

Equilibrium. Quantity supplied equals quantity demanded.

Supply price equals demand price.

All gains from trade realized.

Surplus is maximized.

Example: Tax Incidence. In a competitive market it doesn't matter who is initially responsible for a tax. The equilibrium is where the gap between demand price and supply price is just enough to cover the tax.

Example

Demand

$$P = 25 - .15Q$$

Supply

$$P = .1Q$$

Equilibrium at P = $10 per hour, with Q = 100.

If a tax is imposed at the rate of $2.50 per unit, the equilibrium moves to a gross price of $11.50, a net price of $9 and an equilibrium quantity of 90. It doesn't matter whether the tax is initially put on the buyers or on the sellers.

Similarly, a subsidy moves quantity beyond 100, and the supply price must exceed the demand price by the amount of the subsidy.

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Intro. The aim is to describe a generic consumer. This basically means a set of preference axioms that only weird people would violate: completeness, transitivity.

Also, some slightly stronger properties might be assumed: monotonicity, convexity, continuity.

First preferences; then utility functions.

Preference Relations

A *binary relation* on a set X is a set of ordered pairs of elements of X – a subset of the Cartesian Product $X \times X$

Example. Say there are 30 people in the room. Then there are 870 possible ordered pairs (or 900 if degenerate pairs are allowed). The relation R is defined by saying $(x, y) \in R$ if x has black hair. This defines a rectangle in $X \times X$. Define the relation L by saying $(x, y) \in L$ if x likes y.

A preference relation $\succsim$ is a binary relation: $\succsim \subset X \times X$. Instead of writing $(x, y) \in \succsim$, it is usual to write $x \succsim y$; the interpretation is that x is at least as good as y.

Definition. A preference relation $\succsim$ on X is *rational* if it is complete and transitive:

(1) for all x, y , either $(x, y) \in \succsim$ or $(y, x) \in \succsim$
(2) if $x \succsim y$ and $y \succsim z$ then $x \succsim z$

In the examples above, neither R nor L is complete, and R is transitive, while L is not.

Indifference. $x \sim y$ means $x \succsim y$ and $y \succsim x$.

Strict Preference. $x \succ y$ means $x \succsim y$ and $x \not\precsim y$.

Implications. If $\succsim$ is rational, then $x \succ y$ and $y \succsim z$ implies $x \succ z$. Transitivity implies $x \succsim z$. If $z \succsim x$ then transitivity implies $y \succsim x$, a contradiction.

Transitivity of strict preference is a special case of this result.

Transitivity of the indifference relationship is implied by transitivity of $\succsim$ in both directions.

Definition. A preference relation $\succsim$ on X is *monotone* if $A > B$ implies $A \succ B$ (where $A > B$ means that each component of the vector A is strictly larger than the corresponding component of B).

Definition. A preference relation $\succsim$ on X is *convex* if $A \succsim B$, $C \succsim B$ implies $\alpha A + (1-\alpha)\, C \succsim B$; that is, if all upper contour sets are convex.

Definition. A preference relation $\succsim$ on X is *continuous* if $A^n \succsim B^n$,$A^n \to A$, $B^n \to B$, implies $A \succsim B$; that is, if all upper and lower contour sets are closed.

Example. The *Lexicographic* ordering is not continuous: $(a_1, a_2) \succ (b_1, b_2) \iff a_1 > b_1$ or $a_1 = b_1, a_2 > b_2$.

Indifference Curves. The indifference curve through A divides those bundles that are better than A from those that are worse. These curves slope down (greed), they are not thick (greed), and they don't cross (greed and transitivity).

Rate of Substitution. Δy/Δx along an indifference curve. This is the subjective value of one good relative to another – the rate at which a trade would break even. This relative value varies from one person to another, and it also varies for any given person as the quantities of the consumption goods vary.

As the changes become small, the RS becomes the Marginal Rate of Substitution (MRS), which is the relative value for a small change. Graphically, the MRS is the slope of the indifference curve at a point.

The MRS has nothing to do with market prices. But in the market, each consumer adjusts quantities until the MRS matches the price ratio – otherwise there are better consumption plans available.

Diminishing MRS. As y decreases and x increases along an indifference curve, the slope of the indifference curve decreases in magnitude (i.e. the curve gets flatter). This is an assumption that the subjective value of one good relative to another falls as one good becomes relatively abundant, and the other becomes relatively scarce. The MRS measures the relative value of consumption and leisure, measured form the worker's subjective point of view. This relative value varies from one worker to another, and it also varies for any given worker as C and L vary.

