# Math-Econ Problems

August 19, 2023

Some of these are routine practice problems, to help solidify understanding of the subject. Others call for formal proofs. And some are meant to help develop problem-solving skills – how to tackle unfamiliar problems.

1. (Rational numbers)
   (a) Show that the sum of two rational numbers is a rational number, and the product of two rational numbers is a rational number.
   (b) Show that for $n \in \mathbb{N}$ the sum of $n$ rational numbers is a rational number, and the product of $n$ rational numbers is a rational number.
   (c) Suppose $\{p, r\} \subset \mathbb{Q}$, with $p < r$. Show that there exists $q \in \mathbb{Q}$ with $p < q < r$ .
2. Let $A$ and $B$ be subsets of some set $X$. Prove the following
   (a) $(A \cap B)^c = A^c \cup B^c$
   (b) $(A \cup B)^c = A^c \cap B^c$
   (c) prove that for $n \in \mathbb{N}$,
   $$(A_1 \cup A_2 \cup \cdots \cup A_n)^c = A_1^c \cap A_2^c \cap \cdots \cap A_n^c$$
3. Let $X = \{a, b, c\}$ and $Y = \{x, y, z\}$. Give examples of functions $f : X \Rightarrow Y$ satisfying each of the following, or show that such a function does not exist:
   (a) a function that is neither one-to-one nor onto
   (b) a one-to-one function that is not onto
   (c) a function that is onto but is not one-to-one
   (d) a bijection
4. Prove or disprove the following assertion (i.e. either prove it or give a counterexample). "Suppose $n \in \mathbb{N}$ and $\sqrt{n} \in \mathbb{Q}$. Then $\sqrt{n} \in \mathbb{Z}$."
5. [Francesco Maurolico, 1575]
   Prove that the sum of the first $n$ odd (natural) numbers is $n^2$.
6. It says in the bible that all of the hairs on your head are numbered. So one day you are bored and you decide to count them (this takes a while). Then you get to wondering whether you are a particularly hairy person, or about average. After thinking about this for some time, you conclude that actually everyone has the same number of hairs. Your reasoning is this. Suppose it is true that any group of $n$ people has the same number of hairs. Then it must be true for groups of $n + 1$ people: the first $n$ people in the larger group have the same number of hairs, and the last $n$ people in this group have the same number, and these groups overlap, so all of the people have the same number. And obviously your conclusion is trivially true when $n = 1$. So by induction, it is universally true. RIGHT?

7. [Pythagorean triples]
   Suppose $\{x, y, z\} \subset \mathbb{N}$ with $0 < x < y < z$ and

$$x^2 + y^2 = z^2$$

   show that if $\gcd(x, y, z) = 1$ (i.e. the triple is not just a multiple of a "primitive" triple), then $z$ is an odd number.

8. [Polya]
   Consider the equation

$$10 + 24 + 35 + 6 + 7 + 8 + 9 \quad = \quad 99$$

   The left side uses each of the first ten digits exactly once. Can you find another equation with this property where the right side is 100? Or if not, can you prove that no such equation exists?

9. [Abbott]
   If a set $A$ contains $n$ elements, prove that the number of distinct subsets of $A$ is $2^n$.

10. [Polya]
    Generalize the following result

$$1 + 8 + 27 + 64 = 100$$

11. [Polya]
    Solve the following equation

$$8\left(4^x + 4^{-x}\right) - 54\left(2^x + 2^{-x}\right) + 101 = 0$$

12. Show that every convergent sequence is bounded.

13. If $a_n \to a$ and $b_n \to b$ show that $a_n b_n \to ab$

14. [Abbott]
    Assume $(a_n)$ is a bounded sequence with the property that every convergent subsequence converges to the same limit $a \in \mathbb{R}$. Show that $a_n$ converges to $a$.

15. [Abbott]
    Show that if $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$, and if $\lim x_n = \lim z_n = 1$, then $\lim y_n = 1$.

16. [Abbott]
    Let $(a_n)$ be a bounded sequence.

    (a) Prove that the sequence defined by $y_n = sup\{a_k : k \geq n\}$ converges.

