ROUGH NOTES: FIRST WELFARE THEOREM

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General Equilibrium. Each consumer has endowment ω^i and owns a share θ^i_j in the profits of firm j. The consumer has a preference ordering $\succeq^i$ over alternative consumption vectors.

Each firm has a technology F^j, the set of feasible net production vectors (with inputs measured as negative numbers, and outputs as positive numbers).

If the price vector is p, firm profits are given by

$$\pi^j = p \cdot y^j$$

where the "cdot" notation $\cdot$ means an inner product:

$$p \cdot y = \sum_{k=1}^{q} p_k y_k$$

Consumer i has income $M^i(p)$, given by

$$M^i(p) = p \cdot \omega^i + \sum_{j=1}^{m} \theta^i_j \pi^j$$

The consumer's demand vector $\hat{x}^i(p, M^i(p))$ is at least as good as anything else in the budget set

$$p \cdot x \leq M^i(p) \Longrightarrow \hat{x}\left(p, M^i(p)\right) \succeq^i x$$

The producer's net supply vector is at least as profitable as anything else in the production set

$$y \in F^j \Longrightarrow p \cdot \hat{y}^j(p) \geq p \cdot y$$

An equilibrium price vector p^* must satisfy the following conditions

- consumers optimize

$$p^* \cdot x \leq M^i \Rightarrow \hat{x}^i\left(p^*, M^i\right) \succeq^i x$$

- producers optimize

$$y \in F^j \Rightarrow p^* \cdot \hat{y}^j(p^*) \geq p^* \cdot y$$

- Markets clear (the allocation is feasible)

$$\sum_{i=1}^{m} \hat{x}^i\left(p^*, M^i(p^*)\right) = \sum_{i=1}^{m} \omega^i + \sum_j \hat{y}^j(p^*)$$

Efficiency (Pareto Optimality). An allocation is efficient (Pareto Optimal) if no one can be made better off without making someone worse off.

First Welfare Theorem. Any competitive equilibrium allocation is Pareto Optimal.

If there is an alternative feasible allocation that is a Pareto improvement, the value of aggregate consumption at the equilibrium prices is strictly larger in this alternative allocation (someone is doing strictly better, so the value of this person's consumption bundle might be strictly greater, or it would have been chosen before; and no one is doing worse, and if they could have achieved this by spending less money, then local nonsatiation implies that they could have done better). The value of consumption is the value of net production plus the value of the endowment. But the value of net production can't be higher in the alternative plan, because if it were, some producer was not maximizing profit. And since the value of the endowment is unchanged, this gives a contradiction.

Details. An alternative feasible allocation means consumption *vectors* $\tilde{x}^i$ for each consumer, and production vectors $\tilde{y}^j \in F^j$ for each producer, such that

$$\sum_i \tilde{x}_k^i = \sum_i \omega_k^i + \sum_j \tilde{y}_k^j$$

for all goods $k = 1, 2, \ldots, q$.

A Pareto improvement means that some consumer is better off, and no one is worse off. Say the first consumer is better off (the consumers can be renumbered so that this is the case). Then

$$p^* \cdot \tilde{x}^1 > M^1 (p^*)$$

since otherwise this consumer would have chosen this alternative conumption plan in the competitive equilibrium (contradicting the assumption that it actually was an equilibrium).

And if

$$p^* \cdot \tilde{x}^i < M^i (p^*)$$

then consumer i could have chosen a better plan, since

$$\tilde{x}^i \succeq^i \hat{x}^i \left(p^*, M^i (p^*)\right)$$

so this consumer could have chosen something just as good as the equilibrium plan, with money left over. So

$$p^* \cdot \tilde{x}^i \geq M^i (p^*)$$

for all consumers, with a strict inequality for at least one consumer. Adding over consumers gives

$$p^* \cdot \sum_{i=1}^{n} \tilde{x}^i > \sum_{i=1}^{n} M^i (p^*)$$

and

$$\begin{aligned} M^i (p^*) &= p^* \cdot \omega^i + \sum_j \theta_j^i \pi^j (p^*) \\ &= p^* \cdot \omega^i + \sum_j \theta_j^i p^* \cdot \hat{y}^j (p^*) \end{aligned}$$

so

$$\sum_{i=1}^{n} M^i(p^*) = p^* \cdot \sum_{i=1}^{n} \omega^i + \sum_{j}^{m} \sum_{i=1}^{n} \theta_j^i p^* \cdot \hat{y}^j(p^*)$$
$$= p^* \cdot \sum_{i=1}^{n} \omega^i + \sum_{j}^{m} p^* \cdot \hat{y}^j(p^*)$$

since $\sum_{i=1}^{n} \theta_j^i = 1$, for each producer j (because θ_j^i is the share of firm j owned by consumer i). So now we have

$$p^* \cdot \sum_{i=1}^{n} \tilde{x}^i > p^* \cdot \sum_{i=1}^{n} \omega^i + \sum_{j}^{m} p^* \cdot \hat{y}^j(p^*)$$

Also, for each producer j

$$p^* \cdot \hat{y}^j(p^*) \geq p^* \cdot \tilde{y}^j$$

since $\hat{y}^j(p^*)$ maximizes profit. And then

$$p^* \cdot \sum_{i=1}^{n} \tilde{x}^i > p^* \cdot \sum_{i=1}^{n} \omega^i + \sum_{j}^{m} p^* \cdot \tilde{y}^j$$

or

$$p^* \cdot \left(\sum_{i=1}^{n} \tilde{x}^i - \sum_{i=1}^{n} \omega^i - \sum_{j}^{m} \tilde{y}^j \right) > 0$$

But feasibility means

$$\sum_{i=1}^{n} \tilde{x}^i - \sum_{i=1}^{n} \omega^i - \sum_{j}^{m} \tilde{y}^j = 0$$

which implies

$$p^* \cdot \left(\sum_{i=1}^{n} \tilde{x}^i - \sum_{i=1}^{n} \omega^i - \sum_{j}^{m} \tilde{y}^j \right) = 0$$

So there is a contradiction, meaning that there is in fact no alternative feasible allocation that is superior to the competitive equilibrium in the sense that it makes some people better off without making someone worse off. In other words, the competitive equilibrium is Pareto optimal.

This is an elementary result. The second welfare theorem is deeper.

Second Welfare Theorem. Any Pareto Optimal allocation can be implemented as a competitive equilibrium, given some redistribution of the endowments.

This can be illustrated using an Edgeworth Box for an exchange economy.