Homotheticity. If $A \succsim B$ and $\alpha \geq 0$, then $\alpha A \succsim \alpha B$

This is one version. The standard version (e.g. MWG 3.B.6) is apparently weaker:

If $A \sim B$ and $\alpha \geq 0$, then $\alpha A \sim \alpha B$ (assuming monotonicity)

Does this imply the stronger version?

The preference ordering must be defined on a cone, or else the definition makes no sense.

If preferences are continuous, then the two definitions are surely equivalent

Utility Functions

Utility functions add nothing once the preference ordering is given. A utility function just puts numbers on the indifference sets, in increasing order. But a utility function is a convenient way to summarize the preference ordering.

Definition. A function $u : X \to \mathbb{R}$ is a utility function representing preference relation $\succsim$ if, for all $x, y \in X$,

$$x \succsim y \iff u(x) \geq u(y)$$

If $\succsim$ can be represented by a utility function, then it is complete and transitive.

The utility function assigns numbers to all of the elements of X, and since there is a complete and transitive ordering of these numbers, the result follows.

If u represents $\succsim$, and $f : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function, then $f \circ u$ also represents $\succsim$. For example, $v(x) \equiv exp(u(x))$ is another utility function representing $\succsim$.

Properties of the utility function that hold for any utility representation are called ordinal properties; these are properties of the preference ordering itself.

Properties that hold for some utility representations but not for others are cardinal properties. For example, concavity is a cardinal property (which is defined only when X is a vector space); quasiconcavity is an ordinal property.

If X has a worst element x_0 under $\succsim$, then there is no loss in considering only nonnegative utility functions. Adding any number to a utility function gives a new utility function; so $v(x) \equiv u(x) - u(x_0)$ is a utility function representing $\succsim$.

Fact. *Any continuous rational ordering can be represented by a utility function.*

Example. CES Preferences

Suppose $X \subset \mathbb{R}^n$. The CES utility function is defined as

$$u(x) = \sum_{i=1}^{n} \alpha_i (x_i - \delta_i)^{\rho_i}$$

This makes sense without any restriction if ρ is an integer (positive or negative). More generally, it makes sense for any $\rho \in \mathbb{R}$ if $x \geq \delta$, with $(x_i - \delta_i)^\rho = \exp(\rho \log(x_i - \delta_i))$.

There is no need to assume that ρ_i is the same number for all i (although that is the conventional CES assumption).

Another utility function is defined by

$$v(x) = \sum_{i=1}^{n_1} \alpha_i \frac{(x_i - \delta_i)^{\rho_i} - 1}{\rho_i} + \sum_{i=n_1+1}^{n_2} \alpha_i \log(x_i - \delta_i)$$

for $\rho_i \neq 0$ for $1 \leq i \leq n_1$.

In general, v and u represent different preference orderings. This is a standard example in life-cycle models.

Show the level curves of CES utility functions, for several different values of ρ

Linear Homogeneous Utility functions. If the preference ordering is homothetic, pick any indifference curve (or surface), and assign the utility level 1 to all of the points in this set. Assign 0 as the utility of zero consumption. Then for any consumption plan A, find the point on the ray from the origin to A that has utility level 1. Call this point A_1. Then $A = \lambda A_1$ for some number $\lambda(A) \geq 0$. Define $u(A) = \lambda(A)$. This function is linear homogeneous by construction.

Utility Maximization

Budget Sets. A Budget set is a set of consumption vectors x that are feasible at given prices p and a given level of income I, as follows

$$B = \{x \mid p \cdot x \leq I\}$$

where

$$\begin{aligned} p \cdot x &= \sum_{i=1}^{n} p_i x_i \\ &= p_1 x_1 + p_2 x_2 + \ldots + p_n x_n \end{aligned}$$

The budget line is the upper boundary of this set, where all income is spent, with nothing left over.