    (b) The *limit superior* of $(a_n)$, or $\limsup a_n$, is defined by

$$\limsup a_n = \lim y_n,$$

    where $y_n$ is the sequence from part (a) of this exercise. Provide a reasonable definition for $\liminf a_n$ and briefly explain why it always exists for any bounded sequence.

    (c) Prove that $\liminf a_n \leq \limsup a_n$ for every bounded sequence, and give an example of a sequence for which the inequality is strict.

    (d) Show that $\liminf a_n = \limsup a_n$ if and only if $\lim a_n$ exists. In this case, all three share the same value.

17. [Classic problem; Ben Ames Williams version, 1926]
Five men and a monkey were shipwrecked on a desert island, and they spent the first day gathering coconuts for food. Piled them all up together and then went to sleep for the night. But when they were all asleep one man woke up, and he thought there might be a row about dividing the coconuts in the morning, so he decided to take his share. So he divided the coconuts into five piles. He had one coconut left over, and gave it to the monkey, and he hid his pile and put the rest back together. By and by, the next man woke up and did the same thing. And he had one left over and he gave it to the monkey. And all five of the men did the same thing, one after the other; each one taking the fifth of the coconuts in the pile when he woke up, and each one having one left over for the monkey. And in the morning they divided what coconuts were left, and they came out in five equal shares. Of course each one must have known that there were coconuts missing; but each one was guilty as the others, so they didn't say anything. How many coconuts were there in the beginning?

18. Show that if the function $f : \mathbb{R} \to \mathbb{R}_{++}$ is continuous on an interval $[a, b]$, with $0 \notin [a, b]$, then the reciprocal of this function $\left(\frac{1}{f}\right)$ is bounded on this same interval.

19. Prove that every convergent sequence is a Cauchy sequence.

20. Show that $\cap_{n\in\mathbb{N}} \left(a - \frac{1}{n}, b + \frac{1}{n}\right) = [a, b]$.

21. Suppose that $f : X \to X$ is a continuous bijection on a complete metric space, with the following property
$$d\left(f\left(x\right), f\left(y\right)\right) \quad > \quad cd\left(x, y\right)$$
for all $x, y \in X$, with $c > 1$. Show that $f$ has a unique fixed point, and show how to compute this fixed point.

22. Give an example of a function with the properties specified in problem 21, and write a program that computes the fixed point (and show that your program works).

23. [Abbott] Let
$$f_a\left(x\right) \quad = \quad \begin{cases} x^a & x \geq 0 \\ 0 & x < 0 \end{cases}$$
    (a) For which values of $a$ is $f$ continuous at zero?
    (b) For which values of $a$ is $f$ differentiable at zero? In this case, is the derivative function continuous?
    (c) For which values of $a$ is $f$ twice-differentiable?

24. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a continuous function. Show that the set
$$X = \{x \in \mathbb{R}^n \mid f(x) = 0\}$$
is closed.

25. Suppose $f : \mathbb{R} \to \mathbb{R}$ and $|f(x)| \leq x^2$. Show that $f$ is differentiable at 0.

26. Suppose $(X, d)$ is a metric space and $a \in X$. Prove that the function $f : X \to \mathbb{R}$ defined by $f(x) = d(a, x)$ is continuous.

27. [Polya]
Find $x, y, v, u$ satisfying the following equations
$$\begin{aligned} x + 7y + 3v + 5u &= 16 \\ 8x + 4y + 6v + 2u &= -16 \\ 2x + 6y + 4v + 8u &= 16 \\ 5x + 3y + 7v + u &= -16 \end{aligned}$$

You could of course use the first equation to eliminate $x$, and then substitute in the second and use that to eliminate $y$, and so on. Or you could find a solution by inverting a $4 \times 4$ matrix. You should eschew such mechanical methods.

