Economics 711 Midterm Exam

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Time allowed: 75 minutes

Answer **THREE** questions. Each question has equal weight. Explain your answers carefully, using diagrams where appropriate. Write as if you are trying to convince an intelligent person who does not already know the answers. Be as precise as you can, but remember that an imprecise answer is better than nothing, and intuitive reasoning can sometimes be convincing. If the problem is too hard, answer a simplified version of it, and then try to sketch an argument for the more general version.

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1. Suppose that a preference ordering on $\mathbb{R}^L_+$ can be represented by the utility function

$$u(x) = \sum_{\ell=1}^{L} \frac{\alpha_\ell (x_\ell - \delta_\ell)^{\rho_\ell} - 1}{\rho_\ell}$$

with $\sum_{\ell=1}^{L} \alpha_\ell = 1$, and $\alpha_\ell \geq 0$ and $\rho_\ell < 1$, and $\sum_{\ell=1}^{L} p_\ell \delta_\ell < w$, where $p \cdot x \leq w$ is the budget constraint.

   (a) Find the Frisch demand functions.

   (b) Suppose that $L = 2, \delta_1 = 1, \delta_2 = 2, \alpha_1 = \frac{1}{2}, \alpha_2 = \frac{1}{2}, \rho_1 = \frac{1}{2}, \rho_2 = \frac{3}{4}, p_1 = 1, p_2 = 1, w = 23$. What is the optimal consumption plan?

2. Assume that a firm is risk neutral with respect to profits and that if there is any uncertainty in prices, production decisions are made after the resolution of such uncertainty. Suppose that the firm faces a choice between two alternatives. In the first, prices are uncertain. In the second, prices are non random and equal to the expected price vector in the first alternative. Show that a firm that maximizes expected profits will prefer the first alternative over the second.

3. An expected utility maximizer with constant relative risk aversion and wealth w buys α units of insurance at price q against a loss D that occurs with probability π, where $q > \pi$. Find α.

4. Pick TWO of the following assertions. State whether they are true, false or ambiguous, and explain why.
[Hints: (1) most true-false questions are false; (2) this exam was written by someone who knows (1)].

   (a) A rise in the wage rate implies a rise in the firm's marginal cost curve.

   (b) If a production technology has production isoquants defined by $y^2 = K^2 + L^2$, the cost function has L-shaped contours in the factor prices (i.e. the combinations of v and w which keep $c(v, w, y)$ constant, for fixed y, lie on an L-shaped isocost curve).

   (c) The utility function $u(x_1, x_2) = x_1^2 x_2^2$ is not quasi-concave, so the expenditure function derived from this utility function is not concave in prices.

Answers

1. Suppose that a preference ordering on $\mathbb{R}^2_+$ can be represented by the utility function

$$u(x) = \frac{1}{2}x_1^2 + \ln(x_2)$$

   (a) Suppose the prices are $p_1 = 2$, $p_2 = 1$. If wealth is $w = 4$, what is the optimal consumption plan?

      i. The marginal utility per dollar spent on x_1 is $\frac{x_1}{p_1}$, and for x_2 it is $\frac{1}{p_2 x_2}$. It is clear that x_2 must be positive (because the marginal utility at zero consumption is infinite). If $x_1$is also positive then $\frac{x_1}{p_1} = \frac{1}{p_2 x_2}$, with $p_2 x_2 = w - p_1 x_1$ from the budget constraint, so $\frac{x_1}{p_1} = \frac{1}{w - p_1 x_1}$. When $w = 4$, and $p_1 = 2$, this gives $x_1(4 - 2x_1) = 2$, so $x_1^2 - 2x_1 + 1 = 0$, and $x_1 = 1$, and $x_2 = 2$. So utility is $u = \frac{1}{2} + \log(2)$. But if $x_1 = 0$ then utility is $u = 2\log(2)$, so the optimal plan is $x_2 = 4$ (because $\log(2) > \frac{1}{2}$).

   (b) Suppose that wealth increases to 5. Now what is the optimal consumption plan?

      i. When $w = 5$, if $x_1$is positive then $x_1(5 - 2x_1) = 2$, so $x_1^2 - \frac{5}{2}x_1 + \frac{25}{16} - \frac{25}{16} + 1 = 0$, and $x_1 = 2$, and $x_2 = 1$. So utility is $u = 2$. But if $x_1 = 0$ then utility is $u = \log(5)$. And $e^2 > 5$, because $e > \left(1 + \frac{1}{n}\right)^n$ for all n, and setting $n = 2$ gives $e > \left(1 + \frac{1}{2}\right)^2$, which implies $e^2 > \frac{81}{16} > 5$.