Optimal Choice. Given a budget set B and a preference ordering $\succsim$, the consumer chooses a point $x^* \in B$ that is optimal, meaning that

$$x \in B \implies x^* \succsim x$$

When preferences are represented by the utility function u, this means

$$u(x^*) = \max_{x \in B} u(x)$$

Marginal Utility of Income. If expenditure on good i is reduced by the amount δ, meaning that consumption of this good falls by $\frac{\delta}{p_i}$, this frees up money that can be used to buy $\frac{\delta}{p_j}$ extra units of some other good j. The change in utility is then

$$\Delta u = u(\tilde{x}) - u(x^*)$$

where

$$\tilde{x}_i = x_i^* - \frac{\delta}{p_i}$$
$$\tilde{x}_j = x_j^* + \frac{\delta}{p_j}$$

with $\tilde{x}_k = x_k^*$ for $k \notin \{i, j\}$.

Suppose the utility function is differentiable, and the marginal utility of good i is denoted by $u_i(x)$

$$u_i(x) = \frac{\partial u(x)}{\partial x_i}$$
$$= \lim_{h \to 0} \frac{u(x_1, x_2, \ldots, x_{i+h}, \ldots x_n)}{h}$$

Then

$$\Delta u \approx \frac{\delta}{p_j} u_j(x^*) - \frac{\delta}{p_i} u_i(x^*)$$

so in the limit (for an infinitesimal expenditure change)

$$\lim_{\delta \to 0} \frac{\Delta u}{\delta} = \frac{u_j(x^*)}{p_j} - \frac{u_i(x^*)}{p_i}$$

In other words the change in utility is obtained by comparing the marginal utility per dollar spent on one good with the marginal utility per dollar spent on another good. If there is any difference, then utility rises when expenditure is shifted from one good to the other.

So at an optimum, marginal utility per dollar must be the same for all goods that the consumer spends money on (and higher for these goods than for all goods that the consumer does not spend money on).

If the consumer's income rises by a small amount ΔI, the increase in utility is $\Delta I \frac{u_i(x^*)}{p_i}$ if the extra money is spent on some good i that was already in the consumption plan, and the increase in utility is the same regardless of which of these goods the extra money is spent on, since marginal utility per dollar is the same for all these goods. The *marginal utility of income* λ is then given by

$$\lambda \geq \frac{u_i(x^*)}{p_i}$$

$$x_i^* > 0 \implies \lambda = \frac{u_i(x^*)}{p_i}$$

The consumer ranks goods not according to how desirable they are in absolute terms, but rather according to how desirable they are in relation to how much they cost. Although all consumer goods are consumed by somebody (by definition), any given person consumes only a small number of goods, the others being dominated in terms of marginal utility per dollar.

The MRS for any two goods is the ratio of the marginal utilities of the goods

$$u(x + \Delta x_j) - u(x - \Delta x_i) = 0$$

so

$$u(x) + \Delta x_j u_j(x) \approx u(x) - \Delta x_i u_i(x)$$

and

$$\frac{\Delta x_j}{\Delta x_i} \approx -\frac{u_i(x)}{u_j(x)}$$

Thus if the marginal utility per dollar is the same for goods i and j, then the MRS between these two goods is equal to the price ratio; in other words, the slope of the indifference curve is equal to the slope of the budget line.

First-Order and Second-Order Conditions for a Maximum. Think of moving along the budget line, and computing the utility level. This traces out a function of a single variable (in the two-good case), measuring the distance traveled along the budget line, starting from the vertical (or horizontal) intercept. If this function has a positive slope, the point is not optimal (unless the point is already at the end of the budget line) – by moving a little further along the budget line, a higher utility level is reached. If the function has a negative slope, then a higher utility level is reached by moving back a little (unless the point is at the beginning). So the slope of this function must be zero at an (interior) optimum.

Call this function $f(z)$. Then at any point z_0, the function can be approximated as

$$f(z) \approx f(z_0) + f'(z_0)(z - z_0) + f''(z_0)(z - z_0)^2$$

At a maximal point, it must be that $f'(z_0) = 0$. If this condition holds, the approximation reduces to

$$f(z) \approx f(z_0) + \frac{1}{2} f''(z_0)(z - z_0)^2$$

So if $f''(z_0) > 0$, the value of the function increases as z moves away from z_0 (in either direction). On the other hand if $f''(z_0) < 0$, there is a (local) maximum at z_0.

But what if $f''(z_0) = 0$? For example, suppose $f(z) = (z-1)^3$, with $z_0 = 1$. Then

$$f(z) \approx f(z_0) + \frac{1}{6} f'''(z_0)(z - z_0)^3$$

In this case z_0 is not optimal unless $f'''(z_0) = 0$, and in that case there is a local maximum only if $f''''(z_0) < 0$. And so on ...

