Contraction Mapping Theorem (Banach Fixed Point Theorem)

Let (X, d) be a complete metric space. Suppose $f : X \to X$ and

$$d(f(x), f(y)) < cd(x, y)$$

for all $x, y \in X$, with $c < 1$ (that is, f is Lipschitz, with modulus less than 1). Then f has a unique fixed point, and it is reached by iterating the function from any starting point.

Continuity. Any Lipschitz function is continuous:
suppose $x_n \to a$. For $\varepsilon < 0$ choose N such that $d(x_n, a) < \varepsilon$ for $n \geq N$. Then

$$\begin{aligned} d(f(x_n), f(a)) &< cd(x_n, a) \\ &< \varepsilon \end{aligned}$$

so $f(x_n) \to f(a)$

Cauchy. Let y_n be the sequence defined by iterating (repeatly applying) the function f, starting from some point y_0.

$$y_n = f(y_{n-1})$$

$$\begin{aligned} d(y_2, y_1) &= d(f(y_1), f(y_0)) \\ &< cd(f(y_0), y_0) \end{aligned}$$

$$\begin{aligned} d(y_3, y_2) &= d(f(y_2), f(y_1)) \\ &< cd(y_2, y_1) \\ &< c^2 d_0 \end{aligned}$$

$$d(y_{n+1}, y_n) < c^n d_0$$

$$\begin{aligned} d(y_{n+k}, y_n) &< \left(c^n + c^{n+1} + \dots c^{n+k-1}\right) d_0 \\ &= c^n \left(1 + c + \dots c^{k-1}\right) d_0 \\ &\leq c^n \sum_{i=0}^{\infty} c^i d_0 \\ &= \frac{c^n}{1-c} d_0 \end{aligned}$$

so

$$d(y_m, y_n) < \varepsilon$$

if

$$\frac{c^n}{1-c} d_0 \leq \varepsilon$$

but this holds if n is large enough, because $0 < c < 1$.

So this is a Cauchy sequence, and therefore it converges, because X is a complete metric space.

Uniqueness. There can't be two distinct fixed points, because if x and y are both fixed points then

$$d(f(x), f(y)) < cd(x, y)$$

and then since these are fixed points

$$d(x, y) < cd(x, y)$$

which is impossible unless the distance is zero (meaning $x = y$).

Iteration. It remains only to show that the limit of the iteration sequence is in fact the fixed point of f (regardless of where the sequence starts).

$$y_n \rightarrow y_0$$

so

$$f(y_n) \rightarrow f(y_0)$$

because f is continuous. But

$$f(y_n) = y_{n+1}$$

and

$$y_{n+1} \rightarrow y_0$$

so

$$f(y_0) = y_0$$