   (c) Can you find a price vector and a wealth level such that $x_1(p, w) = 1$?

      i. No. If $x_1 = 1$ is optimal, then $x_2 = \frac{p_1}{p_2}$ and $p_2 x_2 = p_1$(from the marginal utility calculations above), but $p_2 x_2 + p_1 = w$ so $2p_1 = w$ and utility is $\frac{1}{2} + \log\left(\frac{p_1}{p_2}\right)$; but if $x_1 = 0$ then utility is

$$\log(x_2) = \log\left(\frac{w}{p_2}\right) = \log\left(\frac{2p_1}{p_2}\right) = \log(2) + \log\left(\frac{p_1}{p_2}\right)$$

      and this utility is higher, because $\log(2) > \frac{1}{2}$, since $2 > e^{\frac{1}{2}}$, because $4 > e$ so $x_1 = 1$ is not optimal, for any (p, w). Note that this implies that the utility at $x_1 = 1$ can't be recovered from the indirect utility function.

2. (a) Prove Hotelling's Lemma (if $\pi(p)$ is the profit function for the production set Y, and $y(p)$ is the supply function, then $\nabla\pi(p) = y(p)$, where $\nabla\pi(p)$ is the gradient of π – the vector of partial derivatives with respect to the prices).

   (b) Show that a factor of production is inferior if and only if an increase in the price of that factor reduces marginal cost.

3. Prove that a risk averse person buys full insurance if the price is actuarially fair, and buys less than full insurance if the price is not actuarially fair.

4. Pick TWO of the following assertions. State whether they are true, false or ambiguous, and explain why.

   (a) A firm uses 10 units of labor and 20 units of capital to produce 10 units of output. The marginal product of labor is 0.5. If there are constant returns to scale the marginal product of capital must be 0.25.

      i. True. $F(K, L) = F_K(K, L)K + F_L(K, L)L$. So $10 = F_K(K, L)20 + \frac{1}{2}10$, which implies $\frac{5}{20} = F_K(K, L)$

   (b) The utility function $u(x_1, x_2) = x_1^2 + \sqrt{x_2}$ is not quasi-concave, so the expenditure function derived from this utility function is not concave in prices.

      i. It's true that the utility function is not quasi-concave.

      ii. But concavity of the expenditure function has nothing to do with this.
      Let $\bar{p} = \alpha p^1 + (1 - \alpha)p^2$, with $\alpha \in [0, 1]$. Then

$$\begin{aligned} e(\bar{p}, u) &= \bar{p} \cdot h(\bar{p}, u) \\ &= \alpha p^1 \cdot h(\bar{p}, u) + (1 - \alpha)p^2 \cdot h(\bar{p}, u) \\ &\geq \alpha e\left(p^1, u\right) + (1 - \alpha)e\left(p^2, u\right) \end{aligned}$$

      where the inequality follows because $h(\bar{p}, u)$ generates utility u at a cost of $p^1 \cdot h(\bar{p}, u)$ when $p = p^1$ and $e\left(p^1, u\right)$ is the minimal cost of reaching u when $p = p^1$, and similarly for $p = p^2$.

   (c) Constant relative risk aversion implies that the demand for insurance is a decreasing function of wealth.

      i. Constant relative risk aversion means that $u(x) = \frac{x^{1-\rho} - 1}{1 - \rho}$, where x is wealth. If insurance is offered at price q against a loss D that occurs with probability π, with $q > \pi$, the expected utility if α units of insurance are purchased is $\pi u(w - D + \alpha - \alpha q) + (1 - \pi)u(w - \alpha q)$, so at an (interior) optimum it must be that

$\pi(1-q)\,u'(w-D+\alpha-\alpha q) = q(1-\pi)\,u'(w-\alpha q)$. In the CRRA case this gives

$$\begin{aligned}
\frac{\pi(1-q)}{(1-\pi)\,q} &= \frac{u'(w-\alpha q)}{u'(w-D+\alpha-\alpha q)} \\
\left(\frac{\pi(1-q)}{(1-\pi)\,q}\right)^{\frac{1}{\rho}} &= \frac{w-D+\alpha-\alpha q}{w-\alpha q} \\
&= 1-\frac{D-\alpha}{w-\alpha q}
\end{aligned}$$

so

$$\frac{D-\alpha}{w-\alpha q} = 1-\left(\frac{\pi(1-q)}{(1-\pi)\,q}\right)^{\frac{1}{\rho}} = A > 0$$

and

$$\frac{D}{A} = w + \left(\frac{1}{A}-q\right)\alpha$$

If insurance is very expensive, the optimal solution is $\alpha = 0$. This satisfies the first-order condition at $w = \frac{D}{A}$. So in the region where the amount of insurance is positive, $w < \frac{D}{A}$ and $\frac{d\alpha}{dw} < 0$ because $q < 1 < \frac{1}{A}$ (since $A < 1$, and there is no point in buying insurance if $q > 1$, since this would be like paying more than a dollar to get a dollar back with probability π).