Solutions to Question 5.

(a) Clearly, the unique subgame perfect equilibrium outcome is (Barrier, No Entry). The equilibrium strategy is for 1 to pick "Barrier," and for 2 to pick "No Entry" if 1 picked "Barrier" and "Entry" if 1 picked "No Barrier."

(b) The unique subgame perfect equilibrium is the unique Nash equilibrium in the simultaneous move game, which is (No Barrier, Entry).

(c) In a pure strategy equilibrium, firm 1 either errects the barrier or not. Suppose that it does. Given a consistent belief, firm 2 will not enter. But, given this plan by firm 2, it pays firm 1 to deviate. Deviating will affect the probability of the signal being generated, but it does not matter in a pure strategy equilibrium since firm 2 will believe that the (candidate) equilibrium strategy will be played. Therefore, firm 1 does not pick "Barrier" in the equilibrium. Suppose now that firm 1 chooses "No Barrier." Given the consistent belief, firm 2 will choose "Entry." Given this action plan by firm 2, it doesn't pay firm 1 to deviate. In sum, the only pure strategy (perfect Bayesian) equilibrium is (No Barrier, Entry). Obviously, this equilibrium does not depend on the value of σ and never approaches to the one found in (a).

(d) Let p denote the probability with which firm 1 picks "Barrier," and q denote the probability with which firm 2 picks "No Entry" when observing n. When observing b, firm 2 chooses "No Entry" with probability 1. The following three conditions must be satisfied.

First, firm 1 must be indifferent in its action. When firm 1 chooses "Barrier," then with probability σ, the signal observed by firm 2 is b. Thus, its payoff is

$$3\sigma + (1-\sigma)(3q)). \tag{1}$$

Similarly, if firm 1 chooses "No Barrier," then its payoff is

$$\sigma[4q + 2(1-q)] + 4(1-\sigma). \tag{2}$$

Equating (1) to (2), one gets

$$q = \frac{5\sigma - 4}{5\sigma - 3}.$$

Second, firm 2 must at least weakly prefer "No Entry" when observing b. When observing b and given firm 1's strategy, the prosterier belief held by firm 2 that firm 1 chose "Barrier" is obtained (by the Bayes rule) as:

$$\rho \equiv \frac{\sigma p}{\sigma p + (1-\sigma)(1-p)}.$$

Therefore, if firm 2 chooses "No Entry," its payoff is

$$\rho \cdot 0 + (1-\rho) \cdot 0 = 0. \tag{3}$$

If firm 2 chooses "Entry," then it receives:

$$\rho(-1) + 2(1-\rho). \tag{4}$$

For firm 2 to at least weakly prefer "No Entry," (3) must weakly exceed (4), which gives

$$p \geq \frac{2(1-\sigma)}{2-\sigma}. \tag{5}$$

Finally, firm 2 must be indifferent in its action, when observing n. When observing n and given firm 1's equilibrium strategy, firm 2's posterior belief that firm 1 has chosen "No Barrier" is obtained, by the Bayes rule, as:

$$\alpha \equiv \frac{\sigma(1-p)}{\sigma(1-p) + (1-\sigma)p}.$$

If firm 2 chooses "No Entry," its payoff is again 0. If it chooses "Entry," its payoff is

$$2\alpha + (-1)(1-\alpha).$$

Equating this to zero, one gets

$$p = \frac{2\sigma}{1+\sigma}.$$

Observe that this p satisfies (5). Therefore,

$$(p,q) = \left(\frac{2\sigma}{1+\sigma}, \frac{5\sigma-4}{5\sigma-3}\right)$$

is shown to be the mixed strategy equilibrium. Note that the equilibrium approaches to the one found in (a), as σ approaches 1; i.e., the signal becomes perfectly informative. (The second component, q, approaches 1/2. But in equilibrium, p approaches 1, so as σ goes to 1, the probability of $s = n$ goes to zero. That is, the equilibrium outcome approaches the one in (a).)

Solution to Question 6.

(a) A pair (e_1, e_2) is Pareto efficient if and only if it satisfies:

$$\max_{e_1 \geq 0, e_2 \geq 0, w_1, w_2} w_1 - e_1^2$$

subject to

$$w_2 - e_2^2 \geq U_2,$$

and

$$w_1 + w_2 \leq e_1 + e_2 + e_1 e_2.$$

for some $U_2 \in \Re$. The constraint must be binding, since otherwise the value of the objective function can be raised without violating the constraint. Substituting the constraints (with equality) into the objective function, the problem reduces to:

$$\max_{e_1 \geq 0, e_2 \geq 0} e_1 + e_2 + e_1 e_2 - e_1^2 - e_2^2.$$

Its first order conditions then yield $e_1 = e_2 = 1$.

(b) In a Nash equilibrium, individual i solves:

$$\max_{e_i \geq 0} \frac{1}{2}[e_1 + e_2 + e_1 e_2] - e_i^2.$$

Solving the two first order conditions simultaneously, one gets

$$e_1 = e_2 = 1/3.$$

(c) Consider the following strategy profile: start and keep playing the Pareto efficient pair, unless an individual deviates unilaterally, in which case the individuals play the one shot Nash repeatedly. If both individuals pick their Pareto efficient pair, then each individual gets the payoff of 1/2. If an individual deviates, he will choose e_i to maximize

$$\frac{1}{2}[e_i + 1 + e_i] - e_i^2.$$

The deviation effort level is $1/2$, and the maximized deviation payoff is $3/4$. Finally, when the individuals play the one shot Nash, each gets the payoff of $5/18$.

For such a strategy to be subgame perfect, it is necessary and sufficient for the following condition to hold:

$$\frac{1}{2} \geq (1-\delta)\frac{3}{4} + \delta\frac{5}{18}.$$

This condition holds if and only if $\delta \geq 9/17$.

(d) Let $e(\theta)$, $\theta \in (0,1]$, denote the common strategy adopted by each individual in the Bayesian Nash equilibrium. Then, for any $\theta \in (0,1]$, $e(\theta)$ must be the e_i that maximizes

$$\begin{aligned} &\int_0^1 \frac{1}{2}[e_i + e(\theta) + e_i e(\theta)]d\theta - \frac{e_i^2}{\theta} \\ =&\frac{1}{2}[e_i + e^* + e^* e_i] - \frac{e_i^2}{\theta}, \end{aligned}$$

where $e^* = E[e(\theta)]$. The first order condition for this maximization problem gives

$$e(\theta) = \frac{\theta(1+e^*)}{4}. \tag{1}$$

Now, applying the definition of e^* to (1), one gets

$$e^* = \int_0^1 e(\theta)d\theta = \int_0^1 \frac{\theta(1+e^*)}{4}d\theta = \frac{1+e^*}{8},$$

so we conclude that $e^* = 1/7$. Substituting this into (1), we get

$$e(\theta) = \frac{2\theta}{7}.$$