28. Means

    (a) Given a set $\{x_i\}_{i=1}^n \subset \mathbb{R}^n$, the generalized mean $m_\rho$ is defined by

    $$n\frac{(m_\rho)^\rho - 1}{\rho} = \sum_{i=1}^n \frac{x_i^\rho - 1}{\rho}$$

    (b) Prove that $a > g > h$ where $a = m_1$, $g = m_0$ and $h = m_{-1}$are the arithmetic, geometric and harmonic means [alphabetical order]

    (c) Prove that $m_\rho$ is an increasing function of $\rho$, and

    $$\begin{aligned}\lim_{\rho\to-\infty} m_\rho &= \min_i x_i \\ \lim_{\rho\to\infty} m_\rho &= \max_i x_i\end{aligned}$$

29. Consider the function

    $$g(x) = \frac{\log\left(\frac{b+x}{a+x}\right)}{\log\left(\frac{c+x}{a+x}\right)}$$

    defined on $\mathbb{R}_+$, with $0 < a < b < c$. Is this function monotonic?

30. Consider the (CRRA) utility function

    $$u(c) = \theta\frac{(c+\kappa)^{1-\gamma} - 1}{1-\gamma}$$

    with $\gamma > 0$. Suppose the marginal utilities are known at three distinct consumption levels, $c_1, c_2, c_3$. Are the utility parameters $\theta, \gamma, \kappa$ then uniquely determined?

31. Suppose

    $$A = \{f : \mathbb{R} \to \mathbb{R}, f \text{ concave}, f(1) = 1, f(3) = 5, f(4) = 6\}$$

    Solve the following equations

    $$\begin{aligned}\sup\{f(2) \mid f \in A\} &= u \\ \inf\{f(2) \mid f \in A\} &= v\end{aligned}$$

32. Suppose $(X, d)$ is a complete metric space, and $A$ is a closed subset of $X$. Show that $(A, d)$ is a complete metric space.

33. Can the contraction mapping theorem be used to show that the function $f(x) = \frac{1}{x}$ has a fixed point?

34. Suppose $a$ is a limit point of the sequence $\{x_n\} \subset \mathbb{R}_{++} = \{x \in \mathbb{R} \mid x > 0\}$. Can you show that $a > 0$? If not, what restrictions on the sequence $\{x_n\}$ are needed to ensure that $a > 0$?

35. Consider the function

    $$f(x) = 3x^4 - 4x^2 + 2x + 4$$

    Find the maximum of this function on the interval $[0, 1]$.

36. [Sundaram] Find and classify all critical points (local maximum, local minimum, neither) of the following function:

$$f(x,y) = e^{2x}\left(x+y^2+2y\right)$$

For local optima that you find figure out whether they are also global optima.

37. [MIT 14.102] Suppose the function $G:\mathbb{R}^2\to\mathbb{R}$ is defined by

$$G(x,y) = x^2+2y^2-6x-7$$

Find all points on the curve $G(x,y)=0$ around which either $y$ is not expressible as a function of $x$ or $x$ is not expressible as a function of $y$. Compute $y'(x)$ along the curve at the origin.

38. [Abbott] Show that if $f$ is differentiable on an interval with $f'(x)\neq 1$, then $f$ can have at most one fixed point.

39. Suppose $\{a,b\}\subset\mathbb{R}^n$, $p\in\mathbb{R}^n_+$ and $\{x,y\}\subset\mathbb{R}$. Define

$$A(x,y) = \sum_{i=1}^n p_i a_i (a_i+x)\sum_{i=1}^n p_i b_i (b_i+y) - \sum_{i=1}^n p_i b_i (a_i+x)\sum_{i=1}^n p_i a_i (b_i+y)$$

$$B(x,y) = \sum_{i=1}^n\sum_{j>i}^n p_i p_j (a_i b_j - a_j b_i)(a_i b_j - a_j b_i + (a_i-a_j)y - (b_i-b_j)x)$$

(a) Which of the following statements is true

$$A(x,y)\geq B(x,y)$$
$$B(x,y)\geq A(x,y)$$

(b) Use the result from (a) to prove the Cauchy-Schwarz inequality

$$\sum_{i=1}^n p_i a_i^2\sum_{i=1}^n p_i b_i^2 \geq \left(\sum_{i=1}^n p_i a_i b_i\right)^2$$

40. **The Likelihood Ratio Test** (Neyman-Pearson Lemma)

A statistician expects to obtain a vector of data generated by one of two probability distributions, $p$ or $q$. The vector will lie in a finite set $X = \left\{x^1, x^2, ..., x^i ..., x^S\right\}$, called the sample space, and the probabilities associated with the points in this set are either $p_1, p_2, ..., p_i, ..., p_S$ or $q_1, q_2, ..., q_i, ..., q_S$, according to whether $p$ or $q$ is the true distribution.