Indirect Utility Functions. To show quasiconvexity, show that the lower contour set is convex

Let $\bar{p} = \alpha p^1 + (1-\alpha) p^2$ and similarly for $\bar{I}$

Then $v\left(\bar{p}, \bar{I}\right) = u(\tilde{x})$ with $\bar{p} \cdot \tilde{x} = \bar{I}$

If $\alpha p^1 \cdot \tilde{x} > \alpha I^1$ and $(1-\alpha) p^2 \cdot \tilde{x} > (1-\alpha) I^2$ then $\bar{p} \cdot \tilde{x} > \bar{I}$, a contradiction

If $p^1 \cdot \tilde{x} \leq I^1$ then $v(p^1, I^1) \geq v\left(\bar{p}, \bar{I}\right)$

If $p^2 \cdot \tilde{x} \leq I^2$ then $v(p^2, I^2) \geq v\left(\bar{p}, \bar{I}\right)$

Thus $v\left(\bar{p}, \bar{I}\right) \leq \max\left\{v(p^1, I^1), v(p^2, I^2)\right\}$

And if (p^1, I^1) and (p^2, I^2) are both in the lower contour set relative to a utility level v_0

Then $v\left(\bar{p}, \bar{I}\right) \leq \max\left\{v(p^1, I^1), v(p^2, I^2)\right\} \leq v_0$

So $\left(\bar{p}, \bar{I}\right)$ is also in the lower contour set.

This is the point of the Houston-Miami-LA problem.

The level curves of the indirect utility function are convex to the origin (but increasing toward the southwest). That is, the level curves look just like the level curves of the direct utility function, but the lower contour set is northeast of the curve.

Problem. Consumer Location Rankings

See micro problems, number 27.

Ask what the consumption plan would be in Houston, and argue that this plan must be available in either Miami or LA, so no consumer can have a (strict) preference for Houston.

Expenditure Functions.

$$e(p,u) = \min\{p \cdot x \mid U(x) \geq u\}$$

It is clear that the expenditure function is linear homogeneous in prices.

Define the Hicksian demand function as the point where expenditure is minimal (ignoring the possibility that there might be more than one such point)

$$p \cdot h(p,u) = e(p,u)$$

To show that the expenditure function is concave in prices, let $\bar{p} = \alpha p^1 + (1-\alpha)p^2$. Let h be the Hicksian demand function, meaning that $h(p,u)$ is the optimal consumption vector at prices p, if u is the highest utility level that can be reached. Then

$$\begin{aligned} e(\bar{p},u) &= \bar{p} \cdot h(\bar{p},u) \\ &= \alpha p^1 \cdot h(\bar{p},u) + (1-\alpha)p^2 \cdot h(\bar{p},u) \\ &\geq \alpha e\left(p^1,u\right) + (1-\alpha)e\left(p^2,u\right) \end{aligned}$$

where the inequality follows because $h(\bar{p},u)$ generates utility u at a cost of $p^1 \cdot h(\bar{p},u)$ when $p = p^1$ and $e(p^1,u)$ is the minimal cost of reaching u when $p = p^1$, and similarly for $p = p^2$.

Expenditure Function Properties. The expenditure function is concave in p, and the Hicksian demand function satisfies the law of demand, and the derivative of the expenditure function is the Hicksian demand function (Shepard's Lemma). These are all implications of a simple line of reasoning. Consider a change Δp in the price vector (so that the new price vector is $p + \Delta p$). The Hicksian demand function gives the quantities that minimize the expenditure needed to reach a particular utility level at various prices. This implies the following relationships

$$e(p,u) = p \cdot h(p,u) \leq p \cdot h(p+\Delta p,u)$$
$$e(p+\Delta p,u) = (p+\Delta p) \cdot h(p+\Delta p,u) \leq (p+\Delta p) \cdot h(p,u)$$

Now consider alternative estimates of the change in expenditure, Δe, resulting from a change in the price vector. This change is given by

$$\Delta e = e(p+\Delta p,u) - e(p,u)$$

The estimates are obtained by replacing the first term by something larger, yielding an upper bound, or by else replacing the second term by something larger, yielding a lower bound (since this term is being subtracted). Thus (using the relationships given above) the upper bound is

$$\begin{aligned} \Delta e &\leq (p+\Delta p) \cdot h(p,u) - p \cdot h(p,u) \\ &= \Delta p \cdot h(p,u) \end{aligned}$$

and the lower bound is

$$\begin{aligned} \Delta e &\geq (p+\Delta p) \cdot h(p+\Delta p,u) - p \cdot h(p+\Delta p,u) \\ &= \Delta p \cdot h(p+\Delta p,u) \end{aligned}$$

