Compound Interest

If the interest rate is 100% per period, and interest is paid in each of n subperiods (of equal length), with compounding, the amount at the end of one period is

$$x_n = \left(1 + \frac{1}{n}\right)^n$$

Bernoulli showed that this sequence converges to a limit (the case of continuous compounding), and this limit has come to be known as e.

$$\left(1 + \frac{1}{n}\right)^n \to e$$

To show that the sequence converges, it is enough to show that it is increasing, and that it is bounded (by the Monotone Convergence Theorem).

Monotonicity

Successive terms in the sequence can be compared using the binomial expansion formula. For any number b,

$$(1+b)^{n+1} = \sum_{i=0}^{n+1} \frac{(n+1)!}{i!\,(n-i+1)!} b^i$$

so for numbers a and b,

$$\begin{aligned}
(1+b)^{n+1} - (1+a)^n &= b^{n+1} + \sum_{i=0}^{n} \left(\frac{(n+1)!}{i!\,(n-i+1)!} b^i - \frac{n!}{i!\,(n-i)!} a^i \right) \\
&= b^{n+1} + \sum_{i=0}^{n} \frac{n!}{i!\,(n-i)!} \left(\frac{n+1}{n-i+1} b^i - a^i \right)
\end{aligned}$$

Let $b = \frac{1}{n+1}$ and $a = \frac{1}{n}$. Then

$$(1+b)^{n+1} - (1+a)^n = b^{n+1} + \sum_{i=0}^{n} \frac{n!}{i!\,(n-i)!} \left(\frac{1}{n-i+1} \frac{1}{(n+1)^{i-1}} - \frac{1}{n^i} \right)$$

To show that the expression on the right is positive, it is enough to show that the terms in parentheses are all positive, which means that

$$\frac{1}{n-i+1} \frac{1}{(n+1)^{i-1}} \geq \frac{1}{n^i}$$

for $0 \leq i \leq n$. Equivalently, show that

$$\left(\frac{n-i+1}{n}\right) \left(\frac{n+1}{n}\right)^{i-1} \leq 1$$

If $i = 0$, the left side is

$$\left(\frac{n+1}{n}\right)\left(\frac{n+1}{n}\right)^{-1} = 1$$

So if it can be shown that the left side decreases as i increases, the result follows. Let R_i be the ratio of the left side at $i+1$ divided by the left side at i, i.e.

$$R_i = \frac{\left(\frac{n-i}{n}\right)\left(\frac{n+1}{n}\right)^i}{\left(\frac{n-i+1}{n}\right)\left(\frac{n+1}{n}\right)^{i-1}}$$

Then

$$\begin{aligned} R_i &= \frac{(n-i)\left(\frac{n+1}{n}\right)}{(n-i+1)} \\ &= \frac{(n-i)(n+1)}{n(n-i+1)} \end{aligned}$$

If $R_i \leq 1$,the left side of the inequality is decreasing in i, as required. This condition can be written as

$$n(n-i+1) \geq (n-i)(n+1)$$

or

$$n^2 - in + n \geq n^2 - in + n - i$$

And this inequality holds, for $i \geq 0$. This shows that the original inequality holds, for $n \geq 1$ and $0 \leq i \leq n$, so the compound interest sequence $\left(1 + \frac{1}{n}\right)^n$ is increasing.

Boundedness

Each term in the sequence can be expanded as

$$\left(1 + \frac{1}{n}\right)^n = \sum_{i=0}^{n} u_i$$

where

$$u_i = \frac{n!}{i!(n-i)!}\frac{1}{n^i}$$

So

$$\begin{aligned} \frac{u_{i+1}}{u_i} &= \frac{i!(n-i)!}{(i+1)!(n-i-1)!}\frac{n^i}{n^{i+1}} \\ &= \frac{n-i}{i+1}\frac{1}{n} \\ &< \frac{1}{i+1} \\ &\leq \frac{1}{2} \end{aligned}$$

for $i > 0$. Then

$$\begin{aligned} \frac{u_{i+k}}{u_i} &= \frac{u_{i+1}}{u_i}\frac{u_{i+2}}{u_{i+1}}\cdots\frac{u_{i+k}}{u_{i+k-1}} \\ &\leq \frac{1}{2^k} \end{aligned}$$

for $i > 0$. And

$$\begin{aligned} \left(1+\frac{1}{n}\right)^n &= u_0 + u_1 \sum_{k=0}^{n-1} \frac{u_{1+k}}{u_1} \\ &\leq u_0 + u_1 \sum_{k=0}^{n-1} \frac{1}{2^k} \\ &< u_0 + u_1 \sum_{k=0}^{\infty} \frac{1}{2^k} \\ &= 1 + 2 \end{aligned}$$

so the sequence is bounded above (by 3).