The statistician wishes to design a test of the hypothesis $H0$ that the true distribution is $p$. The test involves dividing $X$ into two parts, $R$ and $A$, rejecting $H0$ if the observed data lie in $R$ and accepting $H0$ if the data lie in $A$. Two types of error are possible here: the test might reject H0 when it is actually true, or it might accept $H0$ when it's false.

The statistician is primarily concerned about errors of the first type. Provided that these can be held to an acceptable level it is also desirable to avoid errors of the second type. Specifically, the aim is to maximize the probability that $H0$ will be rejected when it's false (this is called the power of the test), subject to the constraint that the probability of type I error should not exceed some number $\alpha$ (called the size of the test).

(a) Translate this problem into the language of consumer theory. (What is the utility function? What is the budget constraint?).

(b) Solve the problem.

(c) Translate the solution back into language that the statistician can understand. (Note the title of the problem).

41. Consider the utility function

$$u = 2x_1^2 + 4x_2^{1/2}$$

(a) Find the demand functions for goods 1 and 2 as they depend on prices and wealth.

42. Suppose that a preference ordering on $\mathbb{R}_+^L$ can be represented by the utility function

$$u(x) = \theta_L x_L x_1 + \sum_{i=1}^{L-1} \theta_i x_i x_{i+1}$$

where $\theta \in \mathbb{R}_+^L$

(a) Is this preference ordering homothetic?

(b) Is this preference ordering separable? For example, if $x_L$ is fixed at some level $a$, the utility function defines a preference ordering on $\mathbb{R}_+^{L-1}$ , and if $x_L$ is fixed at $b$, the utility function defines another preference ordering on $\mathbb{R}_+^{L-1}$. Are these two orderings actually the same?

43. Maximize the function

$$u(x) = \frac{1}{2}x_1^2 + \ln(x_2)$$

subject to the constraints

$$\begin{aligned} p_1x_1 + p_2x_2 &\leq y \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

where $\{p_1, p_2, y\} \in \mathbb{R}_{++}$

44. Suppose $\{a_n\} = \{(x_n, y_n)\}$ is a sequence in $\mathbb{R}^2$ with the following properties

(a) $x_n$ is decreasing

(b) $y_n$ is increasing

(c) $||a_n||$ is bounded

Does the sequence $\{a_n\}$ converge? Either prove that it does, or give a counterexample.

45. [Pythagorean triples, again]
Suppose $\{x, y, z\} \subset \mathbb{N}$ with $0 < x < y < z$ and

$$x^2 + y^2 = z^2$$

(a) show that if $x$ is odd then $z$ is also odd and $y$ is a multiple of 4.

(b) show that if $\gcd(x, y, z) = 1$ (i.e. the triple is not just a multiple of a "primitive" triple), and if $x$ is even, then both $y$ and $z$ are odd, and $x$ is a multiple of 4.

(c) show that $xyz$ is divisible by 12.

46. Suppose $\mathcal{A}$ is a collection of open sets in a metric space $(X, d)$, and let $G = \cap\{A \mid A \in \mathcal{A}\}$. Then any point $g \in G$ must be in all of the sets in $\mathcal{A}$, and all of these sets are open, so for each $A \in \mathcal{A}$ there is a ball $B(g, r) \subset A$. And the intersection of these balls is not empty, (e.g. by the Nested Interval Property, if $X = \mathbb{R}$). So each point in $G$ can be surrounded by a ball contained within $G$. So $G$ is open.

(a) Is this argument right? If so, give an example.

(b) If the argument is not right, give a counterexample, and explain where the argument goes wrong.

47. Suppose that $f$ is a continuous mapping from a metric space $(X, d_X)$ to a metric space $(Y, d_Y)$, and $A$ is a compact subset of $X$. Prove that $f(A)$ is a compact subset of $Y$.