So

$$\Delta p \cdot h\left(p+\Delta p, u\right) \leq \Delta p \cdot h\left(p, u\right)$$

(since the left side is the lower bound for Δe and the right side is the upper bound). More compactly,

$$\Delta p \cdot \Delta h \leq 0$$

where

$$\Delta h = h\left(p+\Delta p, u\right) - h\left(p, u\right)$$

This is the "law of demand". It says that prices and quantities move in opposite directions (as long as the consumer remains at the same utility level). For example, if the price of a single good rises, then the quantity of that good falls (if no other price changes). If many prices change at the same time, then the pattern of quantity changes may be more complicated, but there is still a sense in which prices must move in opposite directions on average, since the sum of all the price changes multiplied by the quantity changes is negative.

Next, if only a single price changes, with $\Delta p_i > 0$ and $\Delta p_j = 0$ for $j \neq i$ then

$$\Delta p_i h_i\left(p+\Delta p_i, u\right) \leq \Delta e \leq \Delta p_i h_i\left(p, u\right)$$

so

$$h_i\left(p+\Delta p_i, u\right) \leq \frac{\Delta e}{\Delta p_i} \leq h_i\left(p, u\right)$$

Taking limits as the price change becomes small

$$\lim_{\Delta p_i \to 0} h_i\left(p+\Delta p_i, u\right) \leq \lim_{\Delta p_i \to 0} \frac{\Delta e}{\Delta p_i} \leq \lim_{\Delta p_i \to 0} h_i\left(p, u\right)$$

which proves Shepard's Lemma:

$$\frac{\partial e}{\partial p_i} = h_i\left(p, u\right)$$

Concavity of a function means that the function lies below its tangent (at every point). The tangent of the expenditure function at p

$$\Delta e \quad = \quad e\left(p+\Delta p, u\right) - e\left(p, u\right)$$

$$\left(p+\Delta p\right) \cdot \Delta h \leq 0 \leq p \cdot \Delta h$$

$$\Delta e = \left(p+\Delta p\right) \cdot h(p+\Delta p, u) - p \cdot h(p, u)$$

$$\Delta p \cdot h(p+\Delta p, u) \leq \Delta e \leq \Delta p \cdot h(p, u)$$

Example. (Cobb-Douglas)

If the expenditure function is

$$e\left(p_1, p_2\right) = 2u\sqrt{p_1 p_2}$$

then the Hicksian demands are

$$h_i = \frac{\partial e\left(p\right)}{\partial p_i}$$

so

$$\begin{aligned} h_1 &= u\sqrt{\frac{p_2}{p_1}} \\ h_2 &= u\sqrt{\frac{p_1}{p_2}} \end{aligned}$$

so

$$u^2 = h_1 h_2$$

and the utility function is

$$u(x_1, x_2) = \sqrt{x_1 x_2}$$

This is a special case of the CES cost function.

Quasilinear Utility Functions. If the preference ordering can be represented by a "quasi-linear" utility function

$$u(x) = x_1 + f(x_2, x_3, \ldots x_n)$$

then as long as x_1 is positive, the optimal response to a change in income is a change in the quantity of x_1 that absorbs the income change, with no change in consumption of the other goods:

$$\Delta x_1 = \frac{\Delta I}{p_1}$$

This response is optimal since it keeps the marginal utility per dollar constant across all of the goods: the marginal utility of the first good is constant no matter what choice is made, and the marginal utilities of the other goods don't change because the quantities don't change, and no price has changed.

For goods other than the first, there are no income effects here, so the compensated and uncompensated demand functions are identical.

