Constant Returns

The cost function is linear in output in the case of constant returns:

$$c(w,q) = qc(w,1)$$

for all input price vectors w.

To prove this, first note that $c(w,1) = w \cdot z^1$ for some production plan z^1 such that $\left(1, -z^1\right) \in Y$. Then by constant returns the production plan $\left(q, -qz^1\right) \in Y$, and this plan produces q units of output at a cost of $w \cdot \left(qz^1\right) = qc(w,1)$. This proves that $c(w,q) \leq qc(w,1)$. Also, $c(w,q) = w \cdot z^q$ for some production plan z^q such that $(q, -z^q) \in Y$. Then by constant returns the production plan $\left(1, -\frac{1}{q}z^q\right) \in Y$, and this plan produces 1 unit of output at a cost of $\frac{1}{q}c(w,q)$. This proves that $c(w,1) \leq \frac{1}{q}c(w,q)$, so $qc(w,1) \leq c(w,q)$

This implies that average and marginal costs are equal.

CES Cost Function

Average Cost

$$c = AC = \sum_{\ell=1}^{L} \frac{w_\ell z_\ell}{q}$$

Marginal Cost

$$c = MC = \frac{w_\ell}{MP_\ell}$$

Production Function

$$\frac{q^\rho - 1}{\rho} = \sum_{\ell=1}^{L} \frac{\theta_\ell (z_\ell)^\rho - 1}{\rho}$$

Marginal (and average) Products

$$q^{\rho-1} MP_\ell = \theta_\ell (z_\ell)^{\rho-1}$$

$$MP_\ell = \theta_\ell (AP_\ell)^{1-\rho}$$

$$\left(\frac{MP_\ell}{\theta_\ell}\right)^\sigma = AP_\ell$$

so from the marginal cost formula above

$$\begin{aligned} c^{-\sigma} &= \left(\frac{MP_\ell}{\theta_\ell}\frac{\theta_\ell}{w_\ell}\right)^\sigma \\ &= AP_\ell \left(\frac{\theta_\ell}{w_\ell}\right)^\sigma \end{aligned}$$

and the average cost formula can be written as

$$c = \sum_{\ell=1}^{L} \frac{w_\ell}{AP_\ell}$$

so

$$c^{1-\sigma} = \sum_{\ell=1}^{L} \frac{w_\ell}{AP_\ell} AP_\ell \left(\frac{\theta_\ell}{w_\ell}\right)^{\sigma}$$

and

$$c^{1-\sigma} = \sum_{\ell=1}^{L} \theta_\ell \left(\frac{w_\ell}{\theta_\ell}\right)^{1-\sigma}$$

or

$$c^{1-\sigma} = \sum_{\ell=1}^{L} (\theta_\ell)^{\sigma} (w_\ell)^{1-\sigma}$$