Compensating Variations. Suppose the price vector changes from p^0 to p^1, with income fixed at I. After adjusting to these price changes, the consumer ends up on a new indifference curve (which may be higher or lower, depending on which prices rose or fell). At the new prices, the expenditure needed to get back to the original indifference curve is $e(p^1, v(p^0, I))$, and the Compensating Variation is the difference between this and the original income level I.

$$\begin{aligned} CV &= e\left(p^1, v\left(p^0, I\right)\right) - I \\ &= e\left(p^1, u^0\right) - e\left(p^0, u^0\right) \end{aligned}$$

Equivalently, the Compensating Variation is implicitly defined by

$$v\left(p^1, I + CV\right) = u^0 = v\left(p^0, I\right)$$

Example. Suppose $u(x) = x_1 + 2\sqrt{x_2}$. Then marginal utility per dollar is 1 or $\frac{mu_2}{p_2}$,whichever is larger. That is

$$\lambda = \max\left\{1, \frac{1}{p_2\sqrt{x_2}}\right\}$$

with $x_1 = 0$ and $x_2 = \frac{I}{p_2}$ if $\lambda > 1$, so

$$\lambda = \max\left\{1, \frac{1}{\sqrt{p_2 I}}\right\}$$

Suppose $p^0 = (1,1)$ and $p^1 = (1,2)$, and $I = 4$. Then $u^0 = 5$ and $u^1 = 4$ so

$$v\left(p^1, I + CV\right) = 5$$

so $CV = 1$.

These preferences are quasilinear (over the relevant region), so compensation involves changing the quantity of the "numeraire" good, so that the consumer stays at the same utility level.

Consumer Surplus. Graphically, Consumer Surplus is the area of the region between the demand curve and the price. A downward-sloping demand curve means that the demand price for the first unit is higher than for the second unit, and so on. For each unit, the consumer gets a surplus – the difference between the highest price that the consumer would have been willing to pay (the demand price for that unit), and the price actually paid. Calculating this surplus for each unit and adding them up gives the total surplus.

This heuristic version of consumer surplus works fine when income effects are negligible; in particular, it is exactly right in the case of a quasilinear utility function. More generally, the precise definition of consumer surplus refers to the compensated (Hicksian) demand curve.

If the price of good ℓ changes, with other prices fixed, the compensating variation is given by

$$e\left(p_\ell^1, u^0\right) - e\left(p_\ell^0, u^0\right) = \int_{p_\ell^0}^{p_\ell^1} h_\ell\left(p, u^0\right) dp_\ell$$

This calculation gives the change in the area under the demand curve – i.e the change in consumer surplus.

Complements and Substitutes

Gross Substitutes. If a consumer responds to an increase in the price of good i by increasing the amount of good j that is consumed, then it seems natural to say that j is a *substitute* for i. But there are income effects that make this kind of response less natural than it seems. If income is held fixed, the price increase makes the consumer worse off, and this would normally imply that consumption of good j would fall even if the two goods are such that neither is a natural substitute for the other (for example, cowboy boots and liver transplants, or shotguns and cheese). And if good i is a Giffen good, then consumption of good i rises when the price rises, and then if consumption of good j rises as well, one would not think that this indicates that the two goods are substitutes. For example if Big Macs are a Giffen good, and if consumers like to put ketchup on their burgers, then an increase in the price of Big Macs would lead to an increase in ketchup consumption, but this is not because ketchup is a substitute for a burger, but rather because these goods are complementary. And there is another difficulty: what if an increase in the price of i leads to an increase in the consumption of j, but an increase in the price of j leads to an decrease in the consumption of i? Then we would be in the position of saying that j is a substitute for i, but not the other way around, which is not natural at all.

Definition. Two goods are *gross substitutes* if neither is a Giffen good, and if an increase in the price of one leads to an increase in the consumption of the other. Two goods are *gross complements* if neither is a Giffen good, and if an increase in the price of one leads to an decrease in the consumption of the other.

Notice that under this definition, the awkward cases discussed above are just left out of the definition – if there is ambiguity, the goods are not classified as either substitutes or complements. Since the ambiguity is generated by income effects, the other way to deal with the issue is to define substitutes and complements with reference to compensated demand functions. Since these demand functions are the derivatives of the expenditure function, this leads to the following definition

(Net) Substitutes.

Definition. Goods i and j are *substitutes* if

$$\frac{\partial^2 e\left(p,u\right)}{\partial p_i \partial p_j} > 0$$

This means that the goods are substitutes if

$$\frac{\partial h_j\left(p,u\right)}{\partial p_i} > 0$$

Since cross-partial derivatives are symmetric (for smooth functions), this definition is symmetric: if j is a substitute for i, then i is a substitute for j.

If income effects are negligible, then the two definitions are equivalent.